HERMITIAN CURVATURE FLOW

JEFFREY STREETS AND GANG TIAN

Abstract. We define a functional for Hermitian metrics using the curvature of the Chern connection. The Euler-Lagrange equation for this functional is an elliptic equation for Hermitian metrics. Solutions to this equation are related to Kähler-Einstein metrics, and are automatically Kähler-Einstein under certain conditions. Given this, a natural parabolic flow equation arises. We prove short time existence and regularity results for this flow, as well as stability for the flow near Kähler-Einstein metrics with negative or zero first Chern class.

1. Introduction

In this paper we introduce a new curvature evolution equation on compact complex manifolds. Specifically, given (M^{2n}, g, J) a manifold with integrable complex structure J and Hermitian metric g, let ∇ denote the Chern connection of g, which is a metric compatible connection with torsion T [12]. Let Ω denote the curvature of ∇. Define

$$S_{i\bar{j}} = (\operatorname{tr}_\omega \Omega)_{i\bar{j}} = g^{k\bar{l}} \Omega_{k\bar{l}i\bar{j}}$$

and let $s = g^{i\bar{j}} S_{i\bar{j}}$ be the scalar Chern curvature. Furthermore let $w_i = g^{j\bar{k}} T_{ij\bar{k}}$ denote the trace of the torsion. Consider the functional

$$\mathbb{F}(g) = \frac{\int_M \left[s - \frac{1}{4} \left| T \right|^2 - \frac{1}{2} \left| w \right|^2 \right] dV}{\left(\int_M dV \right)^{\frac{n-1}{n}}}. \tag{1}$$

As we will see in section 3 this is the unique functional yielding $\overset{\circ}{S}$ as the traceless component of the second-order terms in the associated Euler-Lagrange equation. Moreover, the form of the Euler-Lagrange equation suggests a flow equation in the same way that Ricci flow is suggested by the usual Hilbert functional. In particular we define an evolution equation

$$\frac{\partial}{\partial t} g = -S + Q \tag{2}$$

where $Q = Q(T)$ is a certain quadratic polynomial in the torsion T of ∇ which is made precise in section 3. We call equation (2) *Hermitian curvature flow* (HCF). Of course now it is known that Ricci flow is indeed the gradient flow of the lowest eigenvalue of a certain Schrödinger operator, although a corresponding statement for HCF is not yet known. It is also possible to write HCF in terms of Hodge-type

Date: January 26, 2009.
The first author was supported by the National Science Foundation via DMS-0703660.
The second author was partly supported by the National Science Foundation via DMS-0703985 and DMS-0804095.

operators. In particular, if $\omega(t)$ denotes the Kähler form of the time varying metric, then it satisfies the equation

$$\frac{\partial}{\partial t}\omega = -\left(\partial_g^*\partial\omega - \partial\partial_g^*\omega - \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g - 2\sqrt{-1}\left(\overline{\partial}_g^*\omega \lrcorner \overline{\partial}\omega\right)\right) + Q'.$$

where here Q' is a distinct fixed quadratic expression in the torsion.

We observe that when a solution $g(t)$ to HCF exists, the metric is Hermitian with respect to the fixed complex structure J for all time. Secondly, we will show that when the initial metric $g(0)$ is Kähler, then the solution $g(t)$ is Kählerian and consequently the solution to HCF is given by Kähler-Ricci flow. Thirdly, we prove that certain static solutions are Kähler-Einstein metrics. It will be a very interesting problem to classify all static solutions. It is possible that they are all Kähler. Hence, in some sense, this new flow evolves Hermitian metrics towards Kähler metrics.

Next we will prove a local existence theorem for HCF and develop some regularity properties for this flow. In particular we derive higher order derivative estimates in the presence of a curvature bound. A consequence of these estimates is the following short-time existence theorem.

Theorem 1.1. *Let (M^{2n}, g_0, J) be a complex manifold with Hermitian metric g_0. There exists a constant $c(n)$ depending only on the dimension such that there exists a unique solution $g(t)$ to HCF for*

$$t \in \left[0, \frac{c(n)}{\max\{|\Omega|_{C^0(g_0)}, |\nabla T|_{C^0(g_0)}, |T|^2_{C^0(g_0)}\}}\right].$$

Moreover, there exist constants C_m depending only on m such that the estimates

$$|\nabla^m \Omega|_{C^0(g_t)}, \left|\nabla^{m+1} T\right|_{C^0(g_t)} \leq \frac{C_m \max\{|\Omega|_{C^0(g_0)}, |\nabla T|_{C^0(g_0)}, |T|^2_{C^0(g_0)}\}}{t^{m/2}}$$

hold for all t in the above interval. Moreover, the solution exists on a maximal time interval $[0, \tau)$, and if $\tau < \infty$ then

$$\limsup_{t \to \tau} \max\{|\Omega|_{C^0(g_t)}, |\nabla T|_{C^0(g_t)}, |T|_{C^0(g_t)}\} = \infty.$$

In some sense, the simplest possible behavior for this flow should occur near Kähler-Einstein metrics, where we expect the flow to be not too much different from Kähler-Ricci flow. To that end we prove a stability result for HCF around Kähler-Einstein metrics with negative or zero first Chern class. Specifically, we show

Theorem 1.2. *Let (M^{2n}, g, J) be a complex manifold with Kähler-Einstein metric g and $c_1(M) < 0$ or $c_1(M) = 0$. Then there exists $\epsilon = \epsilon(g)$ so that if $\widetilde{g}$ is a Hermitian metric on M compatible with J and $|\widetilde{g} - g|_{C^\infty} < \epsilon$ then the solution to HCF with initial condition $\widetilde{g}$ exists for all time and converges to a Kähler-Einstein metric.*

There are two natural directions which motivate defining this flow. First, given all of the success of Ricci flow it is natural to study it on complex manifolds. However, it is usually the case that the Ricci tensor of a Hermitian metric is not $(1,1)$, and thus the Hermitian condition for the metric is not preserved. Thus the Ricci flow is not the best tool for studying complex geometry which is not already

Kähler. The tensor S is a natural $(1,1)$ curvature tensor associated to a Hermitian metric which differs from the Ricci tensor by torsion terms, meaning that it equals the Ricci tensor in the Kähler setting. Moreover, the operator $g \to S(g)$ is strictly elliptic, giving HCF nice existence properties. Thus from this perspective HCF is the right analogue of Ricci flow for Hermitian geometry.

The second motivation, and actually our original motivation for HCF is that it serves as a "holonomy flow." If one looks on the level of the Kähler form and asks for a parabolic flow which preserves the Hermitian condition and is stationary on Kähler manifolds, HCF comes up quite naturally. There is the side effect that one ends up looking not just for Kähler metrics, but *Kähler-Einstein* metrics. Given the excellent existence properties of the Kähler-Ricci flow, this is an acceptable price to pay. Indeed, other natural analytic approaches to this question which strictly look for Kähler metrics among Hermitian metrics (see for instance [19]) yield equations which are not elliptic. In [4] it is shown that if the usual Ricci-type curvature of the Chern connection is a nonzero scalar multiple of the metric, then the metric is automatically Kähler-Einstein. However, this Ricci tensor is not in general $(1,1)$, so from the perspective of Hermitian geometry, especially defining a flow of Hermitian metrics, this condition is not natural.

Here is an outline of the rest of the paper. In section 2 we define all of the relevant objects and notation and provide various curvature formulas. In section 3 we discuss the Hermitian Hilbert functional. Section 4 gives the definition of HCF and provides various equivalent formulations using Hodge-type operators and the Levi-Civita connection. In sections 5-7 we prove existence and regularity properties for HCF. In section 8 we prove the stability result for HCF around Kähler-Einstein metrics. We conclude in section 9 with a discussion of some related questions. Section 10 is an appendix containing various useful calculations related to Hermitian geometry.

Acknowledgments The first author would like to thank Aaron Naber, Yanir Rubinstein, and Jeff Viaclovsky for interesting conversations.

2. Differential Operators on Hermitian Manifolds

Let (M^{2n}, g, J) be a complex manifold with a Hermitian metric g. In particular

$$J : TM \to TM$$

is an integrable almost complex structure, i.e.

$$N_J(X,Y) := [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] = 0$$

for all $X, Y \in TM_p$. Furthermore

$$g(u,v) = g(Ju,Jv).$$

This equation is written in a unitary frame as

$$g_{ij} = g_{\overline{i}\overline{j}} = 0, \qquad g_{i\bar{j}} = g_{\bar{j}i} = \overline{g}_{\bar{i}j}.$$

First we recall the Chern connection ∇. In complex coordinates, the only nonvanishing components of the connection are given by

$$\Gamma_{ij}^{k} = g^{k\bar{l}} \partial_i g_{j\bar{k}}.$$

This connection is compatible with g, but has torsion T. In particular in complex coordinates we have

$$T_{ij}^k = g^{k\bar{l}} \left(\partial_i g_{j\bar{l}} - \partial_j g_{i\bar{l}} \right).$$

Also, there is a natural trace of the torsion

$$w_i = T_{ij}^j. \tag{3}$$

As we will see in the next section w is just a multiple of $\overline{\partial}^* \omega$ but this separate definition will be useful to us. We will also need certain quadratic expressions in the torsion. Specifically let

$$\begin{aligned}
Q^1_{i\bar{j}} &= g^{k\bar{l}} g^{m\overline{n}} T_{ik\overline{n}} T_{\overline{j}\bar{l}m} \\
Q^2_{i\bar{j}} &= g^{k\bar{l}} g^{m\overline{n}} T_{\bar{l}\overline{n}i} T_{km\bar{j}} \\
Q^3_{i\bar{j}} &= g^{k\bar{l}} g^{m\overline{n}} T_{ik\bar{l}} T_{\overline{j}\overline{n}m} \\
Q^4_{i\bar{j}} &= \frac{1}{2} g^{k\bar{l}} g^{m\overline{n}} \left(T_{mk\bar{l}} T_{\overline{n}\bar{j}i} + T_{\overline{n}\bar{l}k} T_{mi\bar{j}} \right).
\end{aligned} \tag{4}$$

Note that each Q^i is a real symmetric $(1,1)$ tensor. The covariant derivatives of torsion also satisfy an identity.

Lemma 2.1. *Given g a Hermitian metric,*

$$\nabla_i T_{jk\bar{l}} + \nabla_k T_{ij\bar{l}} + \nabla_j T_{ki\bar{l}} = T_{ij}^p T_{kp\bar{l}} + T_{jk}^p T_{ip\bar{l}} + T_{ki}^p T_{jp\bar{l}}$$

Proof. We directly compute

$$\begin{aligned}
\nabla_i T_{jk\bar{l}} + \nabla_k T_{ij\bar{l}} + \nabla_j T_{ki\bar{l}} &= \partial_i T_{jk\bar{l}} + \partial_k T_{ij\bar{l}} + \partial_j T_{ki\bar{l}} \\
&\quad - \Gamma_{ij}^p T_{pk\bar{l}} - \Gamma_{ik}^p T_{jp\bar{l}} - \Gamma_{ki}^p T_{pj\bar{l}} \\
&\quad - \Gamma_{kj}^p T_{ip\bar{l}} - \Gamma_{jk}^p T_{pi\bar{l}} - \Gamma_{ji}^p T_{kp\bar{l}} \\
&= T_{ij}^p T_{kp\bar{l}} + T_{jk}^p T_{ip\bar{l}} + T_{ki}^p T_{jp\bar{l}}
\end{aligned}$$

□

Next we collect some useful formulas for the Chern curvature. In particular, let Ω denote the curvature of the Chern connection and let S be the trace, i.e.

$$S_{\alpha\overline{\beta}} = \left(\operatorname{tr}_\omega \Omega\right)_{\alpha\overline{\beta}} = g^{\mu\overline{\nu}} \Omega_{\mu\overline{\nu}\alpha\overline{\beta}}.$$

Further, let

$$s = g^{\alpha\overline{\beta}} S_{\alpha\overline{\beta}}.$$

We will let P denote the trace of the transpose of Ω, namely

$$P_{\alpha\overline{\beta}} = g^{\mu\overline{\nu}} \Omega_{\alpha\overline{\beta}\mu\overline{\nu}}.$$

In the Kähler case $S_{\alpha\overline{\beta}} = P_{\alpha\overline{\beta}}$ is the Ricci curvature and $s = r$ is the scalar curvature.

Lemma 2.2. *Given g a Hermitian metric we have*

$$S_{j\overline{k}} = -g^{l\overline{m}} g_{j\overline{k},l\overline{m}} + g^{l\overline{m}} g^{p\overline{q}} g_{p\overline{k},\overline{m}} g_{j\overline{q},l}$$

Proof. First of all we have

$$\begin{aligned}\Omega_{l\overline{m}j\overline{k}} =& -g_{\overline{k}p}\partial_{\overline{m}}\left(g^{p\overline{q}}\partial_l g_{j\overline{q}}\right)\\ =& -g_{j\overline{k},l\overline{m}} + g^{p\overline{q}}g_{p\overline{k},\overline{m}}g_{j\overline{q},l}\end{aligned}$$

Thus

$$\begin{aligned}S_{j\overline{k}} &= g^{l\overline{m}}\Omega_{l\overline{m}j\overline{k}}\\ &= -g^{l\overline{m}}g_{j\overline{k},l\overline{m}} + g^{l\overline{m}}g^{p\overline{q}}g_{p\overline{k},\overline{m}}g_{j\overline{q},l}\end{aligned}$$

as required. □

Lemma 2.3. (Bianchi Identity) *For $X, Y, Z \in T_x(M)$ we have*

$$\Sigma\{\Omega(X,Y)Z\} = \Sigma\{T(T(X,Y),Z) + \nabla_X T(Y,Z)\}$$

$$\Sigma\{\nabla_X\Omega(Y,Z) + \Omega(T(X,Y),Z)\} = 0$$

Lemma 2.4. *Given g a Hermitian metric we have*

$$P_{i\overline{j}} - S_{i\overline{j}} = g^{k\overline{l}}\left(\nabla_{\overline{l}}T_{ki\overline{j}} + \nabla_i T_{\overline{l}\overline{j}k}\right)$$

Proof. We compute using the Bianchi identity and the symmetries of the torsion

$$\begin{aligned}\Omega_{i\overline{j}k\overline{l}} &= \Omega_{k\overline{j}i\overline{l}} + \nabla_{\overline{j}}T_{ki\overline{l}}\\ &= \Omega_{\overline{j}k\overline{l}i} + \nabla_{\overline{j}}T_{ki\overline{l}}\\ &= \Omega_{\overline{l}k\overline{j}i} + \nabla_k T_{\overline{l}\overline{j}i} + \nabla_{\overline{j}}T_{ki\overline{l}}.\end{aligned}$$

Taking the trace and relabelling indices gives the result. □

Now we focus on Hodge operators associated to g. Let

$$\omega(u,v) = -g(u,Jv)$$

be the Kähler form of g. In local complex coordinates we have

$$\omega = \frac{\sqrt{-1}}{2}g_{i\overline{j}}dz^i \wedge d\overline{z}^j.$$

Let

$$\Lambda^k = \bigoplus_{p+q=k}\Lambda^{p,q}$$

denote the usual decomposition of complex differential two-forms into forms of type (p,q). The exterior differential d decomposes into the operators ∂ and $\overline{\partial}$

$$\begin{aligned}\partial &: \Lambda^{p,q} \to \Lambda^{p+1,q}\\ \overline{\partial} &: \Lambda^{p,q} \to \Lambda^{p,q+1}.\end{aligned}$$

Also the operator d_g^*, the L^2 adjoint of d, decomposes into ∂_g^* and $\overline{\partial}_g^*$

$$\begin{aligned}\partial_g^* &: \Lambda^{p+1}, q \to \Lambda^{p,q}\\ \overline{\partial}_g^* &: \Lambda^{p,q+1} \to \Lambda^{p,q}\end{aligned}$$

Using these operators we can define the complex Laplacians

$$\begin{aligned}\Box_\omega &= \partial_g^*\partial + \partial\partial_g^* : \Lambda^{p,q} \to \Lambda^{p,q}\\ \overline{\Box}_\omega &= \overline{\partial}_g^*\overline{\partial} + \overline{\partial}\overline{\partial}_g^* : \Lambda^{p,q} \to \Lambda^{p,q}\end{aligned}$$

It is well known that the operator $\alpha \to \Box_\omega \alpha$ is a second-order elliptic operator with symbol that of the Laplacian in complex coordinates [12]. Moreover, one has the formula

$$\Delta_{d,g} = \Box_g + \overline{\Box}_g + \text{lower order terms} \tag{5}$$

However, we will be interested in the action of these operators on ω itself, so the terms which are lower-order in (5) become highest order terms in this context. In the lemmas which follow we compute the action of these differential operators explicitly.

Lemma 2.5. *Given g a Hermitian metric we have in complex coordinates*

$$\left(\partial_g^* \omega\right)_{\overline{k}} = \frac{\sqrt{-1}}{2} g^{p\overline{q}} \left(\partial_{\overline{q}} g_{p\overline{k}} - \partial_{\overline{k}} g_{p\overline{q}} \right) \tag{6}$$

$$\left(\overline{\partial}_g^* \omega\right)_j = \frac{\sqrt{-1}}{2} g^{p\overline{q}} \left(\partial_p g_{j\overline{q}} - \partial_j g_{p\overline{q}} \right) \tag{7}$$

Proof. We compute using integration by parts. Given $\alpha \in \Lambda^{0,1}$ we have

$$\begin{aligned}
\left(\partial_g^* \omega, \alpha\right) =&\ (\omega, \partial \alpha)\\
=&\ \int_M g^{\overline{k}l} g^{\overline{i}j} \left(\omega_{j\overline{k}} \overline{\partial \alpha_{i\overline{l}}} \right) \overline{g}\\
=&\ \frac{\sqrt{-1}}{2} \int_M g^{\overline{i}l} \left(\overline{\alpha_{\overline{l},i}} \right) \overline{g}\\
=&\ -\frac{\sqrt{-1}}{2} \int_M \overline{\alpha_{\overline{l}}} \left[\partial_{\overline{i}} \left(g^{\overline{i}l} \overline{g} \right) \right]\\
=&\ -\frac{\sqrt{-1}}{2} \int_M \overline{\alpha_{\overline{l}}} \left(\overline{g} \right) \left[-g^{\overline{i}m} \partial_{\overline{i}} g_{m\overline{n}} g^{\overline{n}l} + g^{\overline{i}l} \frac{1}{\overline{g}} \partial_{\overline{i}} \overline{g} \right].
\end{aligned}$$

This gives the first formula, and the second follows analogously. □

Lemma 2.6. *Given g a Hermitian metric we have in complex coordinates*

$$\left(\partial \partial_g^* \omega\right)_{j\overline{k}} = \frac{\sqrt{-1}}{2} \left[g^{p\overline{q}} \left(g_{p\overline{k},\overline{q}j} - g_{p\overline{q},\overline{k}j} \right) + g^{p\overline{q}} g^{r\overline{s}} g_{r\overline{q},j} \left(g_{p\overline{s},\overline{k}} - g_{p\overline{k},\overline{s}} \right) \right] \tag{8}$$

Proof. In general for $\alpha \in \Lambda^{0,1}$ we have

$$(\partial \alpha)_{j\overline{k}} = \partial_j \alpha_{\overline{k}}.$$

Thus we compute using Lemma 2.5

$$\begin{aligned}
\left(\partial \partial_g^* \omega\right)_{j\overline{k}} =&\ \frac{\sqrt{-1}}{2} \partial_j \left(g^{p\overline{q}} \left(\partial_{\overline{q}} g_{p\overline{k}} - \partial_{\overline{k}} g_{p\overline{q}} \right) \right)\\
=&\ \frac{\sqrt{-1}}{2} \left[g^{p\overline{q}} \left(g_{p\overline{k},\overline{q}j} - g_{p\overline{q},\overline{k}j} \right) - g^{p\overline{m}} g_{\overline{m}n,j} g^{n\overline{q}} \left(g_{p\overline{k},\overline{q}} - g_{p\overline{q},\overline{k}} \right) \right].
\end{aligned}$$

The result follows. □

Lemma 2.7. *Given g a Hermitian metric we have in complex coordinates*

$$\begin{aligned}
\left(\partial_g^* \partial \omega\right)_{j\overline{k}} =&\ \frac{\sqrt{-1}}{2} \left[g^{p\overline{q}} \left(g_{p\overline{k},j\overline{q}} - g_{j\overline{k},p\overline{q}} \right) + g^{p\overline{q}} g^{r\overline{s}} \left(g_{p\overline{s},\overline{q}} - g_{p\overline{q},\overline{s}} \right) \left(g_{j\overline{k},r} - g_{r\overline{k},j} \right) \right.\\
&\left. + g^{p\overline{q}} g^{r\overline{s}} g_{j\overline{q},\overline{s}} \left(g_{p\overline{k},r} - g_{r\overline{k},p} \right) + g^{p\overline{q}} g^{r\overline{s}} g_{p\overline{k},\overline{s}} \left(g_{j\overline{q},r} - g_{r\overline{q},j} \right) \right].
\end{aligned}$$

Proof. First of all we know that

$$(\partial\omega)_{ij\overline{k}} = \frac{\sqrt{-1}}{2}\left(g_{j\overline{k},i} - g_{i\overline{k},j}\right)$$

Now, we use the general formula for ∂^*_ω and compute

$$\begin{aligned}
\left(\partial^*_g\partial\omega\right)_{j\overline{k}} =& - g_{j\overline{p}}g_{\overline{k}q}\left(\frac{\partial}{\partial\overline{z}^m} + \frac{1}{g}\partial_{\overline{m}}g\right)(\partial\omega)^{\overline{m}\overline{p}q}\\
=& -\frac{\sqrt{-1}}{2}\Big[g_{j\overline{p}}g_{\overline{k}q}\frac{\partial}{\partial\overline{z}^m}\left[g^{\overline{m}i}g^{\overline{p}r}g^{\overline{s}q}\left(g_{r\overline{s},i} - g_{i\overline{s},r}\right)\right]\\
&+g^{\overline{m}n}g^{p\overline{q}}g_{p\overline{q},\overline{m}}\left(g_{j\overline{k},n} - g_{n\overline{k},j}\right)\Big]\\
=& \frac{\sqrt{-1}}{2}\Big[g^{p\overline{q}}\left(g_{p\overline{k},j\overline{q}} - g_{j\overline{k},p\overline{q}}\right)\\
&+ g_{j\overline{p}}g_{\overline{k}q}\left(g_{r\overline{s},i} - g_{i\overline{s},r}\right)\left[g^{\overline{m}u}g_{u\overline{v},\overline{m}}g^{i\overline{v}}g^{\overline{p}r}g^{\overline{s}q}\right]\\
&+ g_{j\overline{p}}g_{\overline{k}q}\left(g_{r\overline{s},i} - g_{i\overline{s},r}\right)\left[g^{\overline{m}i}g^{\overline{p}u}g_{u\overline{v},\overline{m}}g^{\overline{v}r}g^{\overline{s}q}\right]\\
&+ g_{j\overline{p}}g_{\overline{k}q}\left(g_{r\overline{s},i} - g_{i\overline{s},r}\right)\left[g^{\overline{m}i}g^{\overline{p}r}g^{\overline{s}u}g_{u\overline{v},\overline{m}}g^{\overline{v}q}\right]\\
&-g^{\overline{m}n}g^{p\overline{q}}g_{p\overline{q},\overline{m}}\left(g_{j\overline{k},n} - g_{n\overline{k},j}\right)\Big]\\
=& \frac{\sqrt{-1}}{2}\Big[g^{p\overline{q}}\left(g_{p\overline{k},j\overline{q}} - g_{j\overline{k},p\overline{q}}\right) + g^{p\overline{q}}g^{r\overline{s}}\left(g_{p\overline{s},\overline{q}} - g_{p\overline{q},\overline{s}}\right)\left(g_{j\overline{k},r} - g_{r\overline{k},j}\right)\\
&+g^{p\overline{q}}g^{r\overline{s}}g_{j\overline{q},\overline{s}}\left(g_{p\overline{k},r} - g_{r\overline{k},p}\right) + g^{p\overline{q}}g^{r\overline{s}}g_{p\overline{k},\overline{s}}\left(g_{j\overline{q},r} - g_{r\overline{q},j}\right)\Big].
\end{aligned}$$

□

Lemma 2.8. *Given g a Hermitian metric we have in complex coordinates*

$$\left(\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g\right)_{j\overline{k}} = \frac{\sqrt{-1}}{2}\left(g^{p\overline{q}}\partial_j\partial_{\overline{k}}g_{p\overline{q}} - g^{p\overline{r}}\partial_j g_{\overline{r}s}g^{s\overline{q}}\partial_{\overline{k}}g_{p\overline{q}}\right)$$

Proof. We compute directly in coordinates

$$\begin{aligned}
\left(\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g\right)_{j\overline{k}} =& \frac{\sqrt{-1}}{2}\partial_j\left(g^{p\overline{q}}\partial_{\overline{k}}g_{p\overline{q}}\right)\\
=& \frac{\sqrt{-1}}{2}\left(g^{p\overline{q}}\partial_j\partial_{\overline{k}}g_{p\overline{q}} - g^{p\overline{r}}\partial_j g_{\overline{r}s}g^{s\overline{q}}\partial_{\overline{k}}g_{p\overline{q}}\right).
\end{aligned}$$

□

Also in this section we introduce canonical coordinates for g. We know that if g is not Kähler then we cannot choose complex coordinates so that all the first derivatives of g vanish. However, we can always ensure that a certain symmetric part of the first derivatives vanishes. This is made clear in the lemma below.

Lemma 2.9. *Given a point $p \in M$, there exist coordinates around p so that*

$$g_{i\overline{j}} = \delta_{ij}$$

and

$$\partial_i g_{j\overline{k}} + \partial_j g_{i\overline{k}} = 0.$$

Proof. Let $\{z_i\}$ be arbitrary complex coordinate functions around p so that $z^i(p) = 0$ for all i. We briefly change our point of view and consider the Hermitian metric h associated to g. The coordinate expression for h takes the form

$$h = h_{ij} dz^i d\overline{z}^j$$

where $h_{ij} = \overline{h}_{ji}$. Without loss of generality by a rotation and rescaling we can assume

$$h_{ij}(p) = \delta_{ij} \tag{9}$$

Define new coordinates $\{w^i\}$ by the equation

$$w^i = z^i + \frac{1}{4} \sum_{j,k} \left(\frac{\partial}{\partial z^k} h_{ij}(p) + \frac{\partial}{\partial z^j} h_{ik}(p) \right) z^j z^k$$

so that

$$dw^i = dz^i + \frac{1}{2} \sum_{j,k} \left(\frac{\partial}{\partial z^k} h_{ij}(p) + \frac{\partial}{\partial z^j} h_{ik}(p) \right) z^j dz^k$$

Note also that (9) still holds in these coordinates. In these new coordinates write

$$h = \widetilde{h}_{ij} dw^i d\overline{w}^j$$

It is clear that

$$\widetilde{h}_{ij} = h_{ij} - \frac{1}{2} \sum_{j,k} \left(\frac{\partial}{\partial z^k} h_{ij}(p) + \frac{\partial}{\partial z^j} h_{ik}(p) \right) z^k + \mathcal{O}(z^2).$$

The claim follows directly by differentiating. □

3. The Hermitian Hilbert Functional

Let (M^{2n}, g, J) be a complex manifold. See section 2 for the definition of various quantities related to the torsion. Consider the functional

$$\mathbb{F}(g) = \frac{\int_M \left[s - \frac{1}{4} \left|T\right|^2 - \frac{1}{2} \left|w\right|^2 \right] dV}{\left(\int_M dV \right)^{\frac{n-1}{n}}}.$$

Lemma 3.1. *Let $g(a)$ be a one-parameter family of Hermitian metrics with variation h. Then*

$$\begin{aligned} \frac{\partial}{\partial a} \mathbb{F}(g) = \left(\int_M dV \right)^{\frac{1-n}{n}} \int_M \Big\langle h, &-S + \frac{1}{2} Q^1 - \frac{1}{4} Q^2 - \frac{1}{2} Q^3 + Q^4 \\ &+ \left(s - \frac{1}{4} \left|T\right|^2 - \frac{1}{2} \left|w\right|^2 - \frac{n-1}{n} \frac{\left(\int_M s - \frac{1}{4} \left|T\right|^2 - \frac{1}{2} \left|w\right|^2 \right) dV}{\int_M dV} \right) g \Big\rangle dV. \end{aligned}$$

Moreover, $\mathbb{F}$ is the unique *second-order functional which yields $\overset{\circ}{S}$ as the leading order term in the traceless part of the variational equation through Hermitian metrics.*

Proof. Combining Lemmas 10.7, 10.8 and 10.9 we see

$$\frac{\partial}{\partial a}\int_M \left[s - \frac{1}{4}|T|^2 - \frac{1}{2}|w|^2\right] dV = \int_M \left[\left\langle h, -S + \frac{1}{2}Q^1 - \frac{1}{4}Q^2 - \frac{1}{2}Q^3 + Q^4\right\rangle + \operatorname{tr} h\left(s - \frac{1}{4}|T|^2 - \frac{1}{2}|w|^2\right)\right] dV.$$

Likewise we compute

$$\frac{\partial}{\partial a}\left(\int_M dV\right)^{\frac{n-1}{n}} = \frac{n-1}{n}\int_M \operatorname{tr} h dV\left(\int_M dV\right)^{-\frac{1}{n}}$$

Combining these two calculations gives the result. The claim of uniqueness is also clear by inspection of the variational formulas in Lemmas 10.7, 10.8 and 10.9. □

Let

$$Q = \frac{1}{2}Q^1 - \frac{1}{4}Q^2 - \frac{1}{2}Q^3 + Q^4 \tag{10}$$

and let $K := S - Q$. Note that

$$k := \operatorname{tr}_g K = s - \frac{1}{4}|T|^2 - \frac{1}{2}|w|^2$$

We can rephrase the above situation in a very simple manner. In particular

$$\mathbb{F}(g) = \int_M k dV$$

and

$$\frac{\partial}{\partial a}\mathbb{F}(g(a)) = \int_M \left\langle h, -K + kg - \frac{n-1}{n}\frac{\int_M k dV}{\int_M dV} g\right\rangle dV \tag{11}$$

which is exactly analogous to the form of the gradient of the normalized Hilbert functional.

Definition 3.2. Given (M^{2n}, g, J) a complex manifold we say that g is *static* if g is critical for $\mathbb{F}$.

Proposition 3.3. *Let (M^{2n}, g, J) be a complex manifold with g static. Then*

$$K - \frac{1}{n}kg = 0.$$

Also k is a constant function. Finally, if $\mathbb{F}(g) \geq 0$ and $\int_M s dV_g \leq 0$ then g is Kähler-Einstein.

Proof. The first property follows immediately by letting $h = K - \frac{1}{n}kg$ in (11). Next let $h = -(\Delta_D k)g$ where Δ_D means the Laplacian with respect to the Levi-Civita connection. Plugging this into (11) yields

$$0 = -\int_M (\Delta_D k) k dV = \int_M |dk|^2.$$

To see the last claim we simply note that together the hypotheses imply

$$0 \leq \mathbb{F}(g) = V^{-\frac{1}{n}}k = V^{-\frac{1}{n}}\left(s - \frac{1}{4}|T|^2 - \frac{1}{2}|w|^2\right) \leq -\frac{1}{V^{\frac{1}{n}}}|T|^2$$

which implies $T \equiv 0$. If the torsion of g vanishes, g is Kähler and moreover K is given by the Ricci tensor of g, so g is Kähler-Einstein. □

To emphasize, S is in a sense the only natural curvature tensor associated to a Hermitian metric which is a symmetric $(1,1)$ tensor and which is a second order elliptic operator. In seeking a functional which yields S as the leading term in the Euler-Lagrange equation, $\mathbb{F}$ above is the *only* choice. We note that the functional $\int_M s dV_g$ was considered in [9]. Our Lemma 10.7 contains the main calculation of that paper. Indeed for this functional one gets automatically that critical points are Kähler-Einstein if the value of the functional is nonzero. However, there the leading term in the Euler-Lagrange equation is P, which is not an elliptic operator on Hermitian metrics. Finally, we remark that $\mathbb{F}$ is *not* the Hilbert functional restricted to Hermitian metrics. Indeed a straightforward calculation (see also [8] line (33)) shows that if r denotes the usual scalar curvature,

$$\int_M r dV_g = \int_M s - \frac{1}{4} \left|T\right|^2.$$

Therefore we see that $\mathbb{F} = \int_M \left(r - \frac{1}{2} \left|w\right|^2 \right) dV_g$ restricted to Hermitian metrics. This bears a certain formal similarity to functionals related to renormalization group flows arising in physical models [13] [16], [17].

4. Hermitian Curvature Flow

In this section we give the definition of Hermitian curvature flow in terms of the Chern curvature. We then provide an equivalent definition using Hodge operators. In all the calculations below Q is defined by (10).

Proposition 4.1. *Let (M^{2n}, g, J) be a Hermitian manifold and let*

$$\Phi(g) := (S - Q)(g).$$

Then Φ is a map

$$\Phi(\omega) : \Re \operatorname{Sym}^{1,1} T^* M \to \Re \operatorname{Sym}^{1,1} T^* M$$

where $\Re \operatorname{Sym}^{1,1} T^* M$ are the real symmetric type $(1,1)$ tensors. Moreover, Φ is a nonlinear second order elliptic operator.

Proof. It follows from Lemma 2.2 that

$$\Phi(g)_{i\bar{j}} = -g^{k\bar{l}} g_{i\bar{j},k\bar{l}} + \mathcal{O}(\partial g)$$

and so Φ is a nonlinear second order elliptic operator since g is positive definite. Also, by definition each of the tensors Q^i is a real symmetric $(1,1)$ tensor and thus Q is. It also follows from Lemma 2.2 that S is as well. Therefore $\Phi(g)$ is a real symmetric $(1,1)$-tensor. The result follows. □

Definition 4.2. Given (M^{2n}, J, g_0) a complex manifold with Hermitian metric g_0. We say that a one-parameter family of Hermitian metrics $g(t)$ is a solution to *Hermitian curvature flow (HCF)* with initial condition g_0 if

$$\begin{aligned} \frac{\partial}{\partial t} g(t) &= -S(g(t)) + Q(g(t)) \\ g(0) &= g_0. \end{aligned}$$

Next we compute a formula for HCF using Hodge operators. Define

$$\Xi = S(\cdot, J\cdot).$$

We will write $\Xi(g)$ using Hodge differentials. Let

$$\Psi(\omega) := \partial_g^* \partial\omega - \partial\partial_g^*\omega - \frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det g - 2\sqrt{-1}\left(\overline{\partial}_g^*\omega \lrcorner \overline{\partial}\omega\right) \tag{12}$$

We choose to isolate this term because as we will see in the calculations below, this term is a real $(1,1)$ form.

Proposition 4.3. *Given g a Hermitian metric, we have*

$$\Xi(g) = \Psi(\omega) - \frac{\sqrt{-1}}{2}\left(2Q^4_{j\overline{k}} + \frac{1}{2}Q^2_{j\overline{k}}\right).$$

Proof. Choose coordinates according to Lemma 2.9 so that at a fixed point $p \in M$

$$\partial_i g_{j\overline{k}} = -\partial_j g_{i\overline{k}}..$$

Using this we have that at the point p

$$\frac{1}{2}T_{ij\overline{k}} = \frac{1}{2}\left(\partial_i g_{j\overline{k}} - \partial_j g_{i\overline{k}}\right) = \partial_i g_{j\overline{k}}$$

Next we compute a formula for Ψ in coordinates. In particular we compute a formula for $\partial\overline{\partial}\log\det g$.

$$\begin{aligned}
\frac{\sqrt{-1}}{2}\left(\partial\overline{\partial}\log\det g\right)_{j\overline{k}} &= \frac{\sqrt{-1}}{2}\partial_j\left(g^{p\overline{q}}\partial_{\overline{k}}g_{p\overline{q}}\right)\\
&= \frac{\sqrt{-1}}{2}\left(g^{p\overline{q}}\partial_j\partial_{\overline{k}}g_{p\overline{q}} - g^{p\overline{r}}\partial_j g_{\overline{r}s}g^{s\overline{q}}\partial_{\overline{k}}g_{p\overline{q}}\right)\\
&= \frac{\sqrt{-1}}{2}\left(g^{p\overline{q}}g_{p\overline{q},j\overline{k}} - \frac{1}{4}g^{p\overline{q}}g^{r\overline{s}}T_{jr\overline{q}}T_{\overline{k}\overline{s}p}\right).
\end{aligned}$$

Next we compute using Lemma 2.5

$$\begin{aligned}
-2\sqrt{-1}\left(\overline{\partial}_g^*\omega \lrcorner \overline{\partial}\omega\right) &= -2\sqrt{-1}g^{p\overline{q}}\left(\overline{\partial}_g^*\omega\right)_p\left(\overline{\partial}\omega\right)_{\overline{q}j\overline{k}}\\
&= \frac{\sqrt{-1}}{2}g^{p\overline{q}}g^{r\overline{s}}T_{rp\overline{s}}T_{\overline{q}\overline{k}j}.
\end{aligned}$$

We now combine these calculations with Lemma 2.6 and Lemma 2.7 to get

$$\begin{aligned}
\Psi(\omega)_{j\overline{k}} = &\frac{\sqrt{-1}}{2}\left[-g^{p\overline{q}}g_{j\overline{k},p\overline{q}} + \frac{1}{2}g^{p\overline{q}}g^{r\overline{s}}\left[2T_{\overline{q}\overline{s}p}T_{rj\overline{k}} + 2T_{pr\overline{q}}T_{\overline{s}\overline{k}j}\right.\right.\\
&\left.\left.+T_{\overline{s}\overline{q}j}T_{rp\overline{k}} + T_{\overline{s}\overline{k}p}T_{rj\overline{q}} - T_{jr\overline{q}}T_{\overline{k}\overline{s}p} + \frac{1}{2}T_{jr\overline{q}}T_{\overline{k}\overline{s}p}\right]\right]\\
= &\frac{\sqrt{-1}}{2}\left[-g^{p\overline{q}}g_{j\overline{k},p\overline{q}}\right.\\
&\left.+\frac{1}{2}g^{p\overline{q}}g^{r\overline{s}}\left[2T_{\overline{q}\overline{s}p}T_{rj\overline{k}} + 2T_{pr\overline{q}}T_{\overline{s}\overline{k}j} + T_{\overline{s}\overline{q}j}T_{rp\overline{k}} + \frac{1}{2}T_{jr\overline{q}}T_{\overline{k}\overline{s}p}\right]\right].
\end{aligned}$$

Likewise we have from Lemma 2.2

$$\Xi(g)_{j\overline{k}} = \frac{\sqrt{-1}}{2}\left[-g^{p\overline{q}}g_{j\overline{k},p\overline{q}} + \frac{1}{4}g^{p\overline{q}}g^{r\overline{s}}T_{jp\overline{s}}T_{\overline{k}\overline{q}r}\right].$$

The result follows by combining these calculations. □

Corollary 4.4. *The HCF equation is equivalent to*

$$\frac{\partial}{\partial t}\omega = -\Psi(\omega) + \frac{\sqrt{-1}}{2}\left(\frac{1}{2}Q^1 + \frac{1}{4}Q^2 - \frac{1}{2}Q^3 + 3Q^4\right).$$

Proof. This follows immediately from the definition of HCF and Proposition 4.3. □

5. Short-Time Existence

Proposition 5.1. *Given (M^{2n}, J, g_0) a compact complex manifold, there exists a unique solution to HCF with initial condition g_0 on $[0, \epsilon)$ for some $\epsilon > 0$.*

Proof. Since the operator $\Phi(g)$ is strictly elliptic by Proposition 4.1 the HCF equation is strictly parabolic, and thus short-time existence and uniqueness follows from standard theory. □

Proposition 5.2. *Given (M^{2n}, J, g_0) a compact complex manifold with Kähler metric g_0, let $g(s)$ denote the solution to HCF with initial condition g_0, which exists on $[0, T)$. Then for all $t \in [0, T)$, $g(t)$ is Kähler, and is a solution to Kähler Ricci flow.*

Proof. Let $\widetilde{g}(t)$ be the solution to Kähler Ricci flow with initial condition g_0. Ricci flow preserves the Kähler condition, thus $\widetilde{g}(t)$ is Kähler for all time, hence $\widetilde{T} = d\widetilde{\omega} = 0$ and $\widetilde{S} = \widetilde{\mathrm{Rc}}$. Together this implies that $\widetilde{g}(t)$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{g}(t) &= -\widetilde{\mathrm{Rc}}\\ &= -\widetilde{S} + \widetilde{Q}\end{aligned}$$

Thus $\widetilde{g}(t)$ is a solution to HCF with initial condition g_0. Since solutions to HCF are unique, it follows that $\widetilde{g}(t) = g(t)$ for all time and hence $g(t)$ is Kähler for all time and solves Kähler-Ricci flow. □

6. Evolution Equations

Lemma 6.1. *For a solution to HCF we have*

$$\begin{aligned}\frac{\partial}{\partial t}\Omega_{i\bar{j}k\bar{l}} = {} & \Delta\Omega_{i\bar{j}k\bar{l}} + g^{m\bar{n}}\left(T_{\bar{n}\bar{j}}^{\bar{p}}\nabla_m\Omega_{i\bar{p}k\bar{l}} + T_{mi}^{p}\nabla_{\bar{j}}\Omega_{p\bar{n}k\bar{l}}\right)\\ & + g^{m\bar{n}}\left(\Omega_{i\bar{j}m}^{p}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{n}\bar{j}}^{\bar{p}}\Omega_{i\bar{p}k\bar{l}} + \Omega_{m\bar{j}k}^{p}\Omega_{i\bar{n}p\bar{l}} + \Omega_{m\bar{j}\bar{l}}^{\bar{p}}\Omega_{i\bar{n}k\bar{p}}\right)\\ & - \Omega_{i\bar{j}k}^{m}\left(S_{m\bar{l}} - Q_{m\bar{l}}\right) - \nabla_{\bar{j}}\nabla_i Q_{k\bar{l}}\end{aligned}$$

Proof. First consider the term Q in the evolution of g. Using Lemma 10.2 we see that this contributes

$$\Omega_{i\bar{j}k}^{m}Q_{m\bar{l}} - \nabla_{\bar{j}}\nabla_i Q_{k\bar{l}}.$$

to the evolution of $\Omega_{i\bar{j}k\bar{l}}$. Next we consider the contribution of the term $-S$ in the evolution of g. Using Lemma 10.2 we see that the evolution $\frac{\partial}{\partial t}g = -S$ yields

$$\frac{\partial}{\partial t}\Omega_{i\bar{j}k\bar{l}} = -\Omega_{i\bar{j}k}^{m}S_{m\bar{l}} + \nabla_{\bar{j}}\nabla_i S_{k\bar{l}}.$$

Now we must apply the second Bianchi identity. We have

$$\begin{aligned}
\nabla_{\bar{j}}\left(\nabla_i S_{k\bar{l}}\right) &= \nabla_{\bar{j}} g^{m\bar{n}}\left(\nabla_i \Omega_{m\bar{n}k\bar{l}}\right)\\
&= \nabla_{\bar{j}} g^{m\bar{n}}\left(\nabla_m \Omega_{i\bar{n}k\bar{l}} + T_{mi}^{p}\Omega_{p\bar{n}k\bar{l}}\right)\\
&= g^{m\bar{n}}\left(\nabla_{\bar{j}}\nabla_m \Omega_{i\bar{n}k\bar{l}} + \nabla_{\bar{j}} T_{mi}^{p}\Omega_{p\bar{n}k\bar{l}} + T_{mi}^{p}\nabla_{\bar{j}}\Omega_{p\bar{n}k\bar{l}}\right)
\end{aligned}$$

Next we commute covariant derivatives to get

$$\begin{aligned}
g^{m\bar{n}}\nabla_{\bar{j}}\nabla_m \Omega_{i\bar{n}k\bar{l}} = g^{m\bar{n}}\Big(&\nabla_m\nabla_{\bar{j}}\Omega_{i\bar{n}k\bar{l}} + \Omega_{m\bar{j}i}^{p}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{j}\bar{n}}^{\bar{p}}\Omega_{i\bar{p}k\bar{l}}\\
&+\Omega_{m\bar{j}k}^{p}\Omega_{i\bar{n}p\bar{l}} + \Omega_{m\bar{j}\bar{l}}^{\bar{p}}\Omega_{i\bar{n}k\bar{p}}\Big).
\end{aligned}$$

Finally, we apply the Bianchi identity again to get

$$\begin{aligned}
g^{m\bar{n}}\nabla_m\nabla_{\bar{j}}\Omega_{i\bar{n}k\bar{l}} &= g^{m\bar{n}}\nabla_m\left(\nabla_{\bar{n}}\Omega_{i\bar{j}k\bar{l}} + T_{\bar{n}\bar{j}}^{\bar{p}}\Omega_{i\bar{p}k\bar{l}}\right)\\
&= \Delta\Omega_{i\bar{j}k\bar{l}} + g^{m\bar{n}}\left(\nabla_m T_{\bar{n}\bar{j}}^{\bar{p}}\Omega_{i\bar{p}k\bar{l}} + T_{\bar{n}\bar{j}}^{\bar{p}}\nabla_m\Omega_{ipk\bar{l}}\right)
\end{aligned}$$

Combining these calculations yields

$$\begin{aligned}
\frac{\partial}{\partial t}\Omega_{i\bar{j}k\bar{l}} &= \Delta\Omega_{i\bar{j}k\bar{l}}\\
&\quad + g^{m\bar{n}}\left(\nabla_m T_{\bar{n}\bar{j}}^{\bar{p}}\Omega_{i\bar{p}k\bar{l}} + T_{\bar{n}\bar{j}}^{\bar{p}}\nabla_m\Omega_{i\bar{p}k\bar{l}} + \nabla_{\bar{j}}T_{mi}^{p}\Omega_{p\bar{n}k\bar{l}} + T_{mi}^{p}\nabla_{\bar{j}}\Omega_{p\bar{n}k\bar{l}}\right)\\
&\quad + g^{m\bar{n}}\left(\Omega_{m\bar{j}i}^{p}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{j}\bar{n}}^{\bar{p}}\Omega_{i\bar{p}k\bar{l}} + \Omega_{m\bar{j}k}^{p}\Omega_{i\bar{n}p\bar{l}} + \Omega_{m\bar{j}\bar{l}}^{\bar{p}}\Omega_{i\bar{n}k\bar{p}}\right) - \Omega_{i\bar{j}k}^{m}S_{m\bar{l}}
\end{aligned}$$

Now we can apply the Bianchi identity to the terms

$$\Omega_{m\bar{j}\bar{n}}^{\bar{p}} = \Omega_{m\bar{n}\bar{j}}^{\bar{p}} + \nabla_m T_{\bar{j}\bar{n}}^{\bar{p}}$$

and

$$\Omega_{m\bar{j}i}^{p} = \Omega_{i\bar{j}m}^{p} + \nabla_{\bar{j}}T_{im}^{p}.$$

Plugging these two in yields

$$\begin{aligned}
\frac{\partial}{\partial t}\Omega_{i\bar{j}k\bar{l}} &= \Delta\Omega_{i\bar{j}k\bar{l}} + g^{m\bar{n}}\left(T_{\bar{n}\bar{j}}^{\bar{p}}\nabla_m\Omega_{i\bar{p}k\bar{l}} + T_{mi}^{p}\nabla_{\bar{j}}\Omega_{p\bar{n}k\bar{l}}\right)\\
&\quad + g^{m\bar{n}}\left(\Omega_{i\bar{j}m}^{p}\Omega_{p\bar{n}k\bar{l}} + \Omega_{m\bar{n}\bar{j}}^{\bar{p}}\Omega_{i\bar{p}k\bar{l}} + \Omega_{m\bar{j}k}^{p}\Omega_{i\bar{n}p\bar{l}} + \Omega_{m\bar{j}\bar{l}}^{\bar{p}}\Omega_{i\bar{n}k\bar{p}}\right)\\
&\quad - \Omega_{i\bar{j}k}^{m}S_{m\bar{l}}.
\end{aligned}$$

Combining this with the above terms gives the result. □

Lemma 6.2. *For a solution to HCF we have*

$$\begin{aligned}
\frac{\partial}{\partial t}T_{ij\bar{k}} = \Delta T_{ij\bar{k}} &+ g^{m\bar{n}}\Big[T_{ji}^{p}\nabla_{\bar{n}}T_{mp\bar{k}} + \nabla_{\bar{n}}T_{mj}^{p}T_{ip\bar{k}} + T_{mj}^{p}\nabla_{\bar{n}}T_{ip\bar{k}}\\
&\qquad +\nabla_{\bar{n}}T_{im}^{p}T_{jp\bar{k}} + T_{im}^{p}\nabla_{\bar{n}}T_{jp\bar{k}}\Big]\\
&+ g^{m\bar{n}}\Big[\Omega_{\bar{n}jm}^{p}T_{ip\bar{k}} + \Omega_{\bar{n}j\bar{k}}^{\bar{p}}T_{im\bar{p}} - \Omega_{\bar{n}im}^{p}T_{jp\bar{k}}\\
&\qquad -\Omega_{\bar{n}i\bar{k}}^{\bar{p}}T_{jm\bar{p}} - \Omega_{p\bar{n}m\bar{k}}T_{ji}^{p}\Big] - T_{ij}^{p}\left(S_{p\bar{k}} - Q_{p\bar{k}}\right)\\
&+ \nabla_i Q_{j\bar{k}} - \nabla_j Q_{i\bar{k}}.
\end{aligned}$$

Proof. First we compute the contribution from the term Q in the evolution of g. In particular using Lemma 10.4 this yields

$$\nabla_i Q_{j\overline{k}} - \nabla_j Q_{i\overline{k}} + T_{ij}^p Q_{p\overline{k}}.$$

Next we focus on the term $-S$. Applying Lemma 10.4 we have.

$$\nabla_j S_{i\overline{k}} - \nabla_i S_{j\overline{k}} - T_{ij}^p S_{p\overline{k}}.$$

Now we rewrite using the Bianchi identity

$$\begin{aligned}\nabla_j S_{i\overline{k}} &= g^{m\overline{n}} \nabla_j \Omega_{m\overline{n}i\overline{k}}\\ &= g^{m\overline{n}} \nabla_j \left(\Omega_{i\overline{n}m\overline{k}} + \nabla_{\overline{n}} T_{im\overline{k}}\right)\end{aligned}$$

and

$$\nabla_i S_{j\overline{k}} = g^{m\overline{n}} \nabla_i \left(\Omega_{j\overline{n}m\overline{k}} + \nabla_{\overline{n}} T_{jm\overline{k}}\right).$$

Combining these yields

$$\begin{aligned}\frac{\partial}{\partial t} T_{ij\overline{k}} = &\, g^{m\overline{n}} \left(\nabla_j \Omega_{i\overline{n}m\overline{k}} - \nabla_i \Omega_{j\overline{n}m\overline{k}} + \nabla_j \nabla_{\overline{n}} T_{im\overline{k}} - \nabla_i \nabla_{\overline{n}} T_{jm\overline{k}}\right)\\ &- T_{ij}^p S_{p\overline{k}}.\end{aligned}$$

Applying the Bianchi identity again yields

$$g^{m\overline{n}} \left(\nabla_j \Omega_{i\overline{n}m\overline{k}} - \nabla_i \Omega_{j\overline{n}m\overline{k}}\right) = - g^{m\overline{n}} T_{ji}^p \Omega_{p\overline{n}m\overline{k}}.$$

Also, we commute derivatives

$$\nabla_j \nabla_{\overline{n}} T_{im\overline{k}} = \nabla_{\overline{n}} \nabla_j T_{im\overline{k}} + \Omega_{\overline{n}ji}^p T_{pm\overline{k}} + \Omega_{\overline{n}jm}^p T_{ip\overline{k}} + \Omega_{\overline{n}j\overline{k}}^{\overline{p}} T_{im\overline{p}}$$

and

$$\nabla_i \nabla_{\overline{n}} T_{jm\overline{k}} = \nabla_{\overline{n}} \nabla_i T_{jm\overline{k}} + \Omega_{\overline{n}ij}^p T_{pm\overline{k}} + \Omega_{\overline{n}im}^p T_{jp\overline{k}} + \Omega_{\overline{n}i\overline{k}}^{\overline{p}} T_{jm\overline{p}}.$$

Finally, using Lemma 2.1 we see

$$\begin{aligned}g^{m\overline{n}} \nabla_{\overline{n}} \left(\nabla_j T_{im\overline{k}} - \nabla_i T_{jm\overline{k}}\right) &= g^{m\overline{n}} \nabla_{\overline{n}} \left(\nabla_m T_{ij\overline{k}} + T_{ji}^p T_{mp\overline{k}} + T_{mj}^p T_{ip\overline{k}} + T_{im}^p T_{jp\overline{k}}\right)\\ &= \Delta T_{ij\overline{k}} + g^{m\overline{n}} \nabla_{\overline{n}} \left(T_{ji}^p T_{mp\overline{k}} + T_{mj}^p T_{ip\overline{k}} + T_{im}^p T_{jp\overline{k}}\right).\end{aligned}$$

Combining these calculations yields

$$\begin{aligned}\frac{\partial}{\partial t} T_{ij\overline{k}} = &\, \Delta T_{ij\overline{k}} + g^{m\overline{n}} \nabla_{\overline{n}} \left(T_{ji}^p T_{mp\overline{k}} + T_{mj}^p T_{ip\overline{k}} + T_{im}^p T_{jp\overline{k}}\right)\\ &+ g^{m\overline{n}} \Big[\Omega_{\overline{n}ji}^p T_{pm\overline{k}} + \Omega_{\overline{n}jm}^p T_{ip\overline{k}} + \Omega_{\overline{n}j\overline{k}}^{\overline{p}} T_{im\overline{p}}\\ &- \Omega_{\overline{n}ij}^p T_{pm\overline{k}} - \Omega_{\overline{n}im}^p T_{jp\overline{k}} - \Omega_{\overline{n}i\overline{k}}^{\overline{p}} T_{jm\overline{p}} - \Omega_{p\overline{n}m\overline{k}} T_{ji}^p\Big] - T_{ij}^p S_{p\overline{k}}.\end{aligned}$$

Using the Bianchi identity we can simplify

$$\begin{aligned}g^{m\overline{n}} \left[\Omega_{\overline{n}ji}^p T_{pm\overline{k}} - \Omega_{\overline{n}ij}^p T_{pm\overline{k}}\right] &= g^{m\overline{n}} \left[\Omega_{\overline{n}ji}^p + \Omega_{i\overline{n}j}^p\right] T_{pm\overline{k}}\\ &= g^{m\overline{n}} \nabla_{\overline{n}} T_{ji}^p T_{pm\overline{k}}.\end{aligned}$$

Plugging this in yields

$$\begin{aligned}\frac{\partial}{\partial t}T_{ij\overline{k}} = \Delta T_{ij\overline{k}} + g^{m\overline{n}}\Big[&T_{ji}^p\nabla_{\overline{n}}T_{mp\overline{k}} + \nabla_{\overline{n}}T_{mj}^pT_{ip\overline{k}} + T_{mj}^p\nabla_{\overline{n}}T_{ip\overline{k}} \\ &+\nabla_{\overline{n}}T_{im}^pT_{jp\overline{k}} + T_{im}^p\nabla_{\overline{n}}T_{jp\overline{k}}\Big] \\ + g^{m\overline{n}}\Big[&\Omega_{\overline{n}jm}^pT_{ip\overline{k}} + \Omega_{\overline{n}j\overline{k}}^{\overline{p}}T_{im\overline{p}} - \Omega_{\overline{n}im}^pT_{jp\overline{k}} \\ &-\Omega_{\overline{n}i\overline{k}}^{\overline{p}}T_{jm\overline{p}} - \Omega_{p\overline{n}m\overline{k}}T_{ji}^p\Big] - T_{ij}^pS_{p\overline{k}}.\end{aligned}$$

Combining this with the terms from Q gives the result. □

7. Higher Derivative Estimates

In this section we will prove derivative estimates for HCF. It will be most convenient to phrase these results in terms of the curvature of the Chern connection. All of the calculations below will be done in canonical coordinates at a fixed point. In particular, in these coordinates any first derivative of g can be expressed in terms of the torsion T, and any second derivative can be expressed in terms of a sum of curvature and torsion.

Lemma 7.1. *Given $(M^{2n}, g(t), J)$ a solution to HCF we have*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k\Omega = \Delta\nabla^k\Omega + \sum_{j=0}^{k}\nabla^jT*\nabla^{k+1-j}\Omega + \sum_{j=0}^{k}\nabla^j\Omega*\nabla^{k-j}\Omega \\ + \sum_{j=0}^{k}\sum_{l=0}^{j}\nabla^lT*\nabla^{j-l}T*\nabla^{k-j}\Omega.\end{aligned}$$

Proof. The case $k=0$ is covered by Lemma 6.1. We directly compute

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k\Omega &= \frac{\partial}{\partial t}\left(\partial+\Gamma\right)*\left(\partial+\Gamma\right)*\dots*\left(\partial+\Gamma\right)\Omega \\ &= \nabla^k\left(\frac{\partial}{\partial t}\Omega\right) + \left(\frac{\partial}{\partial t}\Gamma\right)*\left(\partial+\Gamma\right)\dots*\left(\partial+\Gamma\right)\Omega \\ &\quad + \left(\partial+\Gamma\right)*\left(\frac{\partial}{\partial t}\Gamma\right)*\dots*\left(\partial+\Gamma\right)\Omega + \dots \\ &\quad + \left(\partial+\Gamma\right)*\dots*\left(\partial+\Gamma\right)*\left(\frac{\partial}{\partial t}\Gamma\right)\Omega\end{aligned}$$

We apply Lemma 10.1 to see that

$$\frac{\partial}{\partial t}\Gamma = \nabla\left(\Omega + T^{*2}\right) = \nabla\Omega + T*\nabla T.$$

Plugging this in yields

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k\Omega &= \nabla^k\left(\Delta\Omega + T*\nabla\Omega + \Omega^{*2} + \Omega * T^{*2}\right)\\ &\quad + \sum_{j=0}^{k-1}\nabla^j\left(\nabla\Omega + T*\nabla T\right)*\nabla^{k-1-j}\Omega\\ &= \Delta\nabla^k\Omega + \sum_{j=0}^{k}\nabla^j T*\nabla^{k+1-j}\Omega + \sum_{j=0}^{k}\nabla^j\Omega*\nabla^{k-j}\Omega\\ &\quad + \sum_{j=0}^{k}\sum_{l=0}^{j}\nabla^l T*\nabla^{j-l}T*\nabla^{k-j}\Omega.\end{aligned}$$

□

Lemma 7.2. *Given $(M^{2n}, g(t), J)$ a solution to HCF we have*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k T &= \Delta\nabla^k T + \sum_{j=0}^{k+1}\nabla^j T*\nabla^{k+1-j}T + \sum_{j=0}^{k}\nabla^j T*\nabla^{k-j}\Omega\\ &\quad + \sum_{j=0}^{k-1}\sum_{l=0}^{j}\nabla^l T*\nabla^{j-l+1}T*\nabla^{k-1-j}T.\end{aligned}$$

Proof. The case $k=0$ is covered by Lemma 6.2. We directly compute

$$\begin{aligned}\frac{\partial}{\partial t}\nabla^k T &= \frac{\partial}{\partial t}\left(\partial+\Gamma\right)*\cdots*\left(\partial+\Gamma\right)T\\ &= \nabla^k\left(\frac{\partial}{\partial t}T\right) + \left(\frac{\partial}{\partial t}\Gamma\right)*\left(\partial+\Gamma\right)\cdots*\left(\partial+\Gamma\right)T\\ &\quad + \left(\partial+\Gamma\right)*\left(\frac{\partial}{\partial t}\Gamma\right)*\cdots*\left(\partial+\Gamma\right)T + \dots\\ &\quad + \left(\partial+\Gamma\right)*\cdots*\left(\partial+\Gamma\right)*\left(\frac{\partial}{\partial t}\Gamma\right)T.\end{aligned}$$

Again we apply Lemma 10.1 to yield

$$\begin{aligned}&= \nabla^k\left(\Delta T + \nabla T * T + \Omega * T\right)\\ &\quad + \sum_{j=0}^{k-1}\nabla^j\left(\nabla\Omega + T*\nabla T\right)*\nabla^{k-1-j}T\\ &= \Delta\nabla^k T + \sum_{j=0}^{k+1}\nabla^j T*\nabla^{k+1-j}T + \sum_{j=0}^{k}\nabla^j T*\nabla^{k-j}\Omega\\ &\quad + \sum_{j=0}^{k-1}\sum_{l=0}^{j}\nabla^l T*\nabla^{j-l+1}T*\nabla^{k-1-j}T.\end{aligned}$$

□

Theorem 7.3. *Let $(M^{2n}, g(t), J)$ be a solution to HCF for which the maximum principle holds. Then for each $\alpha > 0$ and every $m \in \mathbb{N}$ there exists a constant C_m*

depending only on m, n and $\max\{\alpha, 1\}$ such that if

$$\begin{aligned} |\Omega|_{C^0(g_t)} &\leq K, \\ |\nabla T|_{C^0(g_t)} &\leq K, \\ |T|^2_{C^0(g_t)} &\leq K \end{aligned} \tag{13}$$

for all $x \in M$ and $t \in \left[0, \frac{\alpha}{K}\right]$, then

$$\begin{aligned} |\nabla^m \Omega|_{C^0(g_t)} &\leq \frac{C_m K}{t^{m/2}}, \\ \left|\nabla^{m+1} T\right|_{C^0(g_t)} &\leq \frac{C_m K}{t^{m/2}} \end{aligned} \tag{14}$$

for all $x \in M$ and $t \in \left(0, \frac{\alpha}{K}\right]$.

Proof. Our proof is by induction on m. First consider $m = 1$. The following evolution equation for $|\Omega|^2$ follows from Lemma 7.1:

$$\frac{\partial}{\partial t} |\Omega|^2 = \Delta |\Omega|^2 - 2 |\nabla \Omega|^2 + T * \nabla \Omega * \Omega + \Omega^{*3} + T^{*2} * \Omega^{*2}. \tag{15}$$

Also from Lemma 7.1 we conclude

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla \Omega|^2 = \Delta |\nabla \Omega|^2 - 2 \left|\nabla^2 \Omega\right|^2 + T * \nabla^2 \Omega * \nabla \Omega \\ + \Omega * \nabla \Omega^{*2} + \nabla T * \nabla \Omega^{*2} + T^{*2} * \nabla \Omega^{*2} \\ + T * \nabla T * \Omega * \nabla \Omega. \end{aligned} \tag{16}$$

Now, we aim to use the term $-2 |\nabla \Omega|^2$ in the evolution of $|\Omega|^2$ to control the evolution of $|\nabla \Omega|^2$. Consider the function

$$F(x, t) := t |\nabla \Omega|^2 + \beta |\Omega|^2$$

where β is a constant to be chosen below. Putting together (15) and (16) gives

$$\begin{aligned} \frac{\partial}{\partial t} F \leq \Delta F - 2t \left|\nabla^2 \Omega\right|^2 + (1 + c_1 t |\Omega| - 2\beta) |\nabla \Omega|^2 \\ + t \big(T * \nabla^2 \Omega * \nabla \Omega + \Omega * \nabla \Omega^{*2} + \nabla T * \nabla \Omega^{*2} \\ + T^{*2} * \nabla \Omega^{*2} + T * \nabla T * \Omega * \nabla \Omega + T * \Omega^{*2} * \nabla \Omega\big) \\ + c_2 \beta \left(|T|^2 |\Omega|^2 + |\Omega|^3\right) + T * \nabla \Omega * \Omega \end{aligned} \tag{17}$$

where all the c_i are universal constants depending only on dimension. We must estimate the different terms in (17). First of all we use the Cauchy-Schwarz inequality and the assumption (13) to conclude

$$\begin{aligned} t T * \nabla^2 \Omega * \nabla \Omega &\leq t c_3 \left(\left|\nabla^2 \Omega\right|\right) (|T| |\nabla \Omega|) \\ &\leq t c_3 \left(\frac{\left|\nabla^2 \Omega\right|^2}{2 c_3} + \frac{c_3 |T|^2 |\nabla \Omega|^2}{2}\right) \\ &\leq \frac{t}{2} \left|\nabla^2 \Omega\right|^2 + c_4 t K |\nabla \Omega|^2. \end{aligned} \tag{18}$$

Similarly we simplify

$$tT * \Omega^{*2} * \nabla\Omega \leq tc_5 \left(\left|\Omega\right|^2\right)\left(\left|T\right|\left|\nabla\Omega\right|\right) \leq c_6 tK^4 + c_6 tK \left|\nabla\Omega\right|^2 \tag{19}$$

and also

$$T * \Omega * \nabla\Omega \leq \frac{c_7}{\beta} K^3 + \beta\left|\nabla\Omega\right|^2. \tag{20}$$

The remaining terms are estimated in an obviously analogous fashion. Plugging these estimates into (17) gives that for $t \in \left[0, \frac{\alpha}{K}\right]$,

$$\frac{\partial}{\partial t} F \leq \Delta F + (1 + c_8\alpha - \beta)\left|\nabla\Omega\right|^2 + c_9(\alpha + \beta) K^3.$$

Choose $\beta \geq \frac{1+c_8\alpha}{2}$ and note that β depends only on the dimension and $\max\{\alpha, 1\}$. Then we have that for $t \in \left[0, \frac{\alpha}{K}\right]$,

$$\frac{\partial}{\partial t} F \leq c_{10}\beta K^3$$

Using that $F(0) \leq \beta K^2$ and applying the maximum principle gives

$$\sup_{x\in M} F(x,t) \leq \beta K^2 + c_{10}\beta K^3 t \leq (1 + c_{10}\alpha)\beta K^2 \leq C_1^2 K^2$$

where again C_1 depends only on n and $\max\{\alpha, 1\}$. Thus

$$\left|\nabla\Omega\right| \leq \sqrt{\frac{F}{t}} \leq \frac{C_1 K}{t^{1/2}}$$

for all $x \in M$ and $t \in \left(0, \frac{\alpha}{K}\right]$. To get the estimate for $\nabla^2 T$ one computes the evolution of a function

$$F(x,t) := t\left(\left|\nabla\Omega\right|^2 + \left|\nabla^2 T\right|^2\right) + \beta\left|\Omega\right|^2$$

and argues as above, the bounds being entirely analogous. This completes the case $m = 1$.

For the induction step we first conclude from Lemma 7.1 the evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}\left|\nabla^k\Omega\right|^2 = \Delta\left|\nabla^k\Omega\right|^2 - 2\left|\nabla^{k+1}\Omega\right|^2 + \sum_{j=0}^{k}\nabla^j T * \nabla^{k+1-j}\Omega * \nabla^k\Omega \\ + \sum_{j=0}^{k}\nabla^j\Omega * \nabla^{k-j}\Omega * \nabla^k\Omega \\ + \sum_{j=0}^{k}\sum_{l=0}^{j}\nabla^l T * \nabla^{j-l} T * \nabla^{k-j}\Omega * \nabla^k\Omega.\end{aligned}$$

We address the first sum in the above equation. We first make the bound

$$\begin{aligned}T * \nabla^{k+1}\Omega * \nabla^k\Omega &\leq c\left|T\right|\left|\nabla^{k+1}\Omega\right|\left|\nabla^k\Omega\right| \\ &\leq \epsilon\left|\nabla^{k+1}\Omega\right|^2 + C(\epsilon)\left|T\right|^2\left|\nabla^k\Omega\right|^2 \\ &\leq \epsilon\left|\nabla^{k+1}\Omega\right|^2 + C(\epsilon) K\left|\nabla^k\Omega\right|^2.\end{aligned} \tag{21}$$

Also we have

$$\begin{aligned}\nabla T * \nabla^k \Omega * \nabla^k \Omega &\le c\,|\nabla T|\,\left|\nabla^k \Omega\right|^2 \\ &\le K\left|\nabla^k \Omega\right|^2.\end{aligned} \tag{22}$$

For the rest of the summand we bound for $j > 0$

$$\begin{aligned}\nabla^j T * \nabla^{k+1-j}\Omega * \nabla^k \Omega &\le c\left|\nabla^j T\right|\left|\nabla^{k+1-j}\Omega\right|\left|\nabla^k \Omega\right| \\ &\le c\frac{K}{t^{(j-1)/2}}\frac{K}{t^{(k+1-j)/2}}\left|\nabla^k \Omega\right| \\ &\le cK\left|\nabla^k \Omega\right|^2 + c\frac{K^3}{t^k}.\end{aligned} \tag{23}$$

A similar calculation yields a bound

$$\begin{aligned}\nabla^j \Omega * \nabla^{k-j}\Omega * \nabla^k \Omega &\le c\left|\nabla^j \Omega\right|\left|\nabla^{k-j}\Omega\right|\left|\nabla^k \Omega\right| \\ &\le c\frac{K}{t^{j/2}}\frac{K}{t^{(k-j)/2}}\left|\nabla^k \Omega\right| \\ &\le cK\left|\nabla^k \Omega\right|^2 + c\frac{K^3}{t^k}.\end{aligned} \tag{24}$$

Next we bound using the inequality $K \le \frac{C}{t}$

$$\begin{aligned}\nabla^l T * \nabla^{j-l} T * \nabla^{k-j}\Omega * \nabla^k \Omega &\le c\left|\nabla^l T\right|\left|\nabla^{j-l} T\right|\left|\nabla^{k-j}\Omega\right|\left|\nabla^k \Omega\right| \\ &\le \frac{K}{t^{(l-1)/2}}\frac{K}{t^{(j-l-1)/2}}\frac{K}{t^{(k-j)/2}}\left|\nabla^k \Omega\right| \\ &\le \frac{K^3}{t^{(k-2)/2}}\left|\nabla^k \Omega\right| \\ &\le \frac{K^2}{t^{(k-1)/2}}\left|\nabla^k \Omega\right| \\ &\le cK\left|\nabla^k \Omega\right|^2 + c\frac{K^3}{t^k}.\end{aligned} \tag{25}$$

Using (21) - (25) we conclude

$$\frac{\partial}{\partial t}\left|\nabla^k \Omega\right|^2 \le \Delta\left|\nabla^k \Omega\right|^2 - \left|\nabla^{k+1}\Omega\right|^2 + CK\left(\left|\nabla^k \Omega\right|^2 + \frac{K^2}{t^k}\right).$$

Furthermore, using completely analogous bounds one can conclude

$$\frac{\partial}{\partial t}\left|\nabla^{k+1} T\right|^2 \le \Delta\left|\nabla^{k+1} T\right|^2 - \left|\nabla^{k+2} T\right|^2 + \frac{1}{2}\left|\nabla^{k+1}\Omega\right|^2 + CK\left(\left|\nabla^{k+1} T\right|^2 + \frac{K^2}{t^k}\right).$$

The extra term $\frac{1}{2}\left|\nabla^{k+1}\Omega\right|^2$ arises from the term $T * \nabla^{k+1}\Omega * \nabla^{k+1} T$. Together these yield, if we set $H_k = \left|\nabla^k \Omega\right|^2 + \left|\nabla^{k+1}\Omega\right|^2$,

$$\frac{\partial}{\partial t}H_k \le \Delta H_k - \frac{1}{2}H_{k+1} + CKH_k + \frac{K^3}{t^k}.$$

This bound is sufficient to carry out the inductive step analogously to the step $k = 1$. The details of this construction are found in [6] page 229-230. □

Corollary 7.4. *There exists a constant $c = c(n)$ such that given (M^{2n}, g, J) a complex manifold with Hermitian metric g, the solution to HCF with initial condition g exists for $t \in \left[0, \frac{c(n)}{\max\{|\Omega|_{C^0}, |\nabla T|_{C^0}, |T|^2_{C_0}\}}\right]$. Moreover the solution exists on a maximal time interval $[0, \tau)$, and if $\tau < \infty$ then*

$$\limsup_{t \to \tau} \max\{|\Omega|_{C^0(g_t)}, |\nabla T|_{C^0(g_t)}, |T|^2_{C^0(g_t)}\} = \infty.$$

Proof. This argument is standard. Using the evolution equations for T, ∇T and Ω it is easy to prove a "doubling-time" estimate for these quantities on the interval stated using the maximum principle. Once this is in place, the derivative estimates follow from Theorem 7.3. These yield bounds on the curvature and torsion and all covariant derivatives on the stated interval, which can be integrated in time to show smooth existence of the flow on that interval.

Finally, if one has that the curvature, torsion and first covariant derivative of torsion are bounded up to a time $\tau < \infty$, one concludes from Theorem 7.3 uniform bounds on the derivatives of curvature and torsion on $[0, \tau]$. These bounds can be integrated in time to get C^k bounds on the metric on this whole time interval, yielding smooth existence up to this time. □

Note now that Theorem 1.1 is a consequence of Proposition 5.1, Theorem 7.3 and Corollary 7.4.

8. Stability

In this section we prove dynamic stability of HCF near a Kähler-Einstein metric with negative or zero first Chern class[1]. By examining the linearized deformation equation we know that Kähler-Einstein metrics are rigid in case where $c_1(M) < 0$. In the case $c_1(M) = 0$, there can be nontrivial deformation of Kähler-Einstein metrics due to variation of Káhler class. There is a general technique for dealing with stability of evolution equations around integrable stationary points [5], [14], [15]. Given the discussion above, our problem falls squarely into the realm of these techniques, and so we adopt them. We note that since the $c_1(M) < 0$ case is rigid, there may be an easier proof for this case, but in the interest of covering the most cases possible with a single proof we choose the more general technique.

Consider the volume-normalized HCF equation

$$\begin{aligned}\frac{\partial}{\partial t} g =& \ - S + Q + \frac{1}{n} \left(\int_M \operatorname{tr}_g (S - Q)\, dV \right) g \\ =: & \ -\mathcal{F}(g).\end{aligned}$$

[1]In an earlier version of this paper we claimed stability for positive first Chern class as well. However, such a stability result needs a further condition on initial metric, since for instance the Kähler class under volume-normalized Ricci flow satisfies $[\omega_t] = c_1(M) + e^t ([\omega_0] - c_1(M))$. Thus even nearby Kähler metrics will not converge under the resulting Kähler Ricci flow. This is reflected in the fact that Kähler-Einstein metrics with positive first Chern class are not linearly stable, which is what causes our proof to fail. We expect that for a given Kähler-Einstein metric g with $c_1(M) > 0$ there exist certain harmonic (1,1)-forms h such that the solution to HCF with initial condition $g + h$ converges modulo automorphisms to a Kähler-Einstein metric. This will be the subject of future work.

We compute the linearization of $\mathcal{F}$ around a Kähler-Einstein metric. Since the tensor $\mathcal{F}(g)$ is only defined for Hermitian metrics we obviously compute the variation of $\mathcal{F}(g)$ through a family of Hermitian metrics.

Proposition 8.1. *Let (M^{2n}, J) be a complex manifold and suppose $g(a)$ is a one-parameter family of unit volume Hermitian metrics compatible with J with*

$$\frac{\partial}{\partial a} g(a)|_{a=0} = h.$$

Moreover suppose $g(0)$ is Kähler-Einstein. Then

$$\frac{\partial}{\partial a}\mathcal{F}(g) = \nabla^*\nabla h - \overset{\circ}{R}(h)$$

where $\overset{\circ}{R}(h)_{k\bar{l}} = h^{i\bar{j}} R_{k\bar{j}i\bar{l}}$.

Proof. Choose complex coordinates which are normal for $g(0)$ at a point $p \in M$. First we note that

$$\begin{aligned} \frac{\partial}{\partial a} T(a) * T(a)\bigg|_{a=0} &= h * T(0) * T(0) + \left(\frac{\partial}{\partial a} T(a)\right) * T(0) \\ &= 0 \end{aligned}$$

since the metric $g(0)$ is Kähler and hence torsion-free. Now using Lemma 2.2

$$\begin{aligned} \frac{\partial}{\partial a} S_{j\bar{k}}\bigg|_{a=0} &= \frac{\partial}{\partial a}\left(g(a)^{l\overline{m}}\left(-\partial_l \partial_{\overline{m}} g(a)_{j\bar{k}} + \partial g(a) * \overline{\partial} g(a)\right)\right)\bigg|_{a=0} \\ &= -h^{l\overline{m}} R_{l\overline{m}j\bar{k}} - g^{l\overline{m}} \partial_l \partial_{\overline{m}} h_{j\bar{k}} \end{aligned}$$

Now $-h^{i\bar{j}} R_{i\bar{j}k\bar{l}} = -\overset{\circ}{R}(h)_{k\bar{l}}$ from the Bianchi identity using that the metric $g(0)$ is Kähler-Einstein. Next, we compute an expression for $\nabla^*\nabla h$ using complex coordinates

$$\begin{aligned} (\nabla^*\nabla h)_{j\bar{k}} &= -g^{l\overline{m}} \nabla_l \nabla_{\overline{m}} h_{j\bar{k}} \\ &= -g^{l\overline{m}}\left(\partial_l \partial_{\overline{m}} h_{j\bar{k}} - \partial_l \Gamma^{\overline{p}}_{\overline{m}\bar{k}} h_{j\overline{p}}\right) \\ &= -g^{l\overline{m}} \partial_l \partial_{\overline{m}} h_{j\bar{k}} - R_{\bar{l}}^{\overline{m}} h_{k\overline{m}} \\ &= -g^{l\overline{m}} \partial_l \partial_{\overline{m}} h_{j\bar{k}} - \frac{1}{n} s h_{k\bar{l}} \end{aligned}$$

where $R = S$ is the Ricci tensor of the Kähler metric $g(0)$ and $s = r = \operatorname{tr}_g S$ is the scalar curvature. Next we compute using Lemma 10.7

$$\begin{aligned} \frac{\partial}{\partial a}\left(\int_M \operatorname{tr}_g S dV\right)\bigg|_{a=0} &= \int_M \left(\langle h, -S\rangle + \operatorname{tr}_g h \operatorname{tr}_g S\right) dV \\ &= \int_M \left(1 - \frac{1}{n}\right)(\operatorname{tr}_g S) \operatorname{tr}_g h dV \\ &= 0 \end{aligned}$$

where the last line follows since $\mathrm{tr}_g S$ is the scalar curvature which is constant and $\int_M \mathrm{tr}_g h dV = 0$ since the volume is fixed through $g(s)$. Thus

$$\begin{aligned} \frac{\partial}{\partial a} \frac{1}{n} \left(\int_M \mathrm{tr}_g S dV \right) g \Big|_{a=0} &= \frac{1}{n} \left(\int_M \mathrm{tr}_g S dV \right) h \\ &= \frac{1}{n} sh. \end{aligned}$$

Putting together these calculations yields the result. □

Definition 8.2. Let $L = L(g_0) = \mathcal{D}_{g_0} \mathcal{F}$ be the linearization of $\mathcal{F}$ at a static metric g_0. We say that g_0 is *linearly stable* if $L \geq 0$.

Definition 8.3. A static metric g_0 is *integrable* if for any solution h of the linearized equation

$$\mathcal{D}_{g_0} \left(\mathcal{F}(g) \right) (h) = 0$$

there exists a path $g(s), s \in (-\epsilon, \epsilon)$ of static metrics where $g(0) = g_0$ and

$$\frac{d}{ds}_{|s=0} g(s) = h$$

In particular this implies that the set of Hermitian metrics g satisfying $\mathcal{F}(g) \equiv 0$ has a smooth manifold structure near g_0.

We note that by the analysis of Koiso's Theorem it follows that Kähler-Einstein metrics are integrable. Indeed, any solution to the linearized deformation equation arises as the variation along a path of Kähler-Einstein metrics, which are static. This can be seen as follows: If $c_1(M) < 0$, Kähler-Einstein metrics are linearly stable. If $c_1(M) = 0$, any infinitesimal deformation of Kähler-Einstein metrics is given by Hermitian symmetric deformation of Einstein metrics which in turn correspond to $(1,1)$-forms, moreover, we have that the eigenvalues L are the eigenvalues of the operator

$$\psi \to \Delta_d \psi - \frac{1}{n} s \psi$$

acting on $(1,1)$-forms ψ ([2] pg. 362). If $s = 0$, nonnegativity follows easily and the kernel of L consists of harmonic $(1,1)$-forms which are simply variations of Kähler-Einstein metrics with vanishing scalar curvature due to the Calabi-Yau theorem. So Kähler-Einstein metrics are integrable in the case that $c_1(M) = 0$. We now proceed with the proof of Theorem 1.2.

Proof. Let (M, g_0, J) be a Kähler-Einstein manifold. Fix h a symmetric two tensor of type $(1,1)$ such that $|h|_{C^\infty} < \epsilon' < \epsilon$ where ϵ' and ϵ are small positive constants to be chosen later. We want to show that solution to the equation

$$\begin{aligned} \frac{\partial}{\partial t} g &= -S + Q(T) + \frac{1}{n} \left(\int_M \mathrm{tr}_g \left(S - Q \right) dV \right) g \\ g(0) &= g_0 + h \end{aligned} \tag{26}$$

exists for all time and converges for ϵ' chosen small enough. Let $h(t) = g(t) - g_0$. First consider

$$\begin{aligned} \frac{\partial}{\partial t} h &= -S + Q + \frac{1}{n}\left(\int_M \mathrm{tr}_g (S - Q)\, dV\right) g \\ &= -(\mathcal{F}(g_0) + \mathcal{D}\mathcal{F}_{g_0}(h) + A(g_0, h)) \\ &= -\mathcal{D}\mathcal{F}_{g_0}(h) + A(g_0, h) \\ &= -L(h) + A(g_0, h) \end{aligned} \tag{27}$$

where A represents the higher order terms in the approximation of $\mathcal{F}$ by $\mathcal{D}S_{g_0}(h)$. Specifically we have the bounds

$$|A(g_0, h)|_{C^k} \leq C\left(|h|_{C^k}\left|\nabla^2 h\right|_{C^{k-2}} + |\nabla h|^2_{C^{k-1}}\right) \tag{28}$$

where the constant C depends on bounds on the geometry of $g(t)$, which we are assuming is staying bounded along the flow anyways since $|g(t) - g_0|_{C^k} < \epsilon$. So, fix $T > 0$ and a small $\epsilon > 0$. We would like to show that for ϵ' small enough as above our solution exists on $[0, T)$ and $|h(t)|_{C^k} < \epsilon$ on this interval. We start with an L^2 growth estimate.

Lemma 8.4. *There exists a uniform (independent of ϵ, ϵ', T) constant C so that if $|h|_{C^k} < \epsilon$ for all $t \in [0, T)$, we have*

$$\int_M |h(t)|^2\, dV_{g_0} \leq e^{C\epsilon t} \int_M |h_0|^2\, dV_{g_0}$$

Proof. Multiplying the final equation in (27) by h and integrating over M gives

$$\frac{\partial}{\partial t}\int_M |h(t)|^2\, dV_{g_0} \leq \int_M (A * h)\, dV_{g_0}$$

since L is negative semidefinite. By straightforward bounds using integration by parts and the assumed C^k bound on h we are able to get the bound

$$\left|\int_M (A * h)\, dV_{g_0}\right| \leq C\epsilon \int_M |h|^2\, dV_{g_0}$$

where C depends only on g_0. The result follows immediately. □

Lemma 8.5. *There exists $\epsilon' = \epsilon'(T, n) << \epsilon$ such that if $|h_0|_{C^\infty} < \epsilon'$, then the solution $g(t)$ exists on $[0, T)$ with $|h(t)|_k < \epsilon$ for all $t \in [0, T)$.*

Proof. We use standard parabolic regularity theory. First we rewrite the evolution equation for h as

$$\frac{\partial}{\partial t} h = \Delta h + \mathrm{Rm}(h) + A(g_0, h) \tag{29}$$

Fix a time $\tau < T$. We first will get an estimate for $\int_0^\tau \int_M |\nabla h|^2\, dV_{g_0} dt$. Take the inner product of (29) with h and integrate over M to get

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_M |h|^2 &= -\int_M |\nabla h|^2 + \int_M \mathrm{Rm} * h^{*2} + \int_M \nabla^2 h * h^{*2} + h * \nabla h^{*2} \\ &\leq -\int_M |\nabla h|^2 + \theta\int_M |\nabla h|^2 + C(\theta)\int_M |h|^2 \\ &\leq -\frac{1}{2}\int_M |\nabla h|^2 + C(\theta)\int_M |h|^2 \end{aligned} \tag{30}$$

Using this bound and integrating over time we conclude

$$\frac{1}{2}\int_0^\tau \int_M |\nabla h|^2 \leq \frac{1}{2}\int_M |h_0|^2 + C(\theta)\tau \sup_{[0,\tau)} \int_M |h(t)|^2$$

Using Lemma 8.4 we see that $\int_0^\tau \int_M |\nabla h|^2$ can be made very small, in particular bounded uniformly in terms of ϵ'. We now show how to get estimates on $\int_0^\tau \int_M \left|\nabla^k h\right|^2$ for all $k > 0$ in terms of the small constant ϵ'. Consider

$$\begin{aligned}
\frac{1}{2}\frac{\partial}{\partial t}\int_M |\nabla h|^2 = & \int_M \left\langle \nabla\left(\Delta h + \mathrm{Rm}(h) + A(g_0, h)\right), \nabla h\right\rangle dV \\
& + \int_M \nabla^2 h * \nabla h * \nabla h + h * \nabla h^{*2} dV \\
\leq & -\int_M \left|\nabla^2 h\right|^2 + C\int_M |\mathrm{Rm}| \, |\nabla h|^2 \\
& + \theta \int_M \left|\nabla^2 h\right|^2 + C(\theta)\int_M |\nabla h|^4 + C\epsilon' \int_M |\nabla h|^2 \, dV \\
\leq & -\frac{1}{2}\int_M \left|\nabla^2 h\right|^2 + C\int_M |\nabla h|^2 .
\end{aligned}$$

This implies the bound

$$\begin{aligned}
\frac{1}{2}\int_0^\tau \left|\nabla^2 h\right|^2 \leq & \frac{1}{2}\int_M |\nabla h_0|^2 + \frac{1}{2}\int_M |h_0|^2 + C\int_0^\tau \int_M |\nabla h|^2 \\
\leq & C\epsilon'.
\end{aligned}$$

Continuing in this fashion we can induct to get a bound of the above form for all covariant derivatives of h. Note that for instance we can now bound

$$\begin{aligned}
\int_0^\tau \int_M \left|\frac{\partial}{\partial t} h\right|^2 \leq & C\left(\int_0^\tau \int_M \left|\nabla^2 h\right|^2 + \int_0^\tau \int_M |h|^2\right) \\
\leq & C\epsilon'.
\end{aligned}$$

It is clear that we can in fact get bounds of the form

$$\int_0^\tau \int_M \left|\frac{\partial^p}{\partial t^p}\nabla^q h\right|^2 \leq C\epsilon'$$

for all $p, q > 0$. One can now apply the Sobolev inequality (with respect to g_0) to conclude C^k bounds on h in terms of ϵ'. These bounds will hold over any time interval where the L^2 norm of h is still small. Since this time can be made arbitrarily large with small choice of ϵ' by Lemma 8.4 the result follows. □

We now improve these estimates to include L^2 decay of h, which will ultimately yield the stated long-time existence and convergence. Say T is a maximal time such that $|h|_k < \epsilon$ on $[0, T)$. Divide the interval $[0, T)$ into intervals of length τ and let N be the integer so that $N\tau < T < (N+1)\tau$. Let $I_j = [j\tau, (j+1)\tau]$. On $M_j := M \times I_j$ define the inner product

$$||f||_{M_j} := \int_{j\tau}^{(j+1)\tau} ||f(t)||_{L^2(g_0)} dt \tag{31}$$

Let π^j denote the orthogonal projection onto ker $\left(\frac{\partial}{\partial t} + L\right)$ with respect to $||f||_{M_j}$. Since L is positive semidefinite we see that π has no positive eigenvalues, but there is still the lingering question of zero eigenvalues. This is where the integrability property comes in. Let $\pi_0^j(h)$ denote the radial component, i.e. the kernel of L. We will show using integrability that there exists a stationary solution g_j on M_j such that $\pi_0^j(g(t)-g_j)$ is very small compared to $g(t)-g_j$. This will allow us to conclude L^2 decay of h and then allow us to conclude convergence.

Lemma 8.6. *Given $\alpha > 0$ there exists $\delta = \delta(n,\tau)$ such that if $\sup_{[\tau_0,\tau_0+\tau]} |h(t)|_k < \delta$ then there exists a Kähler-Einstein metric g_1 such that*

$$\left|\pi_0^j(g-g_1)\right| \leq \alpha(t-\tau_0)\,|g-g_1| \tag{32}$$

and

$$|g_1-g_0|_{C^k} \leq C\sup_I |g-g_0|_{C^k} \tag{33}$$

Proof. Recall from the discussion above that the set of metrics g near g_0 satisfying $\mathcal{F}(g)=0$, call it $\mathcal{U}$, has a natural smooth manifold structure. The tangent space to $\mathcal{U}$ is given by the kernel of L, call it $\mathcal{K}$, which is finite dimensional since L is elliptic. Let $\{B_i\}$ be a basis for $\mathcal{K}$ orthonormal with respect to the L^2 norm induced by g_0. Also using ellipticity, we get a system of eigenvectors $\{E_\lambda\}$ for L orthonormal with respect to the L^2 inner product above. We see that there exist constants r_λ such that $C_\lambda = r_\lambda E_\lambda e^{\lambda t}$ is a basis for $\ker\left(\frac{\partial}{\partial t}+L\right)$ which is orthonormal with respect to the inner product in (31).

Define the map $\Psi:\mathcal{U}\to\mathcal{K}$ by $\Psi(g)=\sum_i \langle g,B_i\rangle B_i$. A simple calculation using the bases described above shows that for $g_1\in\mathcal{U}$, $\Psi(g_1)=\Psi(\pi^j(g_1))=\pi_0^j(g_1)$. Also it is easy to see that the differential of Ψ at g_0 is the identity map, so we can apply the inverse function theorem. Fix the time $\tau_0\in I_j$. If $|g(t)-g_0|_k$ is small enough, then in particular $\pi_0^j(g(\tau_0)-g_0)=\pi_0^j(g(\tau_0))$ can be made small, so that by the argument above there exists $g_1\in\mathcal{U}$ such that

$$\Psi(g_1)=\pi_0^j(g(\tau_0)).$$

Thus in particular using the above equalities we have $\pi_0^j(g_1-g(\tau_0))=0$. Using the evolution equations satisfied by g and g_1 it is clear that one has estimate (32). Also note that we have $g_1=\Psi^{-1}((\pi^j g)_0)$ and $g_0=\Psi^{-1}((\pi^j g_0)_0)$ thus using our bound from the inverse function theorem we get

$$||g_1-g_0||_{M_j}\leq C||\pi^j(g-g_0)||_{M_j}$$

and again using that these are all solutions of the same parabolic equation, we can get the bound

$$|g_1-g_0|_{C^k}\leq C\sup_{I_j}|g-g_0|_{C^k}.$$

□

Lemma 8.7. *Let $I=[\tau_0,\tau_0+\tau]$ and take g_1 as in Lemma 8.6. Then there exists $\epsilon>0$ depending only on g_0 such that if $|h_1(0)|_k<\epsilon$ where $h_1=g-g_1$ then*

$$\sup_{\left[\tau_0+\frac{\tau}{2},\tau_0+\tau\right]}\int_M |g-g_1|^2\,dV_{g_0}\leq e^{-\frac{\tau\lambda}{2}}\sup_{\left[\tau_0,\tau_0+\frac{\tau}{2}\right]}\int_M |g-g_1|^2\,dV_{g_0} \tag{34}$$

where $\lambda=\min\{\lambda_i:\lambda_i$ is an eigenvalue of $L,\lambda_i\neq 0\}>0$.

Proof. Let $h_1(t) = g(t) - g_1$. If $\left|h_1(0)\right|_k < \epsilon$, a calculation like that in Lemma 8.4 combined with the bound on $\pi_0(h_1(t))$ shows that

$$\begin{aligned}\frac{d}{dt}\int_M |h_1|^2 =& \int_M \langle 2Lh_1, h_1\rangle\, dV_{g_0} + \int_M A(h_1, g_0) dV_{g_0}\\ \leq& -2\lambda \int_M \left|h_1 - \pi_0^j(h_1)\right| dV_{g_0} + C\epsilon \int_M |h_1|^2\, dV_{g_0}\\ \leq& \left(-\frac{3}{2}\lambda + C\epsilon\right)\int_M |h_1|^2\\ \leq& -\lambda\int_M |h_1|^2\end{aligned}$$

as long as $\epsilon < \frac{\lambda}{C}$. Thus $\int_M |h_1(t)|^2\, dV_{g_0} \leq e^{-\lambda(t-\tau)}\int_M |h_1(\tau)|^2\, dV_{g_0}$ from which the claim follows immediately. □

We will need one more Lemma, which roughly says that if a solution to (26) is decaying at a certain rate at a particular time then it decayed at that rate earlier in time. This Lemma is inspired by Lemma 5.31 in [5], and the proof is the same.

Lemma 8.8. *There exists a constant $\nu(n,\tau) > 0$ with the following property. Let k be a symmetric two-tensor satisfying the equation*

$$\frac{\partial}{\partial t}k = -Lk + A(g_0, k),$$

and

$$\sup_{[\tau_0, \tau_0+\tau]} |k|_{C^k} < \nu$$

and

$$|\pi_0(k)| \leq \alpha(t - \tau_0)\,|k|$$

where here we mean projection onto the kernel of $\left(\frac{\partial}{\partial t} + L\right)$ restricted to the interval $\left[\tau_0 - \frac{\tau}{2}, \tau_0 + \tau\right]$. Then if

$$\sup_{\left[\tau_0+\frac{\tau}{2}, \tau_0+\tau\right]}\int_M |k|^2\, dV_{g_0} \leq e^{-\frac{\tau\lambda}{2}}\sup_{\left[\tau_0, \tau_0+\frac{\tau}{2}\right]}\int_M |k|^2\, dV_{g_0} \tag{35}$$

then

$$\sup_{\left[\tau_0, \tau_0+\frac{\tau}{2}\right]}\int_M |k|^2\, dV_{g_0} \leq e^{-\frac{\tau\lambda}{2}}\sup_{\left[\tau_0-\frac{\tau}{2}, \tau_0\right]}\int_M |k|^2\, dV_{g_0}. \tag{36}$$

Proof. First note that the the analogous claim where k satisfies the linear equation $\left(\frac{\partial}{\partial t} + L\right)k = 0$ and $\pi_0(k) = 0$ is obvious since L is positive semidefinite and by definition $\pi(k)_0 = 0$. In fact there is decay at the rate λ as opposed to the $\lambda/2$ in the statement above. So, if the claim were false, then for a sequence $\nu_i \to 0$ we would have k_i satisfying the hypothesis but not the conclusion with the bound $|k_i|_{C^k} < \nu_i$. By standard compactness arguments we can parabolically rescale k and extract a subsequence converging to k_∞, which satisfies the initial decay hypothesis but not the conclusion. Moreover, given that A is quadratic in k, it is clear that this k_∞ satisfies the linear equation $\left(\frac{\partial}{\partial t} + L\right)k = 0$ and $\pi_0(k) = 0$, contradicting the above. □

We now proceed with the main proof. Suppose the maximal existence time satisfies $T < \infty$, and subdivide $[0, T]$ into N intervals of length τ labelled I_j as above. For fixed j, let g_j be the metric such that $\pi_0^j(g(t) - g_j) = 0$ on I_j given by Lemma 8.6. Define $h_j := g(t) - g_j$. By Lemma 8.7 and parabolic regularity we have that

$$\sup_{\left[(j+\frac{1}{2})\tau,(j+1)\tau\right]} |h_j| \leq Ce^{-\frac{\tau\lambda}{2}} \sup_{\left[j\tau,(j+\frac{1}{2})\tau\right]} |h_j| . \tag{37}$$

And we can apply Lemma 8.8 inductively to conclude

$$\sup_{[j\tau,(j+1)\tau]} |h_j| \leq e^{-\lambda\tau(j-1)} \sup_{\left[0,\frac{\tau}{2}\right]} |h_j| .$$

This allows us to conclude that on I_j we have

$$\begin{aligned}
\left|\frac{\partial}{\partial t} g\right| &= \left|\frac{\partial}{\partial t}(g - g_j)\right| \\
&\leq C \sup_{I_j} |h_j|_k \\
&\leq C\epsilon e^{-\lambda\tau(j-1)} = \frac{C\epsilon}{p^{j-1}}.
\end{aligned}$$

Now note that simply integrating over time we see that

$$\sup_{I_j} |g - g_0| \leq 2\tau \sup_{I_j \cup I_{j-1}} \left|\frac{\partial}{\partial t} g\right| + \sup_{I_{j-1}} |g - g_0| .$$

Applying this estimate inductively we see that

$$\begin{aligned}
\sup_{I_j} |g - g_0| &\leq 2\tau \sum_{k=1}^{N} \sup_{I_k \cup \cdots \cup I_N} \left|\frac{\partial}{\partial t} g\right| + \sup_{I_0} |g - g_0| \\
&\leq \sum_{k=1}^{\infty} \frac{2\tau C\epsilon}{p^{k-1}} + \sup_{I_0} |h_0| \\
&\leq \frac{2\tau C\epsilon}{p-1} + \sup_{I_0} |h_0| .
\end{aligned} \tag{38}$$

Now we want to choose our constants ϵ, ϵ' and τ to derive a contradiction from this equation. So, choose τ initially so large that

$$\frac{1}{c(n)e^{\tau\lambda}} + \frac{2C\tau}{e^{\tau\lambda} - 1} < \frac{1}{C} e^{-\frac{\tau\lambda}{4}} \tag{39}$$

where $c(n)$ is a fixed large constant and C is a constant depending only on g_0.

Now let $\epsilon = \min\{\delta(n,\tau), \nu(n,\tau), \frac{\lambda}{C_0}\}$ where $\delta(n,\tau)$ is as in Lemma 8.6, $\nu(n,\tau)$ is as in Lemma 8.8, and C_0 is a constant depending only on g_0 and the dimension which we now make explicit. By Lemma 8.4 we can bound the growth of L^2 derivatives of h, and then by Sobolev embeddings we can bound C^k norms. Specifically there exists a constant depending only on g_0 so that

$$|h|_{C^4} < Ce^{C\epsilon\tau} |h(0)|_{C^0}$$

Then let $C_0 := 12C$. Note that if $\epsilon < \frac{\lambda}{C_0}$ and we start our flow with some $h(t_0)$ satisfying $|h(t_0)|_{C^4} < \frac{\epsilon}{C} e^{-\tau\lambda/4}$, the solution exists at least on $[t_0, t_0 + 3\tau)$ and moreover $\sup_{[t_0,t_0+3\tau)} |h(t)|_{C^4} < \epsilon$.

Now again using Lemma 8.4 we see that we may choose ϵ' so that the solution exists on $[0,3\tau]$ and further

$$\sup_{[0,3\tau]} \left|h(t)\right|_k < \frac{\epsilon}{c(n)} e^{-\tau\lambda}.$$

Since $\epsilon < \min\{\delta(n,\tau),\nu(n,\tau)\}$ we can apply (38) to get that

$$\begin{aligned}\sup_{I_N} |g - g_0| &\leq \epsilon \left(\frac{1}{c(n)e^{\tau\lambda}} + \frac{2C\tau}{e^{\tau\lambda} - 1} \right) \\ &\leq \frac{\epsilon}{C} e^{-\frac{\tau\lambda}{4}}.\end{aligned}$$

Thus the solution to HCF with initial metric $g(T-\tau)$ exists on an interval of length 3τ with $\left|h\right|_{C^2} < \epsilon$, contradicting the maximality of T. Thus the solution exists for all time and $\left|g(t) - g_0\right|_k < \epsilon$ for all time. Indeed we have decay

$$|g(t) - g_j| \leq Ce^{-\lambda t}$$

for all $t \in [0, j\tau)$ and for all j. Since $\{g_j\}$ is a sequence of Kähler-Einstein metrics with uniform C^k bounds, we get a convergent subsequence $g_j \to g_\infty$ a critical metric, with exponential convergence $g(t) \to g_\infty$. □

9. Further Questions

It bears mentioning that the Theorem 1.1 and Theorem 1.2 are both true for more general equations. Specifically both results hold for solutions to (2) where Q can be *any* quadratic expression in the torsion.

The HCF is similar in some regards to certain renormalization group flows arising in physics where external fields, say Yang-Mills or B-fields, are added to the pure gravity theory and then arise in the flow equations, see for instance [13], [16], [17]. In these flows the torsion is given as an external field, whereas in HCF everything is defined in terms of the metric. A similar case is studied in [3], [10] where a "holonomy flow" is proposed for closed G_2 structures. Here one evolves the definite three-form σ defining the G_2 structure by the Hodge Laplacian of σ taken with the metric induced by σ. This is a quasilinear equation which bears a certain resemblance to HCF in that it can be written as "Ricci flow plus torsion," where the torsion is defined in terms of the underlying metric. The techniques of our stability theorem likely apply to show stability of this flow near G_2-holonomy spaces with negative semidefinite Lichnerowicz operator.

Finally, Hermitian curvature flow provides a framework for addressing questions on the existence of integrable complex structures. In particular, if one had a complete description of the behavior of this flow for certain geometric conditions and a complete understanding of the limiting objects, one could then describe the manifolds admitting integrable complex structures with Hermitian metrics satisfying the initial geometric conditions. With strong enough convergence results for this flow one could in particular answer the question of the existence of an integrable complex structure on S^6. Since we expect our flow to be moving towards Kähler metrics and we know that S^6 does not support Kähler metrics, we expect the volume normalized flow to develop singularities either at finite time or at infinity. In either case these singularities will have some extra structure and can possibly be classified. It may lead to a contradiction to the assumption on the existence of integrable complex structures on S^6.

10. Appendix: Variational Formulas

In this appendix we collect variational formulas for quantities related to the curvature and torsion of Hermitian metrics.

Lemma 10.1. *Let $g(a)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial a}\Gamma_{ik}^{l} = g^{l\overline{m}}\nabla_i h_{k\overline{m}}$$
$$\frac{\partial}{\partial a}\Gamma_{\overline{ik}}^{\overline{l}} = g^{\overline{l}m}\nabla_{\overline{i}} h_{m\overline{k}}$$

Proof. We compute directly in canonical complex coordinates

$$\begin{aligned}\frac{\partial}{\partial a}\Gamma_{ik}^{l} =& \frac{\partial}{\partial a}g^{l\overline{m}}\left(\partial_i g_{k\overline{m}}\right)\\ =& -h^{l\overline{m}}\partial_i g_{k\overline{m}} + g^{l\overline{m}}\partial_i h_{k\overline{m}}\\ =& g^{l\overline{m}}\left(\partial_i h_{k\overline{m}} - \Gamma_{ik}^{p}h_{p\overline{m}}\right)\\ =& g^{l\overline{m}}\nabla_i h_{k\overline{m}}.\end{aligned}$$

Which gives the first formula and the second follows by conjugation. □

Lemma 10.2. *Let $g(a)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial a}\Omega_{i\overline{j}k}^{l} = -g^{m\overline{l}}\nabla_{\overline{j}}\nabla_i h_{k\overline{m}}$$
$$\frac{\partial}{\partial a}\Omega_{i\overline{j}k\overline{l}} = \Omega_{i\overline{j}k}^{m}h_{m\overline{l}} - \nabla_{\overline{j}}\nabla_i h_{k\overline{l}}$$

Proof. We compute directly

$$\begin{aligned}\frac{\partial}{\partial a}\Omega_{i\overline{j}k}^{l} =& \frac{\partial}{\partial a}\left(-\partial_{\overline{j}}\Gamma_{ik}^{l}\right)\\ =& -\partial_{\overline{j}}\left(g^{l\overline{m}}\nabla_i h_{k\overline{m}}\right)\\ =& g^{l\overline{p}}\partial_{\overline{j}}g_{\overline{p}q}g^{q\overline{m}}\nabla_i h_{k\overline{m}} - g^{l\overline{m}}\partial_{\overline{j}}\nabla_i h_{k\overline{m}}\\ =& -g^{m\overline{l}}\left(\partial_{\overline{j}}\nabla_i h_{k\overline{m}} - \Gamma_{\overline{jm}}^{\overline{p}}\nabla_i h_{k\overline{p}}\right)\\ =& -g^{m\overline{l}}\nabla_{\overline{j}}\nabla_i h_{k\overline{m}}.\end{aligned}$$

This gives the first formula and the second follows easily. □

Lemma 10.3. *Let $g(a)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial a}s = -\Delta\operatorname{tr}h - \left\langle h, S + \operatorname{div}^{\nabla}T - \nabla w\right\rangle.$$

Proof. We compute using Lemma 10.2

$$\begin{aligned}\frac{\partial}{\partial a}s =& \frac{\partial}{\partial a}g^{k\overline{l}}S_{k\overline{l}}\\ =& -h^{k\overline{l}}S_{k\overline{l}} + g^{k\overline{l}}\left[-\Delta h_{k\overline{l}} - h^{i\overline{j}}\Omega_{i\overline{j}k\overline{l}} + h_{k\overline{m}}S_{\overline{l}}^{\overline{m}}\right]\\ =& -\Delta\operatorname{tr}h - \langle h, P\rangle.\end{aligned}$$

The result now follows from Lemma 2.4. □

Lemma 10.4. *Let $g(a)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial a} T_{ij\bar{k}} = \nabla_i h_{j\bar{k}} - \nabla_j h_{i\bar{k}} + T_{ij}^m h_{m\bar{k}}$$
$$\frac{\partial}{\partial a} w = \nabla \operatorname{tr} h - \operatorname{div}^{\nabla} h.$$

Proof. We compute directly

$$\begin{aligned} \frac{\partial}{\partial a} T_{ij\bar{k}} &= \frac{\partial}{\partial a}\left(\partial_i g_{j\bar{k}} - \partial_j g_{i\bar{k}}\right) \\ &= \partial_i h_{j\bar{k}} - \partial_j h_{i\bar{k}} \\ &= \nabla_i h_{j\bar{k}} + \Gamma_{ij}^m h_{m\bar{k}} - \nabla_j h_{i\bar{k}} - \Gamma_{ji}^p h_{p\bar{k}} \\ &= \nabla_i h_{j\bar{k}} - \nabla_j h_{i\bar{k}} + T_{ij}^m h_{m\bar{k}}. \end{aligned}$$

This gives the first formula. For the second we see

$$\begin{aligned} \frac{\partial}{\partial a} w_i &= \frac{\partial}{\partial a} g^{j\bar{k}} T_{ij\bar{k}} \\ &= -h^{j\bar{k}} T_{ij\bar{k}} + g^{j\bar{k}}\left(\nabla_i h_{j\bar{k}} - \nabla_j h_{i\bar{k}} + T_{ij}^m h_{m\bar{k}}\right) \\ &= \nabla_i \operatorname{tr} h - \operatorname{div}^{\nabla} h_i. \end{aligned}$$

□

Lemma 10.5. *Let $g(s)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial s}\left|T\right|^2 = \left\langle h, -2Q^1 + Q^2\right\rangle + 4\left\langle \nabla h, T\right\rangle$$

where

$$\left\langle \nabla h, T\right\rangle = \frac{1}{2} g^{i\bar{j}} g^{k\bar{l}} g^{m\bar{n}}\left(\nabla_i h_{k\bar{n}} T_{\bar{j}\bar{l}m} + T_{ik\bar{n}} \nabla_{\bar{j}} h_{\bar{l}m}\right).$$

Proof. We compute directly

$$\begin{aligned} \frac{\partial}{\partial s}\left|T\right|^2 &= \frac{\partial}{\partial s} g^{i\bar{p}} g^{j\bar{q}} g^{\bar{k}r} T_{ij\bar{k}} T_{\bar{p}\bar{q}r} \\ &= -h^{i\bar{p}} g^{j\bar{q}} g^{\bar{k}r} T_{ij\bar{k}} T_{\bar{p}\bar{q}r} - h^{j\bar{q}} g^{i\bar{p}} g^{\bar{k}r} T_{ij\bar{k}} T_{\bar{p}\bar{q}r} - h^{\bar{k}r} g^{i\bar{p}} g^{j\bar{q}} T_{ij\bar{k}} T_{\bar{p}\bar{q}r} \\ &\quad + g^{i\bar{p}} g^{j\bar{q}} g^{\bar{k}r}\left[\left(\nabla_i h_{j\bar{k}} - \nabla_j h_{i\bar{k}} + T_{ij}^m h_{m\bar{k}}\right) T_{\bar{p}\bar{q}r} + T_{ij\bar{k}}\left(\nabla_{\bar{p}} h_{\bar{q}r} - \nabla_{\bar{q}} h_{\bar{p}r} + T_{\bar{p}\bar{q}}^{\bar{s}} h_{r\bar{s}}\right)\right] \\ &= \left\langle h, -2Q^1 + Q^2\right\rangle + 4\left\langle \nabla h, T\right\rangle. \end{aligned}$$

The result follows. □

Lemma 10.6. *Let $g(s)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial s}\left|w\right|^2 = -\left\langle h, Q^3\right\rangle + 2\left\langle \nabla \operatorname{tr} h - \operatorname{div}^{\nabla} h, w\right\rangle.$$

Proof. In canonical complex coordinates at a point we compute

$$\begin{aligned}
&\frac{\partial}{\partial s} g^{i\bar{j}} g^{m\bar{n}} g^{r\bar{s}} T_{im\bar{n}} T_{\bar{j}\bar{s}r} \\
&= -h^{i\bar{j}} g^{m\bar{n}} g^{r\bar{s}} T_{im\bar{n}} T_{\bar{j}\bar{s}r} - g^{i\bar{j}} h^{m\bar{n}} g^{r\bar{s}} T_{im\bar{n}} T_{\bar{j}\bar{s}r} - g^{i\bar{j}} g^{m\bar{n}} h^{r\bar{s}} T_{im\bar{n}} T_{\bar{j}\bar{s}r} \\
&\qquad + g^{i\bar{j}} g^{m\bar{n}} g^{r\bar{s}} \Big[\left(\nabla_i h_{m\bar{n}} - \nabla_m h_{i\bar{n}} + T_{im}^{p} h_{p\bar{n}} \right) T_{\bar{j}\bar{s}r} \\
&\qquad + \left(\nabla_{\bar{j}} h_{\bar{s}r} - \nabla_{\bar{s}} h_{\bar{j}r} + T_{\bar{j}\bar{s}}^{\bar{q}} h_{\bar{q}r} \right) T_{im\bar{n}} \Big] \\
&= -\left\langle h, Q^3 \right\rangle + 2 \left\langle \nabla \operatorname{tr} h - \operatorname{div}^{\nabla} h, w \right\rangle .
\end{aligned}$$

□

Lemma 10.7. *Let $g(a)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial a} \int_M s dV = \int_M \left[\left\langle h, -S - \operatorname{div}^{\nabla} T + \nabla w \right\rangle + \operatorname{tr} h \left(s - \operatorname{div}^{\nabla} w - |w|^2 \right) \right] dV.$$

Proof. From Lemma 10.3 we see

$$\frac{\partial}{\partial a} \int_M s dV = \int_M \left[-\Delta \operatorname{tr} h + \left\langle h, -S - \operatorname{div}^{\nabla} T + \nabla w \right\rangle + s \operatorname{tr} h \right] dV.$$

Now by Lemma 10.11 we see

$$\int_M -\Delta \operatorname{tr} h dV = \int_M \operatorname{tr} h \left(-\operatorname{div}^{\nabla} w - |w|^2 \right) dV$$

and the result follows. □

Lemma 10.8. *Let $g(a)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\begin{aligned}
\frac{\partial}{\partial a} \int_M |T|^2 dV = \int_M \Big[&\left\langle h, -2Q^1 + Q^2 - 4Q^4 - 4 \operatorname{div}^{\nabla} T \right\rangle \\
&+ (\operatorname{tr} h) |T|^2 \Big] dV.
\end{aligned}$$

Proof. We directly compute using Lemma 10.5

$$\frac{\partial}{\partial a} \int_M |T|^2 dV = \int_M \left[\left\langle h, -2Q^1 + Q^2 \right\rangle + 4 \left\langle \nabla h, T \right\rangle + (\operatorname{tr} h) |T|^2 \right] dV.$$

We now integrate by parts

$$\begin{aligned}
\int_M \langle \nabla h, T\rangle \, dV =& \int_M g^{i\overline{p}} g^{j\overline{q}} g^{\overline{k}r} \nabla_i h_{j\overline{k}} T_{\overline{p}\overline{q}r} dV\\
=& \int_M g^{i\overline{p}} g^{j\overline{q}} g^{\overline{k}r} \left(\partial_i h_{j\overline{k}} - \Gamma_{ij}^s h_{s\overline{k}} \right) T_{\overline{p}\overline{q}r} dV\\
=& -\int_M h_{j\overline{k}} \partial_i \left(g^{i\overline{p}} g^{j\overline{q}} g^{\overline{k}r} T_{\overline{p}\overline{q}r} dV \right) + g^{i\overline{p}} g^{j\overline{q}} g^{\overline{k}r} \Gamma_{ij}^s h_{s\overline{k}} T_{\overline{p}\overline{q}r} dV\\
=& \int_M h_{j\overline{k}} \Big[g^{i\overline{n}} \partial_i g_{m\overline{n}} g^{m\overline{p}} g^{j\overline{q}} g^{\overline{k}r} T_{\overline{p}\overline{q}r} + g^{i\overline{p}} g^{j\overline{n}} \partial_i g_{m\overline{n}} g^{m\overline{q}} g^{\overline{k}r} T_{\overline{p}\overline{q}r}\\
& + g^{i\overline{p}} g^{j\overline{q}} g^{\overline{k}m} \partial_i g_{m\overline{n}} g^{\overline{n}r} T_{\overline{p}\overline{q}r} - g^{i\overline{p}} g^{j\overline{q}} g^{\overline{k}r} \partial_i T_{\overline{p}\overline{q}r}\\
& - g^{i\overline{p}} g^{j\overline{q}} g^{\overline{k}r} T_{\overline{p}\overline{q}r} g^{m\overline{n}} \partial_i g_{m\overline{n}}\\
& - g^{m\overline{p}} g^{n\overline{q}} g^{\overline{k}r} \Gamma_{mn}^j T_{\overline{p}\overline{q}r} \Big] dV\\
=& \int_M \left\langle h, -\operatorname{div}^{\nabla} T - Q^4 \right\rangle dV
\end{aligned}$$

Plugging in this simplification yields the result. □

Lemma 10.9. *Let $g(a)$ be a family of Hermitian metrics compatible with the given complex structure J. Then*

$$\frac{\partial}{\partial a} \int_M |w|^2 \, dV = \int_M \left[\left\langle h, Q^3 + 2\nabla w \right\rangle + \operatorname{tr} h \left[-2 \operatorname{div}^{\nabla} w - |w|^2 \right] \right] dV.$$

Proof. We directly compute using Lemma 10.6 and integrating by parts using Lemmas 10.10 and 10.12

$$\begin{aligned}
\frac{\partial}{\partial a} \int_M |w|^2 \, dV =& \int_M \left[\left\langle h, -Q^3 \right\rangle + 2 \left\langle \nabla \operatorname{tr} h - \operatorname{div}^{\nabla} h, w \right\rangle + (\operatorname{tr} h) |w|^2 \right] dV\\
=& \int_M \Big[\left\langle h, -Q^3 \right\rangle + 2 \operatorname{tr} h \left[-\operatorname{div}^{\nabla} w - |w|^2 \right]\\
& + 2 \left\langle h, \nabla w + Q^3 \right\rangle + (\operatorname{tr} h) |w|^2 \Big] dV
\end{aligned}$$

□

Lemma 10.10. *Given $\phi \in C^\infty(M)$ and $\alpha \in T^{1,0}(M)$ we have*

$$\int_M \langle \nabla \phi, \alpha \rangle = \int_M \phi \left[-\operatorname{div}^{\nabla} \alpha - \langle w, \alpha \rangle \right] dV.$$

Proof. We directly compute

$$\begin{aligned}
\int_M \langle \nabla \phi, \alpha \rangle \, dV =& \int_M g^{i\overline{j}} \partial_i \phi \alpha_{\overline{j}} dV\\
=& -\int_M \phi \partial_i \left[g^{i\overline{j}} \alpha_{\overline{j}} dV \right]\\
=& \int_M \phi \left[g^{i\overline{s}} \partial_i g_{r\overline{s}} g^{r\overline{j}} \alpha_{\overline{j}} - g^{i\overline{j}} \partial_i \alpha_{\overline{j}} - g^{i\overline{j}} \alpha_{\overline{j}} g^{p\overline{q}} \partial_i g_{p\overline{q}} \right] dV\\
=& \int_M \phi \left[-\operatorname{div}^{\nabla} \alpha - \langle w, \alpha \rangle \right] dV.
\end{aligned}$$

□

Lemma 10.11. *Given $\phi \in C^\infty(M)$ we have*

$$\int_M \Delta\phi dV = \int_M \phi \left[\operatorname{div}^\nabla w + |w|^2\right] dV$$

Proof. We directly compute

$$\begin{aligned}
\int_M \Delta\phi dV &= \int_M g^{i\bar{j}} \partial_{\bar{j}} \partial_i \phi dV \\
&= -\int_M \partial_i \phi \partial_{\bar{j}} \left(g^{i\bar{j}} dV\right) \\
&= \int_M \partial_i \phi \left[g^{i\bar{s}} \partial_{\bar{j}} g_{r\bar{s}} g^{r\bar{j}} - g^{i\bar{j}} g^{p\bar{q}} \partial_{\bar{j}} g_{p\bar{q}}\right] dV \\
&= \int_M \partial_i \phi \left[g^{i\bar{j}} g^{p\bar{q}} T_{\bar{q}\bar{j}p}\right] dV \\
&= -\int_M \langle \nabla\phi, w\rangle dV \\
&= \int_M \phi \left[\operatorname{div}^\nabla w + |w|^2\right] dV.
\end{aligned}$$

In the line we applied Lemma 10.10. □

Lemma 10.12. *Given $\beta \in T^{1,0}M$ and $h \in \operatorname{Sym}^{1,1}(M)$ we have*

$$\int_M \left\langle \operatorname{div}^\nabla h, \beta \right\rangle dV = \int_M \left[\langle h, -\nabla\beta - w \otimes \beta\rangle\right] dV$$

Proof.

$$\begin{aligned}
\int_M \left\langle \operatorname{div}^\nabla h, \beta \right\rangle dV &= \int_M g^{\bar{j}l} g^{\bar{k}i} \nabla_{\bar{k}} h_{i\bar{j}} \beta_l dV \\
&= \int_M g^{i\bar{k}} g^{l\bar{j}} \left[\partial_{\bar{k}} h_{i\bar{j}} - \Gamma_{\bar{k}\bar{j}}^{\bar{p}} h_{i\bar{p}}\right] \beta_l dV \\
&= -\int_M h_{i\bar{j}} \partial_{\bar{k}} \left[g^{i\bar{k}} g^{l\bar{j}} \beta_l dV\right] + g^{i\bar{k}} g^{l\bar{j}} \Gamma_{\bar{k}\bar{j}}^{\bar{p}} h_{i\bar{p}} \beta_l dV \\
&= \int_M h_{i\bar{j}} \left[g^{i\bar{s}} \partial_{\bar{k}} g_{r\bar{s}} g^{r\bar{k}} g^{l\bar{j}} \beta_l + g^{i\bar{k}} g^{l\bar{s}} \partial_{\bar{k}} g_{r\bar{s}} g^{r\bar{j}} \beta_l - g^{i\bar{k}} g^{l\bar{j}} \partial_{\bar{k}} \beta_l \right. \\
&\qquad \left. - g^{i\bar{k}} g^{l\bar{j}} g^{r\bar{s}} \partial_{\bar{k}} g_{r\bar{s}} \beta_l\right] dV - g^{i\bar{k}} g^{l\bar{j}} \Gamma_{\bar{k}\bar{j}}^{\bar{p}} h_{i\bar{p}} \beta_l dV \\
&= \int_M \left[\langle h, -\nabla\beta - w \otimes \beta\rangle\right] dV.
\end{aligned}$$

□

References

[1] Aubin, T., *Some nonlinear problems in Riemannian geometry*, Springer-Verlag, Berlin, 1998.
[2] Besse, A., *Einstein manifolds*, Springer-Verlag, Berlin, 1987.
[3] Bryant, R. *Some remarks on G_2-structures*, arXiv:math/0305124.
[4] Catenacci, R., Marzuoli, A., *A note on a hermitian analog of Einstein spaces*, Ann. Inst. Henri Poincaré, vol. 40, no. 2 (1984) 151-157.
[5] Cheeger, J. Tian, G. *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Invent. Math. **118** (1994) 493-571.
[6] Chow, B. Lu, P. Ni, L. *Hamilton's Ricci Flow*, American Mathematical Society Science Press, 2006

[7] DeTurck, D. Kazdan, L. *Some regularity theorems in Riemannian geometry*, Ann. Sci. École Norm. Sup. (4) **14** (1981) no. 3 249-260 MR **83f**:53018.
[8] Gauduchon, P. *Torsion d'une variété hermitienne compacte*. Math. Ann. 267 (1984) 495-518.
[9] Gauduchon, P. *Variation des courbures scalaires en géometrie hermitienne*. C. R. Acad. Sc. Paris Ser. A-B, 290 (1980), no. 7 327-330.
[10] Hitchin, N. *The geometry of three-forms in 6 and 7 dimensions*, J. Differential Geom. **55** (2000), 547-576.
[11] Kobayashi, S. Nomizu, K. *Foundations of Differential Geometry*, vol. 1, Wiley-Interscience, London, 1963.
[12] Morrow, *Complex Manifolds*, American Mathematical Society Science Press, 2006.
[13] Oliynyk, T. Suneeta, V. Woolgar, E. *A gradient flow for worldsheet nonlinear sigma models*, Nucl.Phys. B739 (2006) 441-458.
[14] Sesum, N. *Linear and dynamic stability of Ricci flat metrics*, Duke Mathematical Journal. vol. 133 no. 1 (2006), 1-26.
[15] Simon, L. *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Annals of Mathematics, vol. 118 (1983), 525-571.
[16] Streets, J. *Regularity and expanding entropy for connection Ricci flow*, J. Geom. Phys., 58 (2008) no. 7.
[17] Streets, J. *Singularities of renormalization group flows*, J. Geom. Phys., 59 (2009), no. 1, 8-16.
[18] Tian, G. *Canonical metrics in Kähler geometry*, Lecture Notes in Mathematics, Berlin, 2000.
[19] Vaisman, I. *On some variational problems for 2-dimensional Hermitian metrics*, Ann. Global Anal. Geom. vol. 8, no. 2 (1990), 137-145.

Fine Hall, Princeton University, Princeton, NJ 08544
E-mail address: jstreets@math.princeton.edu

Fine Hall, Princeton University, Princeton, NJ 08544
E-mail address: tian@math.princeton.edu