

A PARABOLIC FLOW OF PLURICLOSED METRICS

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ABSTRACT. We define a parabolic flow of pluriclosed metrics. This flow is of the same family introduced by the authors in [11]. We study the relationship of the existence of the flow and associated static metrics topological information on the underlying complex manifold. Solutions to the static equation are automatically Hermitian-symplectic, a condition we define herein. These static metrics are classified, on K3 surfaces, complex toroidal surfaces, nonminimal Hopf surfaces, surfaces of general type, and Class VII⁺ surfaces. To finish we discuss how the flow may potentially be used to study the topology of Class VII⁺ surfaces.

1. INTRODUCTION

In [11] the authors introduced a class of elliptic equations for Hermitian metrics with associated parabolic flows. A particular equation of this type was singled out as being the unique such elliptic equation arising as the Euler-Lagrange equation of a functional. The purpose of this article is to identify another equation in this same class related to pluriclosed metrics.

Definition 1.1. Let (M^{2n}, g, J) be a complex manifold with Hermitian metric g and Kähler form $\omega = g(J\cdot, \cdot)$. We say that ω is *pluriclosed* if

$$(1.1) \quad \partial\bar{\partial}\omega = 0.$$

A result of Gauduchon [9] says that every conformal class of a Hermitian metric on a complex $2n$ -manifold has a unique element satisfying

$$\partial\bar{\partial}\omega^{n-1} = 0.$$

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We say that a metric satisfying this condition has *null-eccentricity*. In the case $n = 2$ the null-eccentricity condition agrees with the pluriclosed condition. Therefore every complex surface admits pluriclosed metrics. The condition is considerably more restrictive in higher dimensions, though it is still in a sense a determined equation for a complex 6-manifold. Some of the results in this paper will apply to pluriclosed metrics in any dimension, but our main focus will be on complex surfaces.

Before writing the definition of the flow let us briefly recall some definitions related to the Chern connection. In particular, let Ω denote the curvature of the Chern connection ∇ associated to g and let

$$S_{k\bar{l}} = g^{i\bar{j}} \Omega_{i\bar{j}k\bar{l}}.$$

Further, let

$$T_{i\bar{j}\bar{k}} = \partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}}$$

denote the torsion of ∇ and define

$$(1.2) \quad \begin{aligned} Q_{i\bar{j}}^1 &= g^{k\bar{l}} g^{m\bar{n}} T_{ik\bar{n}} T_{j\bar{l}m} \\ Q_{i\bar{j}}^2 &= g^{k\bar{l}} g^{m\bar{n}} T_{l\bar{n}i} T_{km\bar{j}}. \end{aligned}$$

The tensor Q^2 will be used in certain evolution equations we derive later. In this paper we will study the evolution equation

$$(1.3) \quad \begin{aligned} \frac{\partial}{\partial t} g &= -S + Q^1 \\ g(0) &= g_0. \end{aligned}$$

We note here that for certain applications a volume normalized version of (1.3) will be useful. In particular one can add a scalar term to the evolution equation to fix the volume of the manifold. Setting $s = \text{tr}_g S$ and noting that $\text{tr}_g Q^1 = |T|^2$, this equation takes the form

$$(1.4) \quad \begin{aligned} \frac{\partial}{\partial t} g &= -S + Q^1 + \frac{1}{n} \left(\int_M s - |T|^2 \right) g \\ g(0) &= g_0. \end{aligned}$$

These equations are of the form studied in [11], and so the regularity theory developed in that paper applies immediately to (1.3). We will see in section 3 that this flow preserves the pluriclosed condition. Furthermore, as observed in [11], if the initial condition is Kähler the resulting family of metrics is a solution to Kähler-Ricci flow. Moreover, in section 2 we will write this equation using Hodge-type operators. In

particular a solution to (1.3) with pluriclosed initial condition is equivalent to a solution of

$$(1.5) \quad \begin{aligned} \frac{\partial}{\partial t} \omega &= \partial \bar{\partial}^* \omega + \bar{\partial} \partial^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \\ \omega(0) &= \omega_0. \end{aligned}$$

We prove the following basic regularity theorem in section 3.

Theorem 1.2. *Let (M^{2n}, g, J) be a compact complex manifold with pluriclosed metric g . There exists a constant $c(n)$ depending only on n such that there exists a unique solution $g(t)$ to (1.3) for*

$$t \in \left[0, \frac{c(n)}{\max\{|\Omega|_{C^0(g_0)}, |\nabla T|_{C^0(g_0)}, |T|_{C^0(g_0)}^2\}} \right].$$

Moreover, there exist constants C_m depending only on m such that the estimates

$$|\nabla^m \Omega|_{C^0(g_t)}, |\nabla^{m+1} T|_{C^0(g_t)} \leq \frac{C_m \max\{|\Omega|_{C^0(g_0)}, |\nabla T|_{C^0(g_0)}, |T|_{C^0(g_0)}^2\}}{t^{m/2}}$$

hold for all t in the above interval. Moreover, if $g(0)$ is pluriclosed the metric $g(t)$ is pluriclosed for all t and is a solution to (1.5). If furthermore $g(0)$ is Kähler, then $g(t)$ is Kähler for all time and $g(t)$ solves Kähler-Ricci flow.

A further consequence of this regularity is the following basic long-time existence obstruction.

Theorem 1.3. *Let $(M^{2n}, g(t), J)$ be a solution to (1.3). Let τ denote the maximal existence time of the flow. If $\tau < \infty$, then*

$$\limsup_{t \rightarrow \tau} \max \left\{ |\Omega|_{C^0(g_t)}, |\nabla T|_{C^0(g_t)}, |T|_{C^0(g_t)}^2 \right\} = \infty.$$

As it turns out, this regularity result can be significantly improved in the case where $n = 2$. In particular carefully analyzing the evolution of the torsion and its covariant derivative on a complex surface allows us to conclude that a bound on the Chern curvature suffices to show existence of the flow.

Theorem 1.4. *Let $(M^4, g(t), J)$ be a solution to (1.3). Let τ denote the maximal existence time of the flow. If $\tau < \infty$, then*

$$\limsup_{t \rightarrow \tau} |\Omega|_{C^0(g_t)} = \infty.$$

Next we classify static solutions of (1.5) on certain complex surfaces. In particular, given (M^{2n}, g, J) a complex manifold with pluriclosed metric, we say that g is *static* if

$$(1.6) \quad -\partial\bar{\partial}^*\omega - \bar{\partial}\partial^*\omega - \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g = \lambda\omega.$$

We note that if g is Kähler and static then it is Kähler-Einstein. The reason for the sign on the left hand side is to make the sign of λ agree with the usual sign for KE-metrics. On a complex surface the existence and signs of static metrics are closely related to the algebraic and topological structure of the manifold. As we will see in section 4, static metrics with nonzero constant automatically imply the existence of a structure we call Hermitian-symplectic. Let us define this condition and expound on it.

Definition 1.5. Consider (M^{2n}, J) a complex manifold. A *Hermitian-symplectic structure* on M is a real two-form ω such that $d\omega = 0$, and $\omega^{(1,1)} > 0$, i.e. the projection of ω onto $(1, 1)$ -tensors determined by J is positive definite. We say that a complex manifold is *Hermitian-symplectic* if it admits a Hermitian-symplectic structure.

This shows that static metrics do indeed carry a lot of structure with them. It is known that the space of symplectic manifolds is strictly larger than the space of Kähler manifolds. However, we do not know of an example or proof to see if the space of Hermitian-symplectic manifolds is strictly larger than the space of Kähler manifolds. Indeed, one can make the following observation.

Proposition 1.6. *A complex surface is Hermitian-symplectic if and only if it is Kähler.*

Proof. A Kähler structure is automatically Hermitian-symplectic, which shows one direction. Suppose (M^4, J) is a complex surface and ω is a Hermitian-symplectic form on M . Suppose for a contradiction that M is not a Kähler manifold. By the signature theorem the intersection form of a non-Kähler surface is negative definite. However, ω represents an element of $H^2(M)$ with positive self-intersection. This is a contradiction. \square

Therefore, any example of a complex manifold admitting a Hermitian-symplectic structure but no Kähler structure must be of dimension higher than 2. We state this question formally for emphasis.

Question 1.7. Do there exist complex manifolds $(M^{2n}, J), n \geq 3$, such that M carries a Hermitian-symplectic structure ω but no Kähler structure?

Let us now discuss the main results on static metrics. As consequences of the structural results in section 4 we give various classification results for static metrics. In particular, we show that static metrics on K3 surfaces, two-dimensional complex tori, and surfaces of general type must automatically be Kähler-Einstein. Finally we can completely determine the question of existence of static metrics on Class VII⁺ surfaces (i.e. $b_1(M) = 1$ with $b_2(M) > 0$).

Theorem 1.8. *Let (M^4, J) be a complex surface of Class VII⁺. Then M does not admit a static metric.*

This theorem has important consequences in applying equation (1.3) to the study of the topology of class VII⁺ surfaces. We will discuss this in section 6.

Here is an outline of the rest of the paper. In section 2 we give some general background calculations for Hermitian metrics. In section 3 we record the basic existence and regularity results for (1.5), which are mainly consequences of the general regularity theory developed in [11]. Also we record a number of evolution equations for various integral quantities along solutions to (1.5). In section 4 we give the improved regularity results in the case of a complex surface. Next in section 5 we derive certain equations and inequalities satisfied by static metrics on surfaces, and classify static metrics on certain complex surfaces. Finally in section 6 we recall a basic structural theorem for Class VII⁺ surfaces and give the proof of Theorem 1.8.

2. BACKGROUND CALCULATIONS

In this section we fix notation and provide some basic calculations for standard objects related to Hermitian geometry. Fix (M^{2n}, g, J) a complex manifold with Hermitian metric g . Let

$$\omega(u, v) = g(Ju, v)$$

be the Kähler form of g . In local complex coordinates we have

$$\omega = \frac{\sqrt{-1}}{2} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Let

$$\Lambda^k = \bigoplus_{p+q=k} \Lambda^{p,q}$$

denote the usual decomposition of complex differential two-forms into forms of type (p, q) . The exterior differential d decomposes into the

operators ∂ and $\bar{\partial}$

$$\begin{aligned}\partial &: \Lambda^{p,q} \rightarrow \Lambda^{p+1,q} \\ \bar{\partial} &: \Lambda^{p,q} \rightarrow \Lambda^{p,q+1}.\end{aligned}$$

Also the operator d_g^* , the L^2 adjoint of d , decomposes into ∂_g^* and $\bar{\partial}_g^*$

$$\begin{aligned}\partial_g^* &: \Lambda^{p+1,q} \rightarrow \Lambda^{p,q} \\ \bar{\partial}_g^* &: \Lambda^{p,q+1} \rightarrow \Lambda^{p,q}.\end{aligned}$$

Lemma 2.1. *Given g a Hermitian metric we have in complex coordinates*

$$(2.1) \quad (\partial_g^* \omega)_{\bar{k}} = \frac{\sqrt{-1}}{2} g^{p\bar{q}} (\partial_{\bar{q}} g_{p\bar{k}} - \partial_{\bar{k}} g_{p\bar{q}}),$$

$$(2.2) \quad (\bar{\partial}_g^* \omega)_j = \frac{\sqrt{-1}}{2} g^{p\bar{q}} (\partial_j g_{p\bar{q}} - \partial_p g_{j\bar{q}}).$$

Proof. We compute using integration by parts. Given $\alpha \in \Lambda^{0,1}$ we have

$$\begin{aligned}(\partial_g^* \omega, \alpha) &= (\omega, \partial \alpha) \\ &= \int_M g^{\bar{k}l} g^{\bar{i}j} (\omega_{j\bar{k}} \bar{\partial} \alpha_{i\bar{l}}) \bar{g} \\ &= \frac{\sqrt{-1}}{2} \int_M g^{\bar{i}l} (\bar{\alpha}_{l,i}) \bar{g} \\ &= -\frac{\sqrt{-1}}{2} \int_M \bar{\alpha}_{\bar{l}} [\partial_{\bar{i}} (g^{\bar{i}l} \bar{g})] \\ &= -\frac{\sqrt{-1}}{2} \int_M \bar{\alpha}_{\bar{l}} (\bar{g}) \left[-g^{\bar{i}m} \partial_{\bar{i}} g_{m\bar{n}} g^{\bar{n}l} + g^{\bar{i}l} \frac{1}{\bar{g}} \partial_{\bar{i}} \bar{g} \right].\end{aligned}$$

This gives the first formula, and the second follows analogously. \square

Lemma 2.2. *Given g a Hermitian metric we have in complex coordinates*

$$(2.3) \quad (\partial \partial_g^* \omega)_{j\bar{k}} = \frac{\sqrt{-1}}{2} [g^{p\bar{q}} (g_{p\bar{k},\bar{q}j} - g_{p\bar{q},\bar{k}j}) - g^{p\bar{m}} g^{n\bar{q}} g_{\bar{m}n,j} (g_{p\bar{k},\bar{q}} - g_{p\bar{q},\bar{k}})]$$

and

$$(\bar{\partial} \bar{\partial}_g^* \omega)_{j\bar{k}} = \frac{\sqrt{-1}}{2} [g^{p\bar{q}} (g_{j\bar{q},p\bar{k}} - g_{p\bar{q},j\bar{k}}) - g^{p\bar{m}} g^{n\bar{q}} g_{n\bar{m},\bar{k}} (g_{j\bar{q},p} - g_{p\bar{q},j})].$$

Proof. In general for $\alpha \in \Lambda^{0,1}$ we have

$$(\partial \alpha)_{j\bar{k}} = \partial_j \alpha_{\bar{k}}.$$

Thus we compute using Lemma 2.1

$$\begin{aligned} (\partial\bar{\partial}^*\omega)_{j\bar{k}} &= \frac{\sqrt{-1}}{2} \partial_j (g^{p\bar{q}} (\partial_{\bar{q}} g_{p\bar{k}} - \partial_{\bar{k}} g_{p\bar{q}})) \\ &= \frac{\sqrt{-1}}{2} [g^{p\bar{q}} (g_{p\bar{k},\bar{q}j} - g_{p\bar{q},\bar{k}j}) - g^{p\bar{m}} g_{\bar{m}n,j} g^{n\bar{q}} (g_{p\bar{k},\bar{q}} - g_{p\bar{q},\bar{k}})]. \end{aligned}$$

The result follows. The second follows analogously. \square

Lemma 2.3. *Given g a Hermitian metric we have in complex coordinates*

$$\left(\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right)_{j\bar{k}} = \frac{\sqrt{-1}}{2} (g^{p\bar{q}} g_{p\bar{q},j\bar{k}} - g^{p\bar{r}} g^{s\bar{q}} g_{\bar{r}s,j} g_{p\bar{q},\bar{k}})$$

Proof. We compute directly in coordinates

$$\begin{aligned} \left(\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right)_{j\bar{k}} &= \frac{\sqrt{-1}}{2} \partial_j (g^{p\bar{q}} \partial_{\bar{k}} g_{p\bar{q}}) \\ &= \frac{\sqrt{-1}}{2} (g^{p\bar{q}} \partial_j \partial_{\bar{k}} g_{p\bar{q}} - g^{p\bar{r}} \partial_j g_{\bar{r}s} g^{s\bar{q}} \partial_{\bar{k}} g_{p\bar{q}}). \end{aligned}$$

\square

In addition to the Hodge operators we will use two of the Ricci-type curvatures of the Chern curvature. In particular, as we noted in the introduction, define

$$S_{k\bar{l}} = g^{i\bar{j}} \Omega_{i\bar{j}k\bar{l}}.$$

Similarly define

$$P_{i\bar{j}} = g^{k\bar{l}} \Omega_{i\bar{j}k\bar{l}}.$$

Lemma 2.4. *Given (M^{2n}, g, J) a complex manifold one has in complex coordinates the formulas*

$$\begin{aligned} S_{k\bar{l}} &= g^{i\bar{j}} (-g_{k\bar{l},i\bar{j}} + g^{m\bar{n}} g_{k\bar{n},i} g_{\bar{l}m,\bar{j}}) \\ P_{i\bar{j}} &= g^{k\bar{l}} (-g_{k\bar{l},i\bar{j}} + g^{m\bar{n}} g_{k\bar{n},i} g_{\bar{l}m,\bar{j}}). \end{aligned}$$

Proof. These both follow from the general formula for the Chern curvature given by

$$\Omega_{i\bar{j}k\bar{l}} = -g_{k\bar{l},i\bar{j}} + g^{m\bar{n}} g_{k\bar{n},i} g_{\bar{l}m,\bar{j}}.$$

\square

3. EXISTENCE AND REGULARITY

In this section we record some basic existence and regularity results for solutions to (1.3). In particular we will see that this flow preserves the pluriclosed condition, and is equivalent to (1.5) when the initial condition is pluriclosed. For notational convenience set

$$(3.1) \quad \Phi(\omega) = -\partial\bar{\partial}^*\omega - \overline{\partial\bar{\partial}^*}\omega - \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g.$$

Proposition 3.1. *Let (M^{2n}, g, J) be a complex manifold with pluriclosed metric. Let $H = \{g \in \text{Sym}^2(T^*M) \mid g \text{ compatible with } J, \partial\bar{\partial}\omega = 0\}$. Then the operator*

$$\omega \rightarrow \Phi(\omega)$$

is a real quasi-linear second-order elliptic operator when restricted to H .

Proof. Combining Lemmas 2.2, and 2.3 we see

$$\begin{aligned} \Phi(\omega)_{j\bar{k}} &= -\frac{\sqrt{-1}}{2}g^{p\bar{q}} [g_{p\bar{k},\bar{q}j} + g_{j\bar{q},p\bar{k}} - g_{p\bar{q},j\bar{k}}] \\ &\quad + \frac{\sqrt{-1}}{2}g^{p\bar{m}}g^{n\bar{q}} [g_{\bar{m}n,j} (g_{p\bar{k},\bar{q}} - g_{p\bar{q},\bar{k}}) + g_{n\bar{m},\bar{k}} (g_{j\bar{q},p} - g_{p\bar{q},j}) \\ &\quad \quad \quad + g_{p\bar{q},j}g_{\bar{m}n,\bar{k}}] \\ &= -\frac{\sqrt{-1}}{2} [g^{p\bar{q}} (g_{p\bar{k},\bar{q}j} + g_{j\bar{q},p\bar{k}} - g_{p\bar{q},j\bar{k}}) \\ &\quad \quad \quad + g^{p\bar{m}}g^{n\bar{q}} (g_{\bar{m}n,j}g_{p\bar{k},\bar{q}} + g_{n\bar{m},\bar{k}}g_{j\bar{q},p} - g_{p\bar{q},j}g_{\bar{m}n,\bar{k}})]. \end{aligned}$$

This coordinate formula shows that Φ is a real quasi-linear second-order operator. We now compute the symbol of the linearization to show that Φ is elliptic when restricted to H . Fix a point $x \in M$ and suppose the complex coordinates are chosen such that $g_{i\bar{j}}(x) = \delta_{ij}$. Moreover fix a family $g(s)$ of pluriclosed metrics satisfying $\frac{\partial}{\partial s}g(s)|_{s=0} = h$. We can compute the variation of Φ with respect to this family. In particular we have

$$(3.2) \quad \sigma D\Phi(\omega)(h)_{j\bar{k}} = -\sum_{p=1}^n h_{p\bar{k},\bar{p}j} + h_{j\bar{p},p\bar{k}} - h_{p\bar{p},j\bar{k}}.$$

Note that the balancing condition for g passes to the linearization. In particular for $i \neq j$ and $k \neq l$ we have

$$\begin{aligned} 0 &= (\partial\bar{\partial}h(J\cdot, \cdot))_{i\bar{j}\bar{k}\bar{l}} \\ &= h_{i\bar{k},j\bar{l}} + h_{j\bar{l},i\bar{k}} - h_{i\bar{l},j\bar{k}} - h_{j\bar{k},i\bar{l}}. \end{aligned}$$

Suppose $j, k > 1$. Applying the balancing condition we can write

$$h_{1\bar{k},1\bar{j}} = h_{1\bar{1},j\bar{k}} + h_{j\bar{k},1\bar{1}} - h_{j\bar{1},1\bar{k}}.$$

Therefore

$$(3.3) \quad \sigma D\Phi(\omega)(h)_{j\bar{k}} = -h_{j\bar{k},1\bar{1}} - \sum_{p=2}^n h_{p\bar{k},p\bar{j}} + h_{j\bar{p},p\bar{k}} - h_{p\bar{p},j\bar{k}}.$$

Now take the Fourier transform of the linear operator $D\Phi(\omega)$, and further rotate coordinates such that the Fourier variable ξ satisfies $\xi = (1, 0, \dots, 0)$. Now we can compute directly using (3.2)

$$[\sigma D\Phi(\omega)]^\wedge(h)(\xi)_{1\bar{1}} = -|\xi|^2 h_{1\bar{1}}.$$

Likewise for $k > 1$ using (3.2) we compute

$$[\sigma D\Phi(\omega)]^\wedge(h)(\xi)_{1\bar{k}} = -|\xi|^2 h_{1\bar{k}}.$$

Finally for $j, k > 1$ we compute using (3.3)

$$[\sigma D\Phi(\omega)]^\wedge(h)(\xi)_{j\bar{k}} = -|\xi|^2 h_{j\bar{k}}.$$

Therefore Φ is elliptic, and the result follows. \square

Next we express equation (1.5) using the curvature and torsion of the Chern connection. Let w denote the trace of the torsion. In particular we have in coordinates

$$w_i = g^{j\bar{k}} T_{ij\bar{k}}.$$

Proposition 3.2. *Let (M^{2n}, g, J) be a complex manifold with pluriclosed metric. Then*

$$\nabla w = -\operatorname{div}^\nabla T - Q^1.$$

Proof. Note that the pluriclosed condition implies

$$\partial_{\bar{i}} T_{jk\bar{l}} = \partial_{\bar{l}} T_{jk\bar{i}}$$

for any i, j, k, l . We directly compute

$$\begin{aligned} (\nabla w)_{i\bar{j}} &= g^{p\bar{q}} (\nabla_{\bar{j}} T_{ip\bar{q}}) \\ &= g^{p\bar{q}} \left(\partial_{\bar{j}} T_{ip\bar{q}} - \Gamma_{\bar{j}\bar{q}}^{\bar{r}} T_{ip\bar{r}} \right) \\ &= g^{p\bar{q}} \left(\partial_{\bar{q}} T_{ip\bar{j}} - \Gamma_{\bar{j}\bar{q}}^{\bar{r}} T_{ip\bar{r}} \right) \\ &= g^{p\bar{q}} \left(\nabla_{\bar{q}} T_{ip\bar{j}} + \left(\Gamma_{\bar{q}\bar{j}}^{\bar{r}} - \Gamma_{\bar{j}\bar{q}}^{\bar{r}} \right) T_{ip\bar{r}} \right) \\ &= -(\operatorname{div}^\nabla T)_{i\bar{j}} - Q_{i\bar{j}}^1. \end{aligned}$$

\square

Proposition 3.3. *Let (M^{2n}, g, J) be a solution to (1.5) with pluriclosed initial condition. Then*

$$\frac{\partial}{\partial t} g = -S + Q^1.$$

Proof. Lemma 2.2 implies that

$$\begin{aligned} \partial\bar{\partial}^* \omega(J, \cdot) &= -\nabla w \\ \bar{\partial}\partial^* \omega(J, \cdot) &= -\bar{\nabla} w. \end{aligned}$$

Thus composing the pluriclosed flow equation with J yields

$$\frac{\partial}{\partial t} g = -\nabla w - \bar{\nabla} w - P.$$

In general the Bianchi identity implies

$$P = S + \operatorname{div}^\nabla T - \bar{\nabla} w.$$

Applying the result of Proposition 3.2 yields

$$\operatorname{div}^\nabla T = -\nabla w - Q^1.$$

Therefore

$$P = S - \nabla w - \bar{\nabla} w - Q^1$$

and the result follows. \square

Theorem 3.4. *Let (M^{2n}, g, J) be a compact complex manifold with pluriclosed metric g . There exists a constant $c(n)$ depending only on n such that there exists a unique solution $g(t)$ to (1.3) for*

$$t \in \left[0, \frac{c(n)}{\max\{|\Omega|_{C^0(g_0)}, |\nabla T|_{C^0(g_0)}, |T|_{C^0(g_0)}^2\}} \right].$$

Moreover, there exist constants C_m depending only on m such that the estimates

$$|\nabla^m \Omega|_{C^0(g_t)}, |\nabla^{m+1} T|_{C^0(g_t)} \leq \frac{C_m \max\{|\Omega|_{C^0(g_0)}, |\nabla T|_{C^0(g_0)}, |T|_{C^0(g_0)}^2\}}{t^{m/2}}$$

hold for all t in the above interval. Moreover, if $g(0)$ is pluriclosed the metric $g(t)$ is pluriclosed for all t and is a solution to (1.5). If furthermore $g(0)$ is Kähler, then $g(t)$ is Kähler for all time and $g(t)$ solves Kähler-Ricci flow.

Proof. The general regularity theorem of [11] applies to give the existence statements, and the claim that a Kähler initial condition results in a solution to Kähler-Ricci flow. For the convenience of the reader, we briefly recall these arguments. Lemma 2.4 shows that S is a strictly elliptic operator, and so the short-time existence follows from standard

theory. Furthermore, a judicious application of the Bianchi identities shows that

$$(3.4) \quad \begin{aligned} \frac{\partial}{\partial t} \nabla^k \Omega &= \Delta \nabla^k \Omega + \sum_{j=0}^k \nabla^j T * \nabla^{k+1-j} \Omega + \sum_{j=0}^k \nabla^j \Omega * \nabla^{k-j} \Omega \\ &+ \sum_{j=0}^k \sum_{l=0}^j \nabla^l T * \nabla^{j-l} T * \nabla^{k-j} \Omega. \end{aligned}$$

and

$$(3.5) \quad \begin{aligned} \frac{\partial}{\partial t} \nabla^k T &= \Delta \nabla^k T + \sum_{j=0}^{k+1} \nabla^j T * \nabla^{k+1-j} T + \sum_{j=0}^k \nabla^j T * \nabla^{k-j} \Omega \\ &+ \sum_{j=0}^{k-1} \sum_{l=0}^j \nabla^l T * \nabla^{j-l+1} T * \nabla^{k-1-j} T. \end{aligned}$$

One can apply standard estimates to these equations to derive the derivative estimates.

We show that the pluriclosed condition is preserved. Suppose $g(0)$ is pluriclosed. By Proposition 3.1, Φ is a strictly elliptic operator, therefore (1.5) is a strictly parabolic equation and so short-time existence and uniqueness of the solution to (1.5) with initial condition $g(0)$ follows from standard results since M is compact. We may directly compute to see that (1.5) preserves the pluriclosed condition. Recall the equations $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$. Using this and the fact that the Chern form $\frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g$ is closed, we directly compute

$$\begin{aligned} \frac{\partial}{\partial t} \partial\bar{\partial} \omega &= -\partial\bar{\partial} \Phi(\omega) \\ &= \partial\bar{\partial} \left[\left(\partial\partial^* \omega + \bar{\partial}\bar{\partial}^* \omega \right) + \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right] \\ &= -\bar{\partial}\partial\partial\partial^* \omega + \partial\bar{\partial}\bar{\partial}\bar{\partial}^* \omega \\ &= 0. \end{aligned}$$

Finally, using Proposition 3.3 we see that the solution to (1.5) in fact solves (1.3). Since solutions to (1.3) are unique as noted above, it follows that the solution to (1.3) coincides with the solution to (1.5), and therefore $g(t)$ is pluriclosed for all time. \square

A further consequence of the derivative estimates is a basic long-time existence obstruction.

Theorem 3.5. *Let $(M^{2n}, g(t), J)$ be a solution to (1.5). Let τ denote the maximal existence time of the flow. If $\tau < \infty$, then*

$$\limsup_{t \rightarrow \tau} \max \left\{ |\Omega|_{C^0(g_t)}, |\nabla T|_{C^0(g_t)}, |T|_{C^0(g_t)}^2 \right\} = \infty.$$

In the remainder of the section we compute evolution equations of basic integral quantities and observe that in certain situations they function as monotonic quantities along solutions to (1.5).

Proposition 3.6. *Let $(M^{2n}, g(t), J)$ be a solution to (1.5) with pluriclosed initial condition. Then the volume of $g(t)$ satisfies*

$$\frac{\partial}{\partial t} \text{Vol}(g(t)) = 2 \int_M |\partial^* \omega|^2 - d.$$

Proof. We directly compute

$$\begin{aligned} \frac{\partial}{\partial t} \text{Vol}(g(t)) &= \frac{\partial}{\partial t} \int_M dV_g \\ &= \int_M \text{tr}_\omega \Phi(\omega) dV_g \\ &= \int_M \left\langle \omega, \partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \right\rangle dV_g \\ &= 2 \int_M |\partial^* \omega|^2 - d. \end{aligned}$$

□

Next we compute the evolution of the degree of a line bundle. Recall the definition of degree.

Definition 3.7. Let (M^{2n}, g, J) be a Hermitian manifold. Let

$$(3.6) \quad d = \text{deg}(M) := \int_M \langle c_1(M), \omega \rangle = \int_M \left(-\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \right) \wedge \omega^{n-1}.$$

This is often called the degree of the surface. More generally, given \mathcal{L} a line bundle over M , define

$$(3.7) \quad \text{deg}(\mathcal{L}) := \int_M c_1(\mathcal{L}) \wedge \omega^{n-1}.$$

Proposition 3.8. *Let $(M^4, g(t), J)$ be a solution to pluriclosed flow on a complex surface, and let L be a line bundle over M . Then*

$$\frac{\partial}{\partial t} \text{deg}_{g_t}(L) = -c_1(L) \cdot c_1(M).$$

Proof. We compute directly

$$\begin{aligned} \frac{\partial}{\partial t} \deg_{g_t}(L) &= \frac{\partial}{\partial t} \int_M c_1(L) \wedge \omega_t \\ &= \int_M c_1(L) \wedge \left(\partial\bar{\partial}^* \omega + \bar{\partial}\partial^* \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right). \end{aligned}$$

Since $c_1(L)$ is closed,

$$\int_M c_1(L) \wedge \left(\partial\bar{\partial}^* \omega + \bar{\partial}\partial^* \omega \right) = 0$$

by Stokes Theorem. Likewise

$$\begin{aligned} \int_M c_1(L) \wedge \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g &= - \int_M c_1(L) \wedge c_1(M) \\ &= - c_1(L) \cdot c_1(M). \end{aligned}$$

□

Proposition 3.9. *Let $(M^4, g(t), J)$ be a solution to pluriclosed flow on a complex surface. Fix D a divisor on M . Then*

$$\frac{\partial}{\partial t} \int_D \omega = c_1(D) \cdot c_1(M).$$

Proof. We directly compute, applying Stokes Theorem as in the previous proposition,

$$\begin{aligned} \frac{\partial}{\partial t} \int_D \omega &= \int_D \left(\partial\bar{\partial}^* \omega + \bar{\partial}\partial^* \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right) \\ &= \int_M (-c_1(D)) \wedge \left(\partial\bar{\partial}^* \omega + \bar{\partial}\partial^* \omega + \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right) \\ &= \int_M c_1(D) \wedge c_1(M) \\ &= c_1(D) \cdot c_1(M). \end{aligned}$$

□

Note that the time evolutions of the quantities in Propositions 3.8 and 3.9 are both topological in nature. Indeed, the sign of this topological number determines these quantities as monotonically increasing, decreasing, or constant. Once one has a sign for the degree, in certain cases then the time evolution of volume is monotonic as well. These same basic observations are behind most of our classification results for static metrics in the next section.

4. IMPROVED REGULARITY ON SURFACES

In this section we improve the basic regularity theory for (1.3) in the case of a complex surface. In the propositions below we give bounds for $|T|^2$ and $|\nabla T|$ in the presence of a bound on $|\Omega|$. The key evolution equation appears in Proposition 4.11 at the end of this section.

Proposition 4.1. *Let $(M^4, g(t), J)$ be a solution to (1.3). There exists a universal constant c_0 such that if $|\Omega| \leq C$ on $[0, \tau]$ then $|T|^2 \leq \max\{|T|_{C^0(g_0)}^2, c_0 C\}$ on $[0, \tau]$.*

Proof. We will apply the maximum principle to the evolution equation for $|T|^2$. According to Proposition 4.11 we conclude that for a solution to (1.3) one has

$$\frac{\partial}{\partial t} |T|^2 = \Delta |T|^2 - 2 |\nabla T|^2 + \Omega * T^{*2} + \nabla |T|^2 * w - \frac{1}{2} |T|^4.$$

One can replace the term $\operatorname{div}^\nabla T$ by Chern curvature terms using the Bianchi identity. Here the Laplacian is that of the Chern connection, however a calculation in coordinates shows that

$$\Delta f = \Delta_{LC} f - \langle w, \nabla f \rangle.$$

Applying the curvature bound it follows that

$$\frac{\partial}{\partial t} |T|^2 \leq \Delta_{LC} |T|^2 + \nabla |T|^2 * w + C |T|^2 - \frac{1}{2} |T|^4.$$

The result follows by the maximum principle. \square

Proposition 4.2. *Let $(M^4, g(t), J)$ be a solution to (1.3). Suppose the solution to (1.3) exists on $[0, T]$ and $|\Omega| \leq K$ and $|T|^2 \leq K$ on $[0, T]$. There exists a universal constant C such that for any $\frac{1}{K} \leq t \leq T$ one has $|\nabla T|(t) \leq CK$.*

Proof. We first recall some basic evolution inequalities. In particular from (3.4) and (3.5) we conclude

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla T|^2 &\leq \Delta |\nabla T|^2 + C (|T| |\nabla T| |\nabla \Omega| + |\Omega| |\nabla T|^2 + |T|^2 |\nabla T|^2) \\ \frac{\partial}{\partial t} |\Omega|^2 &\leq \Delta |\Omega|^2 - 2 |\nabla \Omega|^2 + C (|\Omega|^3 + |T|^2 |\Omega|^2). \end{aligned}$$

We will derive an estimate for $|\nabla T|$ at $t = \frac{1}{K}$ which implies the general statement of the proposition. Let

$$\Phi(x, t) = t (|\nabla T|^2 + |\Omega|^2) + A |T|^2$$

where A is a constant to be chosen later. Combining the evolution equations above with Proposition 4.11 and applying the assumed bound on curvature and torsion we conclude

$$\begin{aligned} \frac{\partial}{\partial t} \Phi &= \Delta \Phi + Ct |T| |\nabla T| |\nabla \Omega| - 2t |\nabla \Omega|^2 \\ &\quad + (CtK |\nabla T|^2 - 2A |\nabla T|^2) + tK^3 + K^2. \end{aligned}$$

Note that we may apply the Cauchy-Schwarz inequality to conclude

$$Ct |T| |\nabla T| |\nabla \Omega| - 2t |\nabla \Omega|^2 \leq C't |T|^2 |\nabla T|^2 \leq C'tK |\nabla T|^2.$$

As long as $t \leq \frac{1}{K}$ we can therefore choose A large with respect to universal constants so that

$$\frac{\partial}{\partial t} \Phi \leq \Delta \Phi + CK^2.$$

Hence by the maximum principle we conclude $\Phi(\frac{1}{K}) \leq K$ and hence

$$\frac{1}{K} |\nabla T|^2 \left(\frac{1}{K} \right) \leq CK$$

and the result follows. \square

Theorem 4.3. *Let $(M^4, g(t), J)$ be a solution to (1.3). Let τ denote the maximal existence time of the flow. If $\tau < \infty$, then*

$$\limsup_{t \rightarrow \tau} |\Omega|_{C^0(g_t)} = \infty.$$

Proof. Suppose $\limsup_{t \rightarrow \tau} |\Omega|_{C^0(g_t)} = C < \infty$. By Proposition 4.1 the torsion is uniformly bounded up to time τ . We may choose a small $\epsilon > 0$ so that Proposition 4.2 applies for all $t \in [\epsilon, \tau]$ to yield a uniform bound on $|\nabla T|$ in this interval. The result now follows from Theorem 3.5. \square

In the remainder of this section we give a precise calculation of the evolution of $|T|^2$ along a solution to (1.3). Before we begin we record a few algebraic identities which hold for quadratic expressions in the torsion on a complex surface. See (1.2) for the definition of Q^2 .

Lemma 4.4. *Let (M^4, g, J) be a Hermitian surface. Then*

$$\begin{aligned} Q^1 &= \frac{1}{2} |T|^2 g \\ \langle Q^2, Q^1 \rangle &= \frac{1}{2} |T|^4 \\ |Q^1|^2 &= \frac{1}{2} |T|^4 \end{aligned}$$

Proof. Choose complex coordinates at a point so that g is the identity matrix. On a surface there are two nonzero components of T up to symmetry. In particular let

$$a = T_{12\bar{1}}, \quad b = T_{12\bar{2}}.$$

Then we can directly compute

$$Q^1 = \begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} 2a^2 & 2ab \\ 2ab & 2b^2 \end{pmatrix}$$

All of the required identities follow immediately. \square

Also, recall the Bianchi identities for the Chern connection.

Lemma 4.5. (Bianchi Identity) *Let (M^{2n}, g, J) be a Hermitian manifold and let ∇ denote the Chern connection associated to (g, J) . For $X, Y, Z \in T_x(M)$ we have*

$$\begin{aligned} \Sigma\{\Omega(X, Y)Z\} &= \Sigma\{T(T(X, Y), Z) + \nabla_X T(Y, Z)\} \\ \Sigma\{\nabla_X \Omega(Y, Z) + \Omega(T(X, Y), Z)\} &= 0 \end{aligned}$$

We start our calculation by using the general calculation for the evolution of T :

Lemma 4.6. *For a solution to $\frac{\partial}{\partial t}g = -S + Q^1$ we have*

$$\begin{aligned} \frac{\partial}{\partial t} T_{ij\bar{k}} &= \Delta T_{ij\bar{k}} + g^{m\bar{n}} [T_{ji}^p \nabla_{\bar{n}} T_{mp\bar{k}} + \nabla_{\bar{n}} T_{mj}^p T_{ip\bar{k}} + T_{mj}^p \nabla_{\bar{n}} T_{ip\bar{k}} \\ &\quad + \nabla_{\bar{n}} T_{im}^p T_{jp\bar{k}} + T_{im}^p \nabla_{\bar{n}} T_{jp\bar{k}}] \\ &\quad + g^{m\bar{n}} [\Omega_{\bar{n}jm}^p T_{ip\bar{k}} + \Omega_{\bar{n}j\bar{k}}^{\bar{p}} T_{im\bar{p}} - \Omega_{\bar{n}im}^p T_{jp\bar{k}} \\ &\quad - \Omega_{\bar{n}i\bar{k}}^{\bar{p}} T_{jm\bar{p}} - \Omega_{\bar{p}\bar{n}m\bar{k}} T_{ji}^p] - T_{ij}^p (S_{p\bar{k}} - Q_{p\bar{k}}^1) \\ &\quad + \nabla_i Q_{j\bar{k}}^1 - \nabla_j Q_{i\bar{k}}^1. \end{aligned}$$

Proof. This evolution equation is derived using the Bianchi identities for the curvature of the Chern connection. This calculation is found in [11] Lemma 6.2. \square

We would like to simplify this equation in the case of a solution to (1.3) on a surface. In a series of lemmas below we simplify the inner product of each term in the above lemma with T .

Lemma 4.7. *Let (M^4, g, J) be a Hermitian surface. Then*

$$g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} \left(\nabla_i Q_{j\bar{k}}^1 - \nabla_j Q_{i\bar{k}}^1 \right) T_{\bar{m}\bar{n}p} = \langle \nabla |T|^2, w \rangle.$$

Proof. From Lemma 4.4 we know $Q^1 = \frac{1}{2}|T|^2 g$. Thus using metric compatibility of the connection we conclude

$$\begin{aligned} g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} \left(\nabla_i Q_{j\bar{k}}^1 - \nabla_j Q_{i\bar{k}}^1 \right) T_{\bar{m}\bar{n}p} &= \frac{1}{2} g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} \left(g_{j\bar{k}} \nabla_i |T|^2 - g_{i\bar{k}} \nabla_j |T|^2 \right) T_{\bar{m}\bar{n}p} \\ &= \langle \nabla |T|^2, w \rangle. \end{aligned}$$

□

Lemma 4.8. *Let (M^4, g, J) be a Hermitian surface. Then*

$$g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} T_{ji}^q \left(\nabla_{\bar{s}} T_{r\bar{q}\bar{k}} - \Omega_{q\bar{s}r\bar{k}} \right) T_{\bar{m}\bar{n}p} = \frac{1}{2} s |T|^2.$$

Proof. We apply the Bianchi identity to conclude

$$\nabla_{\bar{s}} T_{r\bar{q}\bar{k}} - \Omega_{q\bar{s}r\bar{k}} = \Omega_{\bar{s}r\bar{q}\bar{k}}.$$

Plugging this in yields

$$\begin{aligned} g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} T_{ji}^q \left(\nabla_{\bar{s}} T_{r\bar{q}\bar{k}} - \Omega_{q\bar{s}r\bar{k}} \right) T_{\bar{m}\bar{n}p} &= g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} T_{ji}^q \left(\Omega_{\bar{s}r\bar{q}\bar{k}} \right) T_{\bar{m}\bar{n}p} \\ &= \langle S, Q^1 \rangle \\ &= \frac{1}{2} |T|^2 s. \end{aligned}$$

The result follows. □

Lemma 4.9. *Let (M^4, g, J) be a Hermitian surface. Then*

$$g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} \left(\left(\nabla_{\bar{s}} T_{rj}^q + \Omega_{\bar{s}jr}^p \right) T_{i\bar{q}\bar{k}} - \left(\nabla_{\bar{s}} T_{ri}^p + \Omega_{\bar{s}ir}^q \right) T_{j\bar{q}\bar{k}} \right) T_{\bar{m}\bar{n}p} = -s |T|^2.$$

Proof. First we observe the symmetry between the pairs of terms and just compute one of them. Applying the Bianchi identity we conclude

$$\nabla_{\bar{s}} T_{rj}^q + \Omega_{\bar{s}jr}^q = \Omega_{\bar{s}rj}^q$$

Plugging this in yields

$$\begin{aligned} g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} \left(\nabla_{\bar{s}} T_{rj}^q + \Omega_{\bar{s}jr}^p \right) T_{i\bar{q}\bar{k}} T_{\bar{m}\bar{n}p} &= g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} \Omega_{\bar{s}rj}^q T_{i\bar{q}\bar{k}} T_{\bar{m}\bar{n}p} \\ &= -\langle S, Q^1 \rangle \\ &= -\frac{1}{2} s |T|^2. \end{aligned}$$

□

Lemma 4.10. *Let (M^4, g, J) be a Hermitian surface. Then*

$$\begin{aligned} g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} \left(T_{ir}^q \left(\nabla_{\bar{s}} T_{j\bar{q}\bar{k}} + \Omega_{\bar{s}j\bar{k}q} \right) - T_{jr}^q \left(\nabla_{\bar{s}} T_{i\bar{q}\bar{k}} + \Omega_{\bar{s}i\bar{k}q} \right) \right) T_{\bar{m}\bar{n}p} \\ = \langle Q^2, S + \operatorname{div}^\nabla T \rangle. \end{aligned}$$

Proof. We note that the two pairs of terms are the same using the skew-symmetry of T . We compute the first one. Note

$$\begin{aligned} g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} T_{ir}^q (\nabla_{\bar{s}} T_{jq\bar{k}} + \Omega_{\bar{s}j\bar{k}q}) T_{\bar{m}n\bar{p}} &= g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} T_{ir}^q (\Omega_{j\bar{s}q\bar{k}} - \nabla_{\bar{s}} T_{qj\bar{k}}) T_{\bar{m}n\bar{p}} \\ &= g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} T_{ir}^q \Omega_{q\bar{s}j\bar{k}} T_{\bar{m}n\bar{p}}. \end{aligned}$$

We want to reexpress this last term using the fact that $n = 2$. We go through the possibilities for the indices, assuming g is the identity matrix at the point we are computing at. First suppose $i = 1$, then $m = 1, r = s = n = j = 2$. The resulting term is

$$g^{p\bar{k}} T_{12}^q \Omega_{q\bar{2}2\bar{k}} T_{1\bar{2}p}.$$

Likewise next assume $i = 2$, which implies $m = 1, r = s = n = j = 1$. The resulting term is

$$g^{p\bar{k}} T_{21}^q \Omega_{q\bar{1}1\bar{k}} T_{2\bar{1}p}.$$

So, let

$$A_{i\bar{j}} = g^{k\bar{l}} \Omega_{k\bar{j}i\bar{l}}.$$

It is clear from the above calculations that

$$g^{i\bar{m}} g^{j\bar{n}} g^{\bar{k}p} g^{r\bar{s}} T_{ir}^q \Omega_{q\bar{s}j\bar{k}} T_{\bar{m}n\bar{p}} = \frac{1}{2} \langle Q^2, A \rangle.$$

Finally we want to apply the Bianchi identity once more to simplify the tensor A . In particular we see

$$\begin{aligned} A_{i\bar{j}} &= g^{k\bar{l}} \Omega_{k\bar{j}i\bar{l}} \\ &= g^{k\bar{l}} \Omega_{\bar{j}k\bar{l}i} \\ &= g^{k\bar{l}} (\Omega_{l\bar{k}j\bar{i}} + \nabla_k T_{l\bar{j}i}) \\ &= S_{i\bar{j}} + \operatorname{div}^\nabla T_{\bar{j}i}. \end{aligned}$$

The result now follows. \square

Proposition 4.11. *Let $(M^4, g(t), J)$ be a complex surface with $g(t)$ a solution to (1.3). Then*

$$\frac{\partial}{\partial t} |T|^2 = \Delta |T|^2 - 2 |\nabla T|^2 + 2 \langle \nabla |T|^2, w \rangle + \langle Q^2, S + 2 \operatorname{div}^\nabla T \rangle - \frac{1}{2} |T|^4$$

Proof. We start with the basic calculation

$$\frac{\partial}{\partial t} |T|^2 = 2 \left\langle \frac{\partial}{\partial t} T, T \right\rangle + \langle 2Q^1 + Q^2, S - Q^1 \rangle.$$

Next we plug the results of Lemmas 4.7 - 4.10 into Lemma 4.6 and apply Lemma 4.4 to conclude

$$\begin{aligned}
\frac{\partial}{\partial t} |T|^2 &= 2 \langle \Delta T, T \rangle + 2 \langle \nabla |T|^2, w \rangle - s |T|^2 \\
&\quad + 2 \langle Q^2, S + \operatorname{div}^\nabla T \rangle - 2 \langle Q^2, S - Q^1 \rangle + \langle 2Q^1 + Q^2, S - Q^1 \rangle \\
&= \Delta |T|^2 - 2 |\nabla T|^2 + 2 \langle \nabla |T|^2, w \rangle + \langle Q^2, S + 2 \operatorname{div}^\nabla T \rangle \\
&\quad + \langle Q^2, Q^1 \rangle - 2 |Q^1|^2 \\
&= \Delta |T|^2 - 2 |\nabla T|^2 + 2 \langle \nabla |T|^2, w \rangle + \langle Q^2, S + 2 \operatorname{div}^\nabla T \rangle - \frac{1}{2} |T|^4
\end{aligned}$$

as required. \square

5. STATIC METRICS ON SURFACES

In this section we derive certain identities satisfied by static metrics. As applications of these identities we will classify static metrics on K3 surfaces, two-dimensional complex tori, surfaces of general type, and nonprimary Hopf surfaces.

Definition 5.1. Let (M^{2n}, g, J) be a complex manifold with pluri-closed metric. We say that g is *static* if

$$\Phi(\omega) = \lambda \omega$$

for some constant λ , and

$$\operatorname{Vol}(g) = 1.$$

We have ruled out the scaling ambiguity by fixing the volume to be 1.

Proposition 5.2. *Let (M^4, g, J) be a complex surface with a static metric. Then*

$$d - 2\lambda = 2 \int_M |\partial^* \omega|^2.$$

Proof. We begin by taking the wedge product of the static equation with ω and integrating. Since $\int_M \omega \wedge \omega = 2$ this yields

$$\begin{aligned}
-2\lambda &= \int_M \left(\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \right) \wedge \omega \\
&= \int_M \left\langle \left(\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \right), \omega \right\rangle \\
&= \int_M |\partial^* \omega|^2 + \int_M |\bar{\partial}^* \omega|^2 - d.
\end{aligned}$$

Since $|\partial^* \omega|^2 = |\bar{\partial}^* \omega|^2$ the result follows. \square

Proposition 5.3. *Let (M^4, g, J) be a complex surface with static metric, and let L be a line bundle on M . Then*

$$c_1(M) \cdot c_1(L) = \lambda \deg L.$$

Proof. Let Ω be a (closed) form representing $c_1(L)$. Take the wedge product of the static equation with Ω and integrate. This yields

$$\int_M \left(-\partial\bar{\partial}^*\omega - \bar{\partial}\bar{\partial}^*\omega - \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g \right) \wedge \Omega = \lambda \int_M \omega \wedge \Omega.$$

By definition the integral on the right hand side is the degree of L . As for the left hand side, since Ω is closed

$$\int_M \left(\partial\bar{\partial}^*\omega + \bar{\partial}\bar{\partial}^*\omega \right) \wedge \Omega = 0$$

by Stokes Theorem. The remaining term is $c_1(M) \cdot c_1(L)$. \square

There is a further quadratic identity we have for static metrics. First we require a certain reverse Hölder type inequality ([4] Lemma 4). We include the proof for the reader's convenience.

Lemma 5.4. ([4] Lemma 4) *If $\psi \in \Lambda_{\mathbb{R}}^{1,1}$ satisfies $\partial\bar{\partial}\psi = 0$, then*

$$\left(\int_M \omega \wedge \psi \right)^2 \geq \left(\int_M \omega^2 \right) \left(\int_M \psi^2 \right),$$

and moreover equality holds if and only if $\psi = c\omega + i\partial\bar{\partial}g$ for some constant c and some $g \in C^\infty(M)$.

Proof. Let $c := \frac{\int_M \omega \wedge \psi}{\int_M \omega^2}$. Since the function solving $f dV = \omega \wedge \psi - c\omega$ is L^2 -orthogonal to the constants, which are the only kernel of $f \rightarrow *i\partial\bar{\partial}(\omega f)$, it follows that there exists a function ρ such that

$$\omega \wedge (\psi - c\omega - i\partial\bar{\partial}\rho) = 0.$$

It follows that the form $\psi - c\omega - i\partial\bar{\partial}\rho$ is antiselfdual, hence

$$\begin{aligned} 0 \leq \|\psi - c\omega - i\partial\bar{\partial}\rho\|^2 &= - \int_M (\psi - c\omega - i\partial\bar{\partial}\rho)^{\wedge 2} \\ &= - \int_M \psi^{\wedge 2} + \frac{\left(\int_M \psi \wedge \omega \right)^2}{\int_M \omega^2}. \end{aligned}$$

The proposition follows from this inequality. \square

Proposition 5.5. *Let (M^4, J, g) be a compact complex surface with static pluriclosed metric. Then*

$$\begin{aligned} c_1^2 - 2\lambda d + \frac{1}{2}d^2 &\geq 0, \\ -c_1^2 + \frac{1}{2}d^2 &\geq 0 \end{aligned}$$

with equality in either case if and only if either g is Kähler-Einstein or $c_1(M) = 0$.

Proof. We directly compute

$$\begin{aligned} 2\lambda^2 &= \int_M \lambda\omega \wedge \lambda\omega \\ &= \int_M \left(\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega + \frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g \right)^{\wedge 2} \\ &= c_1^2(M) + \int_M \left(\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega \right)^{\wedge 2} \end{aligned}$$

The last line follows by Stokes Theorem since $\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g$ is closed. Now we apply Lemma 5.4 to conclude

$$\begin{aligned} \int_M \left(\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega \right)^{\wedge 2} &\leq \frac{1}{2} \left(\int_M \omega \wedge \left(\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega \right) \right)^2 \\ &= 2 \left(\int_M |\partial^*\omega|^2 \right)^2 \\ &= \frac{1}{2} (2\lambda - d)^2 \\ &= 2\lambda^2 - 2\lambda d + \frac{1}{2}d^2. \end{aligned}$$

The second to last line follows from Proposition 5.2. Plugging this into the above yields the first inequality, and the second follows using $\lambda d = c_1(M)^2$, which follows from Proposition 5.3. To characterize the equality case we note by Lemma 5.4 that it occurs if and only if

$$\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega = c\omega + \partial\bar{\partial}g.$$

Plugging this into the static equation yields

$$\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g + \partial\bar{\partial}g = -(\lambda + c)\omega.$$

The left hand side is closed, so if $\lambda + c \neq 0$ then ω is Kähler, hence Kähler-Einstein. Otherwise we have that $\frac{\sqrt{-1}}{2}\partial\bar{\partial}\log\det g$ is represented by an exact form, therefore $c_1(M) = 0$. \square

Proposition 5.6. *Consider a K3 surface or torus (M^4, g, J) with static metric g . Then g is in fact Kähler and Ricci-flat.*

Proof. Note that $c_1(M) = 0$. Since M has a Kähler Ricci-flat metric g_0 , it follows that

$$\begin{aligned} d(M, g) &= \int_M -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \wedge \omega \\ &= \int_M -\frac{\sqrt{-1}}{2} \partial \bar{\partial} \left(\log \frac{\det g}{\det g_0} + \log \det g_0 \right) \wedge \omega \\ &= 0. \end{aligned}$$

The last follows by integrating by parts, using that $\partial \bar{\partial} \omega = 0$. Also, the Kähler form of g_0 , ω_0 , is closed. Taking the wedge product of the static equation with ω_0 and integrating yields

$$\begin{aligned} \lambda \int_M \omega \wedge \omega_0 &= \int_M \left(\partial \partial^* \omega + \bar{\partial} \bar{\partial}^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \right) \wedge \omega_0 \\ &= 0. \end{aligned}$$

However, since ω and ω_0 are both positive $(1, 1)$ forms one has $\int_M \omega \wedge \omega_0 > 0$, therefore $\lambda = 0$. Therefore by Proposition 5.2 we conclude

$$\int_M |\partial^* \omega|^2 = d - 2\lambda = 0,$$

therefore g is Kähler and hence Kähler-Einstein. Since $c_1(M) = 0$ this means g is Ricci-flat. \square

Proposition 5.7. *Let (M^4, g, J) be a surface of general type with static metric. Then g is Kähler-Einstein.*

Proof. Since (M^4, J) is of general type we have $c_1^2 > 0$, and also one has $d < 0$ since the canonical bundle is ample. It follows from Proposition 5.3 that $\lambda < 0$. By Proposition 5.5 we conclude $d^2 \geq 2c_1^2$, therefore $d \leq -\sqrt{2c_1^2}$. Returning to Proposition 5.3 we conclude

$$c_1^2 = \lambda d \geq -\lambda \sqrt{2c_1^2}.$$

Therefore $\lambda \geq -\sqrt{\frac{c_1^2}{2}}$. It follows that

$$d - 2\lambda \leq -\sqrt{2c_1^2} + 2\sqrt{\frac{c_1^2}{2}} = 0.$$

Now it follows from Proposition 5.2 that $\partial^* \omega = 0$ therefore $\partial \omega = 0$ and the metric is Kähler, hence Kähler-Einstein. \square

Proposition 5.8. *Let (M^4, g, J) be a complex surface with static metric and suppose $\Sigma \subset M$ is a holomorphic curve such that $[\Sigma] = 0 \in H^2(M, \mathbb{R})$. Then $\lambda = 0$.*

Proof. We simply integrate the static equation along the curve Σ to yield

$$\begin{aligned} \lambda \int_{\Sigma} \omega &= \int_{\Sigma} \left(-\partial\bar{\partial}^* \omega - \bar{\partial}\partial^* \omega - \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right) \\ &= \int_M c_1(\Sigma) \wedge \left(-\partial\bar{\partial}^* \omega - \bar{\partial}\partial^* \omega - \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det g \right) \\ &= c_1(\Sigma) \cdot c_1(M). \end{aligned}$$

The last line follows applying Stokes Theorem since $c_1(\Sigma)$ is closed. However, since Σ is null-homologous, $c_1(\Sigma) = 0$, therefore the right hand side is zero. Since $\int_{\Sigma} \omega > 0$, therefore $\lambda = 0$. \square

Proposition 5.9. *Let (M^4, J) be a Hopf surface blown up at $p > 0$ points. Then M admits no static metrics.*

Proof. We start by observing that any Hopf surface admits a holomorphic curve which is homologous to zero. A general Hopf surface is defined by taking the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the group G generated by the map

$$(z_1, z_2) \rightarrow (\alpha_1 z_1, \alpha_2 z_2)$$

where $0 < |\alpha_1| \leq |\alpha_2| < 1$. It follows from [1] Proposition 18.2 that the space $H_{\alpha} = (\mathbb{C}^2 \setminus \{0\})/G$ is either an elliptic fibration over \mathbb{P}^1 or contains exactly two irreducible curves. Thus any case contains at least one holomorphic curve, and since all H_{α} are homeomorphic to $S^3 \times S^1$ this curve is necessarily null-homologous since $H_2(S^3 \times S^1, \mathbb{R}) = 0$. It is clear that this curve still exists, and is still null-homologous, if we blow-up the Hopf surface at finitely many points. It follows from Proposition 5.8 that $\lambda = 0$ for our static metric. By Proposition 5.3 we yield $c_1^2(M) = \lambda d = 0$. However, if we have blown up at p points we have $c_1^2(M) = -p$, a contradiction. Thus the result follows. \square

We now come to a key observation on static metrics. In particular, we find that manifolds admitting static metrics with nonzero constant admit Hermitian-symplectic structures, which were defined in the introduction.

Proposition 5.10. *Let (M^{2n}, g, J) be a compact complex manifold with static metric. If $\lambda \neq 0$, then M is a Hermitian-symplectic manifold*

and specifically ω is the $(1, 1)$ -part of a Hermitian-symplectic form $\tilde{\omega}$. Furthermore, one has

$$\int_M \tilde{\omega}^{\wedge n} > 0.$$

Proof. Consider the real two-form

$$\tilde{\omega} = \omega - \frac{1}{\lambda} \left(\bar{\partial} \partial^* \omega + \partial \bar{\partial}^* \omega \right).$$

We compute using the static equation

$$\begin{aligned} d\tilde{\omega} &= d \left(\omega - \frac{1}{\lambda} \left(\bar{\partial} \partial^* \omega + \partial \bar{\partial}^* \omega \right) \right) \\ &= -\frac{1}{\lambda} d \left(d\partial^* \omega + d\bar{\partial}^* \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g \right) \\ &= 0. \end{aligned}$$

Thus $\tilde{\omega}$ is a Hermitian-symplectic form. The claim that $\int_M \tilde{\omega}^{\wedge n} > 0$ is a standard fact whose proof we outline below, following calculations of [5]. In particular, set

$$\tilde{\omega} = \phi + H + \bar{\phi}$$

where $H = \omega$ is the $(1, 1)$ part of $\tilde{\omega}$ and ϕ is the $(2, 0)$ part of $\tilde{\omega}$, with $\bar{\phi}$ then being the $(0, 2)$ part. Let $\{\theta^j\}$ denote a basis for $(1, 0)$ forms at a point $x \in M$ which diagonalizes the metric ω . We note

$$\begin{aligned} \tilde{\omega}^{\wedge n} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{(k!)^2 (n-2k)!} i^r H_{j_1 \bar{j}_1} \dots H_{j_r \bar{j}_r} \cdot \\ &\quad \theta^{j_1} \wedge \bar{\theta}^{\bar{j}_1} \wedge \dots \wedge \theta^{j_r} \wedge \bar{\theta}^{\bar{j}_r} \wedge \phi^{\wedge k} \wedge \bar{\phi}^{\wedge k}. \end{aligned}$$

Note that the term $k = 0$ in the above summand is *strictly* positive since all of the coefficients $H_{j\bar{j}}$ are strictly positive and the resulting form is then a positive multiple of the volume form of ω . It remains to show that the terms $k > 0$ are nonnegative. Fix a particular term in the summand, and assume (without loss of generality by relabelling) that $\{j_i = 2k + i\}$. One can directly compute ([5] pg. 845) that

$$\phi^{\wedge k} \wedge \bar{\phi}^{\wedge k} = i^{2k} f \theta^1 \wedge \bar{\theta}^1 \wedge \dots \wedge \theta^{2k} \wedge \bar{\theta}^{2k}.$$

where $f \geq 0$. Thus the terms in the above summand with $k > 0$ are nonnegative multiples of the volume form of ω , and the result follows. \square

6. CLASS VII SURFACES

A minimal compact complex surface S is of Kodaira's Class VII if $b_1(S) = 1$. The standard examples of surfaces of class VII with $b_2(S) = 0$ are the Hopf surfaces, defined in section 4. As we note in the following example, the standard Hopf surface admits a static metric with $\lambda = 0$.

Example 6.1. Consider the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the group Γ generated by the map

$$(z_1, z_2) \rightarrow \left(\frac{1}{2}z_1, \frac{1}{2}z_2 \right).$$

The standard complex structure on \mathbb{C}^2 descends to the quotient manifold, which is homeomorphic to $S^3 \times S^1$. Furthermore, setting $\rho = \sqrt{z_1\bar{z}_1 + z_2\bar{z}_2}$, the Kähler form

$$\omega = \frac{1}{\rho^2} \partial\bar{\partial}\rho^2$$

is Γ -invariant and compatible with the standard complex structure, and therefore defines a metric on the quotient. This metric is pluriclosed, although it should be noted that this is not true for higher dimensional Hopf surfaces. Call the associated metric g . One can directly compute that

$$S_g = g$$

and

$$Q_g^1 = g$$

so that

$$S_g - Q_g^1 = 0.$$

Thus g is a static metric with $\lambda = 0$. This constant is of course correctly predicted by Proposition 5.8.

It may be the case that every class VII surface with $b_2(S) = 0$ admits a static metric. The situation when $b_2(S) > 0$ is completely different though, with none admitting static metrics. We say that a class VII surface is in class VII⁺ if $b_2(S) > 0$. We recall a basic structural theorem for class VII⁺ surfaces.

Theorem 6.2. ([6] *Theorem 1.8*) *Let S be a (not necessarily minimal) compact complex surface such that $b_1(S) = 1$, and $b_2(S) = n > 0$. Then there exist n exceptional line bundles $L_j, j = 0, \dots, n-1$, unique up to torsion by a flat line bundle $F \in H^1(S, \mathbb{C}^*)$ such that*

- $E_j = c_1(L_j), 0 \leq j \leq n-1$ is a \mathbb{Z} -basis of $H^2(S, \mathbb{Z})$.
- $K_S L_j = -1$ and $L_i L_j = -\delta_{jk}$
- $K_S = L_0 + \cdots + L_{n-1} \in H^2(M, \mathbb{Z})$

Proof. In fact we have only stated the part of ([6] Theorem 1.8) which we will need. We just sketch a couple of the ideas. One applies Donaldson's Theorem [8] to diagonalize the intersection form on the torsion-free projection of $H^2(M, \mathbb{Z})$. Since the Kodaira dimension of a class VII⁺ surface is $-\infty$, we have that the geometric genus $p_g(M) = 0$. By [1] Theorem 2.7 (iii) we know that $b^+(M) = 2p_g(M) = 0$. Therefore $b^+(M) = 0$ and the intersection form of M is negative definite. Since $p_g = h^2(S, \mathcal{O}_S) = 0$, each element of the basis is realized as $c_1(L)$, and so the first claim follows. The second and third and fourth follow from applications of Riemann-Roch. For the proof of the final claim see [6]. \square

Proposition 6.3. *Class VII⁺ surfaces admit no static metrics.*

Proof. Let $n = b_2(M) > 0$. It follows from Theorem 6.2 that

$$c_1(M)^2 = K_S \cdot K_S = -n.$$

Therefore it follows from Proposition 5.3 that

$$-n = \lambda d.$$

Therefore $\lambda \neq 0$. Proposition 5.10 implies that M carries the structure of a symplectic manifold given by $\tilde{\omega}$. Moreover, $\int_M \tilde{\omega}^{\wedge 2} > 0$, so that $\tilde{\omega}$ is a cohomology class with positive self-intersection. However, as we remarked in the proof of Theorem 6.2, the intersection form on M is negative definite. Thus we have arrived at a contradiction. \square

Proposition 6.3 is an important first step in applying solutions to (1.3) to studying the topology of class VII⁺ surfaces. Recall that surfaces of class VII with $b_2 = 0$ are classified. In particular they are biholomorphic to either a Hopf surface or an Inoue surface (a free quotient of $\mathbb{C} \times \mathbb{H}$ by a properly discontinuous affine action). This result is typically called Bogomolov's Theorem [2], [3], although the first complete proofs appear to have been given independently in [10] and [12]. As for surfaces of class VII⁺, (i.e. $b_2 > 0$), in deep recent work Teleman [13], [14] has confirmed the conjectural list for $b_2 = 1, 2$ using Donaldson theory. The dedicated effort of many authors, culminating in the theorem of Dloussky-Oeljeklaus-Toma [7], has reduced the problem of classifying class VII⁺ surfaces to finding b_2 rational curves in a given minimal surface of class VII⁺, and this is what Teleman shows for $b_2 = 1, 2$. What Theorem 1.8 says is that equation (1.3)

must encounter some nontrivial singularities on a class VII^+ surface. In principle, these singularities should be closely related to curves on the surface. Indeed, such behavior is seen for the Kähler-Ricci flow, where the flow exhibits blowdowns along complex subvarieties [15]. If sufficiently many curves can be shown to exist as singularities of (1.3), one could finish the classification of class VII surfaces. Much work remains to see to what extent singularities of (1.3) can be analyzed.

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