

RICCI YANG-MILLS FLOW ON SURFACES

JEFFREY STREETS

ABSTRACT. We study the behaviour of the Ricci Yang-Mills flow for $U(1)$ bundles on surfaces. We show that existence for the flow reduces to a bound on the isoperimetric constant. In the presence of such a bound, we show that on S^2 , if the bundle is nontrivial, the flow exists for all time. For higher genus surfaces the flow always exists for all time. The volume normalized flow always exists for all time and converges to a constant scalar curvature metric with the bundle curvature F parallel. Finally, in an appendix we classify all gradient solitons of this flow on surfaces.

1. INTRODUCTION

Fix (M^n, g) a Riemannian manifold. Suppose $L \rightarrow M$ is the total space of a $U(1)$ -bundle over M , and A is a connection on this bundle with curvature F . This F is a purely imaginary two-form on M which represents the first Chern class of the line bundle associated to L . In what follows we will often not refer to the total space of the bundle and focus attention on M, g and A , and furthermore identify F with a real valued two-form. We say that a family $(M, g(t), A(t))$ is a solution to Ricci Yang-Mills flow (RYM-flow) if

$$(1) \quad \begin{aligned} \frac{\partial}{\partial t} g_{ij} &= -2 \operatorname{Rc}_{ij} + g^{kl} F_{ik} F_{jl} \\ \frac{\partial}{\partial t} A &= -d^* F. \end{aligned}$$

This equation was studied in [9] in the hope that by introducing connections A where the curvature F has special properties then the flow would have behaviour simpler than that of the Ricci flow. Moreover, this system arises naturally in physics as the renormalization group flow for a certain nonlinear sigma model. Also, a recent paper of Lebrun [8] shows an interesting connection between solutions to the static equation, known as the Einstein-Maxwell equation, to the existence of extremal Kähler metrics in dimension 4. Finally, we mention that this equation has generated interest as a tool for better understanding magnetic flows on surfaces [7]. By examining homogeneous solutions, the following conjecture is plausible:

Conjecture 1. *Let (M^{2n}, g) be a Riemannian manifold. Let $L \rightarrow M$ be the total space of a $U(1)$ bundle over M . Given A a connection on L satisfying*

$$[F^{\wedge n}] \neq 0$$

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then the solution to the Ricci Yang-Mills flow with initial condition (g, A) exists for all time.

We mention a related conjecture for odd-dimensional manifolds in the conclusion. In this paper we examine this conjecture in the case $n = 1$. We show that the regularity of the flow can be reduced to showing a bound on the Sobolev constant of the manifold. Recall that the Sobolev constant of a Riemannian surface (M^2, g) is the smallest constant C_S such that the inequality

$$(2) \quad \left(\int_M |f - \bar{f}|^2 dV_g \right)^{\frac{1}{2}} \leq C_S \int_M |\nabla f|$$

holds for any function $f \in C^1(M)$, where \bar{f} is the average value of f . It is known that this constant is equivalent to other Sobolev constants, and moreover is equivalent to the isoperimetric ratio [2].

Theorem 2. *Let g be a Riemannian metric on an oriented surface M , and let $L \rightarrow M$ denote the total space of a $U(1)$ -bundle over M with connection A . Let $(g(t), A(t))$ be the solution to RYM flow with this condition. If the solution goes singular at time $T < \infty$, then either*

$$\lim_{t \rightarrow T} \text{Vol}(g(t)) = 0$$

or

$$\lim_{t \rightarrow T} C_S(g(t)) = \infty.$$

If $(g(t), A(t))$ is the solution to volume-normalized RYM flow and it goes singular at time $T < \infty$, then

$$\lim_{t \rightarrow T} C_S(g(t)) = \infty.$$

In other words, in the presence of volume and Sobolev constant bounds the solution to RYM flow on a surface is nonsingular.

We of course expect the isoperimetric constant to stay bounded along the flow. In fact, such a bound for the Ricci flow on S^2 was shown by Hamilton [6]. Such a bound for solutions to RYM flow is as yet unclear. We are moreover able to completely describe the limiting behaviour of infinite-time solutions with no hypotheses. The overall situation is described in the main theorem below.

Main Theorem. *Suppose solutions to RYM flow satisfy a Sobolev constant bound. In other words, given (M^2, g, A) a solution to RYM flow on $[0, T]$, one has $C_S(g(t)) < C(T)$ for all $t \leq T$. Let g be a Riemannian metric on an oriented surface M , and let $L \rightarrow M$ denote the total space of a $U(1)$ -bundle over M with connection A .*

- (1) *If $M \cong S^2$ and $[F] \neq 0$, the solution to RYM flow with initial condition (g, A) exists for all time. If the volume stays finite at infinity, the solution converges to the round metric with F parallel. Moreover, the volume normalized flow exists for all time and converges to the round metric with F parallel.*
- (2) *If $M \cong S^2$ and $[F] = 0$, the solution to volume normalized RYM flow with initial condition (g, A) exists for all time and converges to the round metric with $F \equiv 0$.*

- (3) *If $\chi(M) \leq 0$, the solution to RYM flow with initial condition (g, A) exists for all time and the volume-normalized flow exists for all time and converges to a constant curvature metric with F parallel.*

In fact, all of the convergence statements for flows existing for all time hold without the Sobolev constant bound hypothesis. Notice that two important questions are left unresolved. In particular, we do not know if the unnormalized equation on S^2 with $[F] \neq 0$ has a volume bound and hence converges at infinity. Also, it would be interesting to know the complete behaviour on S^2 when $[F] = 0$. We conjecture that the solution goes singular in finite time, converging to a round point with $F \equiv 0$. Since understanding gradient solitons may play a role in resolving these issues, we provide a classification.

Proposition 3. *If g is a gradient soliton on a closed surface Σ^2 then g has constant curvature and F is parallel.*

The proof of long time existence in the presence of the Sobolev constant bound generalizes the corresponding proof for Ricci flow found by Struwe [11]. We first reduce to a flow on a conformal factor u and a connection A , and indeed we show that a certain energy functional generalizing the Liouville energy for the conformal factor to include the Yang-Mills coupling is monotonically decreasing along a solution to RYM flow. Using this and a further a priori integral estimate we are able to bound the H^2 norms of u and A , and thus prove long time existence. The Moser-Trudinger inequality plays a key role in the proof as well. Given the long time existence, we are able to show that the Calabi energy remains bounded, and thus apply the compactness result of Xiuxiong Chen [3] to show convergence at infinity. We note that Andrea Young has independently obtained stability results for the Ricci Yang-Mills flow on a surface [13].

In section 2 we rewrite the RYM flow equation on a surface in terms of a conformal factor and introduce the volume normalized equation. Section 3 contains certain a-priori integral estimates, section 4 completes the proof of Theorem 2, and section 5 has the proofs of convergence, completing the proof of the Main Theorem. Section 6 is a concluding discussion, and section 7 is an appendix containing the classification of Ricci Yang-Mills solitons on surfaces.

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2. REDUCTION TO CONFORMAL FLOW

In this section we show that the metric component of the Ricci Yang-Mills flow on a surface is a conformal flow. We already know that on a surface $\text{Rc} = \frac{1}{2}Rg$. On a surface the term $g^{kl}F_{ik}F_{jl}$ is a scalar multiple of the metric as well.

Lemma 4. *Given (M^2, g) a Riemannian surface and $F \in \wedge^2 T^*M$, $g^{kl}F_{ik}F_{jl} = \frac{1}{2}|F|_g^2 g_{ij}$.*

Proof. Fix a point $x \in M$ and choose normal coordinates for g at x . In these coordinates we have $F(x) = \lambda(x)dx^1 \wedge dx^2$. Clearly then

$$g^{kl}F_{ik}F_{jl}(x) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{pmatrix} = \lambda^2(x)g(x).$$

Since the left and right hand sides are both tensors it follows that there exists a function $\lambda(x)$ so that $g^{kl}F_{ik}F_{jl} = \lambda^2(x)g_{ij}$. Taking the trace of this equation gives $\lambda^2(x) = \frac{1}{2}|F|_g^2$. \square

Using this lemma the RYM-flow on a surface becomes the system of equations

$$(3) \quad \begin{aligned} \frac{\partial}{\partial t}g &= -Rg + \frac{1}{2}|F|_g^2g \\ \frac{\partial}{\partial t}A &= -d^*F \end{aligned}$$

Furthermore, if $g = e^u g_0$ where g_0 is a fixed metric of constant curvature R_0 and unit volume, then we can write

$$R = e^{-u}(R_0 - \Delta u)$$

where Δ is with respect to the metric g_0 . Thus we can write the RYM-flow as the system

$$(4) \quad \begin{aligned} \frac{\partial}{\partial t}u &= e^{-u} \left(\Delta u - R_0 + \frac{1}{2}e^{-u}|F|^2 \right) \\ \frac{\partial}{\partial t}A &= -d^*F \end{aligned}$$

where in this equation the norm $|F|^2$ is taken with respect to g_0 . Note that since A is a connection on a $U(1)$ bundle F may be thought of as just a usual (closed) 2-form on M^2 . Thus we derive the evolution equation

$$(5) \quad \begin{aligned} \frac{\partial}{\partial t}F &= \frac{\partial}{\partial t}(dA) \\ &= -dd_g^*F \\ &= \Delta_{d,g}F \\ &= \Delta_g F \end{aligned}$$

where $\Delta_{d,g}$ is the Laplace-Beltrami operator of g . Note that the curvature term in the Böchner formula for n -forms always vanishes on n -manifolds [12], thus the last line follows where Δ_g is the rough Laplacian of g . We take the time here to mention an important convention in this paper. Any metric which is used without further decoration will be the fixed background metric. Any time we use the time-dependent metric $g(t)$ we will decorate the quantity with a g .

We will also need a certain volume-normalized system. Note that

$$\int_M R_g dV_g - \frac{1}{2} \int_M |F|_g^2 dV_g = R_0 - \frac{1}{2} \int_M e^{-u} |F|^2 dV$$

Thus consider

$$(6) \quad \begin{aligned} \frac{\partial}{\partial t}u &= e^{-u} \Delta u + R_0(1 - e^{-u}) + \frac{1}{2} \left(e^{-2u} |F|^2 - \frac{\int_M e^{-u} |F|^2 dV}{\int_M e^u dV} \right) \\ \frac{\partial}{\partial t}A &= -d^*F \end{aligned}$$

The volume of $g(t)$ remains constant under this evolution equation. Note that this system does *not* differ from the unnormalized equation by a rescaling in space and time. This is a consequence of the fact that $\frac{\partial}{\partial t}g$ does not have homogeneous scaling. In particular, the term $\frac{1}{2}|F|_g^2 g$ has inverse scaling with respect to the metric while $R_g g$ has neutral scaling with the metric. Also, F still obeys (5) with respect to this time dependent metric.

3. INTEGRAL ESTIMATES

In this section we will prove a-priori integral estimates for the RYM-flow on a surface. First we define a functional which is monotonic for solutions to RYM-flow

$$(7) \quad \mathcal{F}(u, A) := \int_M \left(|du|^2 + e^{-u} |F|^2 \right) dV + 2R_0 \int_M u dV$$

where the norms and volume form are those of the background metric g_0 .

Proposition 5. *Given $(M^2, u(t), A(t))$ a solution to (4) we have*

$$(8) \quad \frac{d}{dt} \mathcal{F}(u(t), A(t)) = -2 \int_M e^u |u_t|^2 dV - 2 \int_M |\nabla^g F|_g^2 dV_g$$

Proof. First we compute

$$\begin{aligned} \frac{d}{dt} \int_M |du|^2 dV &= 2 \int_M \left\langle d \left(e^{-u} \left(\Delta u - R_0 + \frac{1}{2} e^{-u} |F|^2 \right) \right), du \right\rangle dV \\ &= - \int_M \left(2e^{-u} (\Delta u)^2 - 2R_0 e^{-u} \Delta u + e^{-2u} |F|^2 \Delta u \right) dV \\ &= 2R_0 \int_M e^{-u} |du|^2 dV - 2 \int_M e^{-u} (\Delta u)^2 dV - \int_M e^{-2u} |F|^2 \Delta u dV \end{aligned}$$

Next we use the equation $\int_M e^{-u} |F|^2 dV = \int_M |F|_g^2 dV_g$ and compute using (3) and (5)

$$\begin{aligned} \frac{d}{dt} \int_M |F|_g^2 dV_g &= -2 \int_M |\nabla^g F|_g^2 dV_g + \int_M \left(R_g - \frac{1}{2} |F|_g^2 \right) |F|_g^2 dV_g \\ &= -2 \int_M |\nabla^g F|_g^2 dV_g - \int_M e^{-2u} (\Delta u - R_0) |F|^2 dV \\ &\quad - \frac{1}{2} \int_M e^{-3u} |F|^4 dV \end{aligned}$$

Next we have

$$\begin{aligned} 2R_0 \frac{d}{dt} \int_M u dV &= 2R_0 \int_M e^{-u} \Delta u - R_0 e^{-u} + \frac{1}{2} e^{-2u} |F|^2 dV \\ &= 2R_0 \int_M e^{-u} |du|^2 - R_0 e^{-u} + \frac{1}{2} e^{-2u} |F|^2 dV. \end{aligned}$$

Combining these calculations gives

$$\begin{aligned}
& \frac{d}{dt} \mathcal{F}(u(t), A(t)) \\
&= 4R_0 \int_M e^{-u} |du|^2 dV - 2 \int_M |\nabla^g F|_g^2 dV_g - 2R_0^2 \int_M e^{-u} dV \\
&\quad + 2R_0 \int_M e^{-2u} |F|^2 dV - 2 \int_M e^{-u} (\Delta u)^2 dV \\
&\quad - 2 \int_M e^{-2u} |F|^2 \Delta u dV - \frac{1}{2} \int_M e^{-3u} |F|^4 dV \\
&= -2 \int_M e^u \left| e^{-u} \Delta u - e^{-u} R_0 + \frac{1}{2} e^{-2u} |F|^2 \right|^2 dV - 2 \int_M |\nabla^g F|_g^2 dV_g \\
&= -2 \int_M e^u |u_t|^2 dV - 2 \int_M |\nabla^g F|_g^2 dV_g
\end{aligned}$$

as required. \square

Proposition 6. *Given $(M^2, u(t), A(t))$ a solution to (6) we have*

$$(9) \quad \frac{d}{dt} \mathcal{F}(u(t), A(t)) = -2 \int_M e^u |u_t|^2 dV - 2 \int_M |\nabla^g F|_g^2 dV_g$$

Proof. Adding a constant to u clearly does not affect the evolution of $\int_M |du|^2 dV$. Next in computing the evolution of $\int_M e^{-u} |F|^2 dV$ we pick up

$$\left(\frac{1}{2} \int_M e^{-u} |F|^2 dV - R_0 \right) \int_M e^{-u} |F|^2 dV.$$

Thus from the previous proposition we compute

$$\begin{aligned}
\frac{d}{dt} \mathcal{F}(u(t), F(t)) &= -2 \int_M e^u \left| e^{-u} \Delta u - e^{-u} R_0 + \frac{1}{2} e^{-2u} |F|^2 \right|^2 dV \\
&\quad - 2 \int_M |\nabla^g F|_g^2 dV_g + 2R_0^2 - 2R_0 \int_M e^{-u} |F|^2 dV \\
&\quad + \frac{1}{2} \left(\int_M e^{-u} |F|^2 dV \right)^2 \\
&= -2 \int_M e^u |u_t|^2 - 2 \int_M |\nabla^g F|_g^2 dV_g
\end{aligned}$$

\square

Corollary 7. *Given $(M^2, u(t), A(t))$ a solution to (4) or (6) we have*

$$\mathcal{F}(u(t), A(t)) \leq \mathcal{F}(u(0), A(0))$$

Proof. This follows from the above lemmas. \square

Lemma 8. *Let $(S^2, g(t), A(t))$ be a solution to the Ricci Yang-Mills flow on S^2 satisfying $[F] \neq 0$. Then there exists a constant $C = C(g_0, |[F]|)$ so that the inequality*

$$(10) \quad \text{Vol}(g(t)) \geq C$$

holds for all time that the flow exists.

Proof. First note that on a Riemannian surface (M, g) , any $F \in \wedge^2 T^*M$ satisfies $F = \pm |F| dV$. This implies the inequality

$$\begin{aligned} 0 &< |[F]| \\ &= \left| \int_M F \right| \\ &\leq \int_M |F|_g dV_g \\ &\leq \left(\int_M |F|_g^2 dV_g \right)^{\frac{1}{2}} \text{Vol}(g)^{\frac{1}{2}}. \end{aligned}$$

Using this and the Gauss-Bonnet Theorem we compute the evolution equation

$$\begin{aligned} \frac{d}{dt} \text{Vol}(g(t)) &= - \int_M R dV + \frac{1}{2} \int_M |F|^2 dV \\ &\geq -4\pi + \frac{|[F]|^2}{2 \text{Vol}(g(t))}. \end{aligned}$$

If $\text{Vol}(g(t)) \leq \frac{|F|^2}{8\pi}$ then $\frac{\partial}{\partial t} \text{Vol}(g(t)) \geq 0$ and the result follows. \square

Lemma 9. *Given $(M^2, g(t), A(t))$ a solution to (4) there exists a constant $C > 0$ depending on $(g(0), A(0))$ so that the inequality*

$$(11) \quad \text{Vol}(g(t)) \leq \text{Vol}(g(0)) + Ct$$

holds for any $t > 0$.

Proof. Using Proposition 5 we estimate

$$\begin{aligned} \frac{d}{dt} \text{Vol}(g(t)) &= \frac{d}{dt} \int_M dV_g \\ &= \int_M \left(-R + \frac{1}{2} |F|_g^2 \right) dV_g \\ &= -2\pi\chi(M) + \frac{1}{2} \int_M |F|_g^2 dV_g. \end{aligned}$$

However, using the fact that \mathcal{F} is bounded and the Liouville energy is bounded below in any conformal class we see

$$\begin{aligned} \int_M |F|_g^2 dV_g &= \mathcal{F}(u(t), A(t)) - \int_M (|du|^2 + 2R_0 u) dV \\ &\leq \mathcal{F}(u(0), A(0)) + C. \end{aligned}$$

Therefore

$$\frac{d}{dt} \text{Vol}(g(t)) \leq C$$

and the result follows. \square

Lemma 10. *Given $(M^2, g(t), A(t))$ a solution to (4) or (6), on any finite time interval $[0, T]$ there exists a constant C depending only on $(u(0), A(0))$ and T so that*

$$\|\nabla u\|_{L^2} \leq C, \quad \int_M e^{-u} |F|^2 dV \leq C$$

Proof. In the case $R_0 \leq 0$ using Jensen's inequality and the volume bound we easily conclude that

$$\int_M u \leq \log \int_M (e^u dV) \leq C.$$

Thus we have an a-priori lower-bound for \mathcal{F} and the result follows. For the case $R_0 > 0$ we must modify our flow by an explicit Möbius transformation as in [11]. Specifically we solve for $\phi(t)$ a family of conformal diffeomorphisms of the sphere such that $h(t) = \phi(t)^* g(t) = e^{v(t)} g_0$ satisfies

$$\int_M x dV_h = 0$$

for all time, where x is the position vector in \mathbb{R}^3 . Note that these diffeomorphisms are certainly different from those obtained for fixing the conformal gauge of Ricci flow. Since \mathcal{F} is diffeomorphism invariant it follows from Proposition 6 (in the volume normalized case) that \mathcal{F} is uniformly bounded for the diffeomorphism-modified flow and thus in particular

$$\int_M |dv|^2 dV + 2R_0 \int_M v dV < C.$$

Then using Aubin's result [1] and the volume bound we conclude

$$\|v\|_{H^1}^2 \leq C$$

One now easily gets an C^1 bound on the diffeomorphism parameter ϕ as in [11] Lemma 6.2 which gives the requisite bounds. \square

Lemma 11. *Given $(M^2, g(t), A(t))$ a solution to (4) or (6), on any finite time interval $[0, T]$ for any k we have*

$$(12) \quad \sup_{0 \leq t < T} \int_M e^{k|u|} dV < \infty$$

Proof. Since the volume is bounded on any finite time interval by Lemma 9 and $\|\nabla u\|_{L^2}$ is bounded on a finite time interval by Lemma 10, the result follows from the Moser-Trudinger inequality. \square

4. LONG TIME EXISTENCE

In this section we will use the integral estimates of the previous section and the assumed bound on the Sobolev constant to get an H^2 bound for both u and A . These bounds prove Theorem 2 and the existence statements of the Main Theorem. In the next section we will use the gradient property to get the convergence statements of the Main Theorem. We point out that a general short-time existence theorem for RYM flow was shown in [9] using the DeTurck gauge fixing procedure for both the Ricci flow and the Yang Mills flow together. Our bounds will apply to any flow whose volume is bounded over any finite time interval. In particular these estimates work to show long time existence for the volume normalized flow, and the unnormalized flow in the cases when $\chi(M) \leq 0$ and when $\chi(M) > 0$, $[F] \neq 0$ by Lemma 8. We will explicitly work with the unnormalized flow.

We will make use of the multiplicative Sobolev inequality

$$(13) \quad \|f\|_{L^4}^2 \leq C \|f\|_{L^2} \|f\|_{H^1} \leq C \|f\|_{H^1}^2.$$

The constant of this inequality is equivalent to the Sobolev constant as we have defined it using Hölder's inequality. Also we use an inequality of Calderón-Zygmund type:

$$(14) \quad \int_M |\nabla^2 f|^2 dV \leq C \int_M |\Delta f|^2 dV.$$

We will have occasion to write certain terms using the metric g for notational convenience, and we will mostly apply the Sobolev inequality with respect to the fixed background metric. There is one term which requires the use of the Sobolev inequality for g , and we treat it explicitly. Also, we will make repeated use of Lemmas 10 and 11.

We start with a preliminary observation. Since $\mathcal{F}(u(t), A(t))$ is continuous, non-increasing and bounded below, given $\epsilon > 0$ there is a $\tau > 0$ so that given any $0 \leq t_0 < t_1 \leq T$ such that $t_1 - t_0 < \tau$ we have

$$(15) \quad \mathcal{F}(u(t_0), A(t_0)) - \mathcal{F}(u(t_1), A(t_1)) \leq \epsilon$$

In particular for such times one has the estimate

$$(16) \quad \int_{t_0}^{t_1} \int_M |\nabla^g F|_g^2 dV_g \leq \int_{t_0}^{t_1} \frac{\partial}{\partial t} \mathcal{F}(u(t), A(t)) \leq \epsilon$$

which follows from Proposition 5. Consider the calculation

$$\begin{aligned} \frac{d}{dt} \int_M e^u |\nabla^g F|_g^2 dV_g &= 2 \int_M e^u \langle \nabla_i^g \Delta_g F, \nabla_i^g F \rangle_g dV_g - \int_M (e^u)_t |\nabla^g F|_g^2 dV_g \\ &= 2 \int_M e^u \langle \nabla_j^g \nabla_j^g \nabla_i^g F + (R_g + F^{*2}) * \nabla^g F + \nabla u_t * F, \nabla_i^g F \rangle_g dV_g \\ &\quad - \int_M (e^u)_t |\nabla^g F|_g^2 dV_g \\ &= -2 \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g + \int_M e^u \nabla u * \nabla^g \nabla^g F * \nabla^g F \\ &\quad + \int_M e^u (R_g + F^{*2}) * \nabla^g F * \nabla^g F dV_g + \int_M e^u \nabla u_t * F * \nabla^g F \\ &\quad - \int_M (e^u)_t |\nabla^g F|_g^2 dV_g \\ &\leq - \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g \\ &\quad + C \int_M \left(e^u |\nabla u|_g^2 + \Delta u + |R_0| + e^u |F|_g^2 \right) |\nabla^g F|_g^2 dV_g \\ &\quad + C \int_M e^u |\nabla u_t|_g |F|_g |\nabla^g F|_g dV_g. \end{aligned}$$

In the second line we commuted derivatives and in the third line integrated by parts. First we estimate

$$(17) \quad C |R_0| \int_{t_0}^{t_1} \int_M |\nabla^g F|_g^2 dV_g dt \leq C$$

by (16). Now we estimate

$$\begin{aligned}
\int_M e^u |\nabla u|_g^2 |\nabla^g F|_g^2 dV_g &= \int_M e^u |\nabla u|^2 |\nabla^g F|_g^2 dV \\
&\leq \|\nabla u\|_{L^4}^2 \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^4}^2 \\
&\leq C \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^2} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{H^1} \\
&\leq C \sup_{t_1 \leq t < t_2} \|u\|_{H^2} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^2} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{H^1}
\end{aligned}$$

Which implies

$$\begin{aligned}
(18) \quad &\int_{t_0}^{t_1} \int_M e^u |\nabla u|_g^2 |\nabla^g F|_g^2 dV_g \\
&\leq C \sup_{t_0 \leq t < t_1} \|u\|_{H^2} \left(\int_{t_0}^{t_1} \int_M |\nabla^g F|_g^2 dV_g \right)^{\frac{1}{2}} \cdot \left(\int_{t_0}^{t_1} \int_M |\nabla e^{\frac{u}{2}} |\nabla^g F|_g|^2 dV \right)^{\frac{1}{2}} \\
&\leq C\epsilon \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2 + C\epsilon \int_{t_0}^{t_1} \int_M e^u \left(|\nabla^g \nabla^g F|_g^2 + |\nabla u|_g^2 |\nabla^g F|_g^2 \right) dV_g \\
&\leq C\epsilon \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2 + C\epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g.
\end{aligned}$$

Analogously to the above estimate we get

$$\begin{aligned}
\int_M e^u |F|_g^2 |\nabla^g F|_g^2 dV_g &\leq \left(\int_M e^{2u} |F|_g^4 dV \right)^{\frac{1}{2}} \left(\int_M e^{2u} |\nabla^g F|_g^4 dV \right)^{\frac{1}{2}} \\
&\leq C \left\| e^{\frac{u}{2}} |F|_g \right\|_{L^2} \left\| e^{\frac{u}{2}} |F|_g \right\|_{H^1} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^2} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{H^1} \\
&\leq C \left\| e^{\frac{u}{2}} |F|_g \right\|_{H^1} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^2} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{H^1}
\end{aligned}$$

Integrating this in time and arguing as in (18) yields

$$\begin{aligned}
(19) \quad &\int_{t_0}^{t_1} \int_M e^u |F|_g^2 |\nabla^g F|_g^2 dV_g \\
&\leq C\epsilon \sup_{t_0 \leq t < t_1} \int_M e^u |\nabla^g F|_g^2 dV_g + C\epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g.
\end{aligned}$$

Also we have the estimate

$$\begin{aligned}
\int_{t_0}^{t_1} \int_M |\nabla^g F|_g^2 \Delta u dV_g &\leq \int_{t_0}^{t_1} \|\Delta u\|_{L^2} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^4}^2 \\
&\leq C\epsilon \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2 + C\epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g.
\end{aligned}$$

Turning to the final term, we see

$$\begin{aligned}
& \int_{t_0}^{t_1} \int_M e^u |\nabla u_t|_g |F|_g |\nabla^g F|_g dV_g dt \\
&= \int_{t_0}^{t_1} \int_M e^{-u} |\nabla u_t| |F| |\nabla^g F| dV \\
&\leq \int_{t_0}^{t_1} \left(\int_M e^u |\nabla u_t|^2 dV dt \right)^{\frac{1}{2}} \left(\int_M e^{-3u} |F|^2 |\nabla^g F|^2 dV \right)^{\frac{1}{2}} dt \\
&\leq C\epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla u_t|^2 dV dt + \int_{t_0}^{t_1} \int_M e^u |F|_g^2 |\nabla^g F|_g^2 dV_g dt \\
&\leq C\epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla u_t|^2 dV dt + C\epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g dt.
\end{aligned}$$

where in the last line we applied (19). Combining these estimates gives

$$\begin{aligned}
(20) \quad & \int_{t_0}^{t_1} \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g + \sup_{t_0 \leq t < t_1} \int_M e^u |\nabla^g F|_g^2 dV_g \\
& \leq C\epsilon \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2 + C \int_M e^u |\nabla^g F|_g^2 dV_g(t_0) + C
\end{aligned}$$

Now we turn to estimating u . Our bounds here are directly adopted from section 6 of [11]. First we have

$$\frac{\partial}{\partial t} e^u - \Delta u = -R_0 + \frac{1}{2} e^{-u} |F|^2$$

Multiplying this equation by $-\Delta u_t$ and integrating gives

$$\begin{aligned}
& \int_M e^u |\nabla u_t|^2 dV + \frac{1}{2} \frac{\partial}{\partial t} \int_M |\Delta u|^2 dV \\
& \leq \frac{1}{2} \int_M e^u |\nabla u_t|^2 dV + C \int_M e^u |\nabla u|^2 |u_t|^2 dV - \int_M e^{-u} |F|^2 \Delta u_t dV
\end{aligned}$$

Integrating in time and using the estimate

$$\|u\|_{L^2}^2 \leq C \int_M e^{2|u|} dV \leq C(T),$$

which follows from Jensen's inequality and Lemma 11, we conclude

$$\begin{aligned}
(21) \quad I &:= \int_{t_0}^{t_1} \int_M e^u |\nabla u_t|^2 dV dt + \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2 \\
&\leq C \int_{t_0}^{t_1} \int_M e^u |\nabla u|^2 (|u_t|^2 + 1) dV dt \\
&\quad - \int_{t_0}^{t_1} \int_M e^{-u} |F|^2 \Delta u_t dV dt + \|u(t_0)\|_{H^2} + C.
\end{aligned}$$

Since e^u is bounded in L^2 we deduce from the Sobolev inequality

$$(22) \quad \int_M e^u |\nabla u|^2 dV \leq C \|\nabla u\|_{L^4}^2 \leq C(T) \|u\|_{H^2}^2 \leq C(T) \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2$$

Similarly, using the Sobolev inequality (13) and the a-priori bound on $\|\nabla u\|_{L^2}$ we are able to bound

$$\begin{aligned}
\int_M e^u |\nabla u|^2 |u_t|^2 dV &= 4 \int_M |\nabla u|^2 |(e^{\frac{u}{2}})_t|^2 dV \\
(23) \quad &\leq C \|\nabla u\|_{L^4}^2 \|(e^{\frac{u}{2}})_t\|_{L^4}^2 \\
&\leq C \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \|(e^{\frac{u}{2}})_t\|_{L^2} \|(e^{\frac{u}{2}})_t\|_{H^1} \\
&\leq C \|(e^{\frac{u}{2}})_t\|_{L^2} \|(e^{\frac{u}{2}})_t\|_{H^1} \sup_{t_0 \leq t < t_1} \|u\|_{H^2}
\end{aligned}$$

We need to estimate the time integral of the first two terms in the above expression. First of all it is clear that

$$(24) \quad \int_{t_0}^{t_1} \|(e^{\frac{u}{2}})_t\|_{H^1}^2 dt \leq C \int_{t_0}^{t_1} \int_M e^u (|\nabla u_t|^2 + |\nabla u|^2 |u_t|^2 + |u_t|^2) dV dt$$

To estimate the other integral, we use (8) to compute

$$\begin{aligned}
\|(e^{\frac{u}{2}})_t\|_{L^2}^2 &= \int_M e^u |u_t|^2 dV \\
&= -\frac{d}{dt} \mathcal{F}(u(t), A(t)) - \int_M |\nabla^g F|_g^2 dV_g \\
&\leq -\frac{d}{dt} \mathcal{F}(u(t), A(t)).
\end{aligned}$$

Thus we can conclude

$$(25) \quad \int_{t_0}^{t_1} \|(e^{\frac{u}{2}})_t\|_{L^2}^2 dt \leq \mathcal{F}(u(t_0), A(t_0)) - \mathcal{F}(u(t_1), A(t_1))$$

Thus integrating (23) in time, applying Hölder's inequality and using (24) and (25) gives

$$\begin{aligned}
II &:= \int_{t_0}^{t_1} \int_M e^u |\nabla u|^2 |u_t|^2 dV dt \\
&\leq C (\mathcal{F}(u(t_0), A(t_0)) - \mathcal{F}(u(t_1), A(t_1)) + t_1 - t_0)^{1/2} (I + II + C)
\end{aligned}$$

Now, recall from (15) that we can choose $t_1 - t_0$ small enough that

$$C (\mathcal{F}(u(t_0), A(t_0)) - \mathcal{F}(u(t_1), A(t_1)) + t_1 - t_0)^{1/2} \leq \epsilon \leq \frac{1}{2}$$

which implies

$$II \leq 2\epsilon I + C$$

Thus from (21) and (22) we conclude

$$(26) \quad I \leq C (t_1 - t_0 + \epsilon) I + C \|u(t_0)\|_{H^2}^2 - \int_{t_0}^{t_1} \int_M e^{-u} |F|^2 \Delta u_t dV dt + C(T)$$

We now turn to the last term in this expression. First of all by integration by parts and the Cauchy-Schwarz inequality we have

$$(27) \quad \int_{t_0}^{t_1} \int_M e^{-u} |F|^2 \Delta u_t dV dt \leq \epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla u_t|^2 dV dt \\ + C \int_{t_0}^{t_1} \int_M \left(e^u |\nabla u|^2 |F|_g^4 + e^{2u} |\nabla^g F|_g^2 |F|_g^2 \right) dV dt.$$

We have already bounded the last term in the above inequality. The first term in the second line above is the one which finally requires the bound on the Sobolev constant of g . We start with an application of Hölder's inequality and the Sobolev inequality with respect to g_0 .

$$(28) \quad \int_M e^u |\nabla u|^2 |F|_g^4 dV \leq \|\nabla u\|_{L^4}^2 \left\| e^{\frac{u}{2}} |F|_g^2 \right\|_{L^4}^2 \\ \leq C \|\nabla u\|_{L^2} \|\nabla u\|_{H^1} \left\| e^{\frac{u}{2}} |F|_g^2 \right\|_{L^2} \left\| e^{\frac{u}{2}} |F|_g^2 \right\|_{H^1} \\ \leq C \sup_{t_0 \leq t < t_1} \|u\|_{H^2} \left\| e^{\frac{u}{2}} |F|_g^2 \right\|_{L^2} \left\| e^{\frac{u}{2}} |F|_g^2 \right\|_{H^1}.$$

Now we note

$$\left\| e^{\frac{u}{2}} |F|_g^2 \right\|_{L^2} = \left(\int_M e^u |F|_g^4 dV \right)^{\frac{1}{2}} \\ = \left\| |F|_g \right\|_{L^4(g)}^2 \\ \leq C_S(g) \left\| |F|_g \right\|_{L^2(g)} \left\| |F|_g \right\|_{H^1(g)} \\ \leq C \left(\int_M |\nabla^g F|_g^2 dV_g \right)^{\frac{1}{2}} \\ = C \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^2}.$$

Plugging this into (28) yields

$$(29) \quad \int_M e^u |\nabla u|^2 |F|_g^4 dV \leq C \sup_{t_0 \leq t < t_1} \|u\|_{H^2} \left\| e^{\frac{u}{2}} |\nabla^g F|_g \right\|_{L^2} \left\| e^{\frac{u}{2}} |F|_g^2 \right\|_{H^1}.$$

Integrating this in time and arguing as in line (18) yields

$$\int_{t_0}^{t_1} \int_M e^u |\nabla u|^2 |F|_g^4 dV dt \\ \leq C \epsilon \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2 + C \epsilon \int_{t_0}^{t_1} \int_M \left[e^u |\nabla u|^2 |F|_g^4 + e^{2u} |\nabla^g F|_g^2 |F|_g^2 \right] dV dt \\ \leq C \epsilon \sup_{t_0 \leq t < t_1} \|u\|_{H^2}^2 + C \epsilon \sup_{t_0 \leq t < t_1} \int_M e^u |\nabla^g F|_g^2 dV_g + C \epsilon \int_{t_0}^{t_1} \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g.$$

where in the last line we rearranged terms and applied (19). Thus plugging this into (27), applying (19) again and plugging the result into (26) gives

$$(30) \quad I \leq C(t_1 - t_0 + \epsilon)I + C\|u(t_0)\|_{H^2}^2 + C\epsilon \left(\sup_{t_0 \leq t < t_1} \int_M e^u |\nabla^g F|_g^2 dV_g + \int_{t_0}^{t_1} \int_M e^u |\nabla^g \nabla^g F|_g^2 dV_g \right) + C(T).$$

Combining this with (20) and choosing ϵ small with respect to universal constants gives

$$\sup_{t_0 \leq t < t_1} \left(\|u\|_{H^2}^2 + \int_M e^u |\nabla^g F|_g^2 dV_g \right) \leq C \left(\|u(t_0)\|_{H^2} + \int_M e^u |\nabla^g F|_g^2 dV_g(t_0) \right) + C(T).$$

Thus we can cover $[0, T]$ by finitely many intervals of length τ to yield an H^2 bound for u and an H^1 -type bound for F on any finite time interval. It is easy to see that we now also have a bound on $\|F\|_{H^1}$. Now we may choose a sequence of times $t_n \rightarrow T$ and choose divergence-free gauges for the connections $A(t_n)$. Our H^1 bound for F then yields an H^2 bound for A and so we can conclude that both A and u have uniform $C^{\frac{1}{2}}$ bound up to time T . Using this and the form of the evolution equations we can apply parabolic Schauder estimates at this point to conclude C^∞ convergence at $t = T$. This completes the proof of Theorem 2 and the existence statements of the Main Theorem.

5. CONVERGENCE RESULTS

We will apply the concentration-compactness result of Chen [3] to show convergence of the volume-normalized flow. Again we note that we do not require the isoperimetric constant bound here, these statements hold for any long-time solution of RYM flow on a surface with bounded volume. The statement we use is taken from [11].

Theorem 12. ([11] *Theorem 3.1*). *Let $g_n = e^{u_n} g_0$ be a family of smooth conformal metrics on a surface M with unit volume and bounded Calabi energy. Then either the sequence $\{u_n\}$ is bounded in $H^2(M, g_0)$ or there exist points $\{x_1, \dots, x_L\} \in M$ and a subsequence $\{u_n\}$ such that for any $\rho > 0$ and any i we have*

$$\liminf_{n \rightarrow \infty} \int_{B_\rho(x_i)} |K_n| dV_{g_n} \geq 2\pi$$

where K_n is the Gauss curvature of g_n . Moreover, there holds

$$2\pi L \leq \limsup_{n \rightarrow \infty} (Ca(g_n) + C_0)^{\frac{1}{2}} < \infty$$

and either $u_n \rightarrow -\infty$ and $n \rightarrow \infty$ locally uniformly on $M/\{x_1, \dots, x_L\}$ or $\{u_n\}$ is locally bounded in $H^2(M, g_0)$ away from $\{x_1, \dots, x_L\}$.

First consider the case $\chi(M) \leq 0$. In this case the energy \mathcal{F} is bounded below. Thus, as a consequence of Proposition 6 we have that

$$\liminf_{t \rightarrow \infty} \frac{d}{dt} \mathcal{F}(g(t), F(t)) = 0$$

Thus choose a sequence of times $\{t_n\}, t_n \rightarrow \infty$ so that

$$\lim_{n \rightarrow \infty} \frac{d}{dt} \mathcal{F}(g(t_n), F(t_n)) = 0$$

It is clear from (9) that for this sequence we further have

$$(31) \quad \lim_{n \rightarrow \infty} \int_M |\nabla^{g_n} F|_{g_n}^2 dV_{g_n} = 0$$

Our goal is to show that the Calabi energy is bounded. To do that we expand the inner product in (9). We note that intuitively since $\int_M |\nabla^g F|_g^2 dV_g$ is very small, one expects that $|F|^2$ is roughly parallel, so that the inner product should split. We carry out estimates to that effect. First note

$$(32) \quad \begin{aligned} & -\frac{1}{2} \frac{d}{dt} \mathcal{F}(g(t), A(t)) \\ &= \int_M e^{-u} (\Delta u)^2 dV - 2R_0 \int_M e^{-u} \Delta u dV + \int_M e^{-2u} \Delta u |F|^2 dV \\ & \quad - R_0 \int_M e^{-2u} |F|^2 dV + R_0 \int_M e^{-u} |F|^2 dV \\ & \quad + \frac{1}{4} \int_M e^{-3u} |F|^4 dV - \frac{1}{4} \left(\int_M e^{-u} |F|^2 \right)^2 \\ & \leq \epsilon \end{aligned}$$

Also we clearly have

$$\begin{aligned} -2R_0 \int_M e^{-u} \Delta u dV &= -2R_0 \int_M e^{-u} |du|^2 dV \\ &\geq 0 \end{aligned}$$

Combining these facts, and using the uniform bound on $\int_M e^{-u} |F|^2 dV$ yields

$$(33) \quad \int_M e^{-u} (\Delta u)^2 dV + \int_M e^{-2u} \Delta u |F|^2 dV + \frac{1}{4} \int_M e^{-3u} |F|^4 dV \leq C$$

We now show that the middle term here must be small which gives us the desired bound on the Calabi energy. In the two estimates below we will use the notation g to refer to a metric in the sequence $g(t_n)$ to simplify notation. Fix a small $\epsilon > 0$ and choose a large n so that $\int_M |\nabla^g F|_g^2 dV_g < \epsilon$. At this time we can estimate

$$\begin{aligned} \int_M e^{-2u} |F|^2 \Delta u dV &= \int_M |F|_g^2 \Delta_g u dV_g \\ &= - \int_M \langle \nabla^g |F|_g^2, \nabla u \rangle_g dV_g \\ &\leq C \int_M |\nabla^g F|_g |F|_g |\nabla u|_g dV_g \\ &= C \int_M \left(e^{\frac{u}{2}} |\nabla^g F|_g \right) \left(e^{-\frac{3u}{4}} |F| \right) \left(e^{-\frac{u}{4}} |\nabla u| \right) dV \\ &\leq C \left(\int_M |\nabla^g F|_g^2 dV_g \right)^{1/2} \left(\int_M e^{-3u} |F|^4 dV \right)^{1/4} \left(\int_M e^{-u} |\nabla u|^4 dV \right)^{1/4} \end{aligned}$$

Now we estimate using the Calderón-Zygmund inequality

$$\begin{aligned}
\left(\int_M e^{-u} |\nabla u|^4 dV \right)^{1/4} &\leq C \left(\int_M |\nabla u|^8 dV \right)^{1/8} \\
&\leq C \|\nabla u\|_{H_1^{8/5}} \\
&\leq C \|\nabla^2 u\|_{L^{8/5}} \\
&\leq C \|\Delta u\|_{L^{8/5}} \\
&= C \left(\int_M e^{\frac{4}{5}u} \left(e^{-\frac{4}{5}u} |\Delta u|^{8/5} dV \right) \right)^{5/8} \\
&\leq C \left(\int_M e^{-u} |\Delta u|^2 dV \right)^{\frac{1}{2}}
\end{aligned}$$

Plugging this into the above calculation gives

$$\int_M e^{-2u} |F|^2 \Delta u dV \leq C\epsilon \left(1 + \int_M e^{-3u} |F|^4 dV + \int_M e^{-u} |\Delta u|^2 dV \right)$$

where C is a universal constant. Thus plugging this back into (33) we can conclude that for ϵ small enough we have a uniform bound on

$$\int_M e^{-u} (\Delta u)^2 dV.$$

This implies that the Calabi energy, given by

$$Ca(g) := \int_M |K_g - \bar{K}_g|^2 dV_g = \int_M e^{-u} |\Delta u|^2 dV - C_0$$

is bounded at these times. Using the bound on $\|e^u\|_{L^2}$ we have

$$\begin{aligned}
\int_{B_\rho(x)} |K_{g(t_n)}| dV_{g_{t_n}} &\leq (Ca(g_{t_n}) + C_0)^{\frac{1}{2}} \left(\int_{B_\rho(x)} e^u dV_{g_0} \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{B_\rho(x)} dV_0 \right)^{\frac{1}{2}}.
\end{aligned}$$

This bound rules out the bubbling possibility of Theorem 12, and so we conclude a uniform H^2 bound on u for this sequence. Also we have an H^1 bound for F as in the previous section so we can take a convergent subsequence, which is in fact smoothly converging to a limit (u_∞, A_∞) . By (31) we know that F is covariant constant. Thus $|F|^2 = \int_M |F|_g^2 dV_g$ and so the limiting metric has constant scalar curvature. This shows that a subsequence converges as required, but using the nonincreasing property of \mathcal{F} , it is clear that in fact the whole flow itself must be converging to this metric.

For the case $\chi(M) > 0$ we consider the gauge-fixed flow introduced in Lemma 10. Here again the energy \mathcal{F} is bounded below so we can argue as above to show that the Calabi energy for the gauge-fixed flow is bounded for a subsequence approaching infinity. Two terms are bounded differently. In particular we have

$$2R_0 \int_M e^{-u} \Delta u dV \leq C + \epsilon \int_M e^{-u} (\Delta u)^2$$

and also

$$R_0 \int_M e^{-2u} |F|^2 \leq C + \epsilon \int_M e^{-3u} |F|^4 dV.$$

Once the Calabi energy is bounded we argue as above using Theorem 12 to show that a subsequence of the gauge-fixed flow converges. Since the diffeomorphism parameter is defined in terms of the varying metric and we now have uniform control and convergence of this metric, these diffeomorphisms also converge thus the solution $(g(t), A(t))$ also converges. Since F is parallel in the limit it follows that the limiting metric must have constant scalar curvature.

It is clear that if one assumes a uniform upper bound on volume for a solution to unnormalized RYM flow on S^2 with $[F] \neq 0$, then using Proposition 5, the arguments we have given above apply to allow us to conclude convergence to the round metric with F parallel. This completes the proof of the main theorem.

6. CONCLUSIONS

Our description of RYM flow in S^2 is encouraging, showing that a purely topological condition changes the qualitative behaviour of this equation. We have to remember however that the Ricci flow on S^2 always encounters a global, type I singularity. Indeed, an easy argument akin to lemma 8 could show a lower volume bound for any type I singularity when $[F^{\wedge n}] \neq 0$. This rules out such global singularities, but says nothing yet about local singularities, which are of course the main problem for Ricci flow in higher dimensions.

We can also make the following conjecture for odd-dimensional manifolds related to Conjecture 1:

Conjecture 13. *Let (M^{2n+1}, g) be a Riemannian manifold and $L \rightarrow M$ the total space of a $U(1)$ bundle with connection A satisfying $[F^{\wedge n}] \neq 0$. Then the solution to RYM-flow exists for all time.*

It may be possible to use the detailed description of Ricci flows on three-manifolds to attack this problem, in particular consider the case of $M^3 = S^2 \times S^1$. An argument like Lemma 8 can show that the minimal volume of an immersed S^2 representing the nonzero homology class can never drop to zero. Thus one does not expect a neckpinch singularity. However, at this point, even showing no local collapsing around a singularity is quite difficult, as Perelman's proofs do not generalize in an obvious way. Thus the structure of singularities is still poorly understood.

Besides the lack of a bound on the isoperimetric constant, there are other questions our main theorem leaves unanswered. In particular, one would like to prove that the volume of the unnormalized equation stays bounded on S^2 when $[F] \neq 0$. Also, it is likely the case that when $[F] = 0$ the flow encounters a singularity in finite time which converges to a round point with $F \equiv 0$. Since resolving these questions may likely make use of the gradient property of Ricci Yang-Mills flow, we have included the classification of Ricci Yang-Mills solitons in dimension 2.

It would also be interesting to know if the flow converges exponentially. This is likely true and would follow from a modification of the argument in [11]. Finally, it would be interesting to see if an a-priori estimate on the gradient of u may be obtained similarly to the Alexandrov reflection. Here the presence of the Yang-Mills term in the evolution of u makes the usual proof break down.

7. APPENDIX: GRADIENT SOLITONS ON SURFACES

In this section we classify all solitons on surfaces. We note that the proof of the corresponding result for Ricci flow using the Kazdan-Warner identity does not work, but we can easily adapt the proof of Chen, Lu, and Tian ([4], [5]).

Definition 14. Given (M, g) a Riemannian manifold and $L \rightarrow M$ an S^1 bundle over M with connection A , we say that (g, A) is a *gradient RYM soliton* if

$$(34) \quad \text{Rc} - \frac{1}{2}\eta + \nabla^2 f + \lambda g = 0$$

$$(35) \quad d^*F = \nabla f \lrcorner F$$

Proposition 15. *Ricci Yang-Mills flow is the gradient flow of the lowest eigenvalue of the Schrodinger operator*

$$(36) \quad -4\Delta + R - \frac{1}{4}|F|^2.$$

Moreover, this eigenvalue is constant in time if and only if the solution is a gradient Ricci Yang-Mills soliton.

Proof. The proof of this proposition is adapted directly from the corresponding proof for the Ricci flow. It can be found in [9], and was discovered independently by Andrea Young [13]. A discussion of this and other gradient properties of RYM flow will appear in a subsequent paper [10]. \square

Lemma 16. ([4] Lemma 1) *Let (Σ, g) be a two dimensional complete Riemannian manifold with non trivial Killing vector field X . If X vanishes at $O \in \Sigma$ then (Σ, g) is rotationally symmetric*

Proposition 17. *If g is a gradient soliton on a closed surface Σ^2 then g has constant curvature and F is parallel.*

Proof. The gradient soliton equations on a surface are

$$(37) \quad \left(R - \frac{1}{2}|F|^2 \right) g_{ij} = c g_{ij} + \nabla_i \nabla_j f$$

$$d^*F = \nabla f \lrcorner F$$

for some constant $c \in \mathbb{R}$. As in the case of Ricci solitons we have that ∇f is a conformal vector field. If J is the complex structure on $T\Sigma$ defined by counterclockwise rotation then $J(\nabla f)$ is a Killing vector field, which vanishes at some point since Σ is closed. Thus by lemma 16 g is rotationally symmetric. In particular we have

$$g = dr^2 + \phi(r)^2 d\theta^2, \quad 0 \leq r \leq A < \infty, \quad 0 \leq \theta \leq 2\pi$$

The gradient soliton equation now implies that F is rotationally symmetric also. In particular we set $F = \psi(r)dr \wedge d\theta$ and in particular $\frac{1}{2}|F|^2 = \frac{\psi^2}{\phi^2}$. Now the metric component of the gradient soliton equation becomes the pair of equations

$$(38) \quad -\frac{\phi''}{\phi} = c + \frac{\psi^2}{\phi^2} + f'', \quad -\frac{\phi''}{\phi} = c + \frac{\psi^2}{\phi^2} + \frac{\phi' f'}{\phi}$$

Combining these two equations gives $f'' = \frac{\phi'}{r}\phi$. We can integrate this to give $f' = a\phi$ for a constant a . Thus

$$(39) \quad -\frac{\phi''}{\phi} = c + \psi + a\phi'$$

Next, the Yang-Mills component of the gradient soliton equation becomes the pair of equations

$$(40) \quad \frac{\phi' \psi}{\phi} = 0, \quad \psi' = \psi f'$$

So, multiplying (39) by $\phi\phi'$, using that $\phi'\psi = 0$ and integrating over $[0, A]$ gives

$$-c \frac{(\phi')^2}{2} \Big|_0^A = \frac{\phi^2}{2} \Big|_0^A + a \int_0^A \phi(\phi')^2 dr$$

Since the metric is smooth we have $\phi(0) = \phi(A) = 0$ and $\phi'(0) = -\phi'(A) = 1$ so that $a = 0$. Thus f is constant. By (40) we see that $\psi' = 0$, so that F is parallel. \square

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FINE HALL, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
E-mail address: jstreets@math.princeton.edu