THE GRADIENT FLOW OF $\int_M |\mathrm{Rm}|^2$ WITH SMALL INITIAL ENERGY

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Abstract. We investigate the low-energy behavior of the gradient flow of the L^2 norm of the Riemannian curvature on a four-manifold. In particular we show that if the initial energy is chosen small enough with respect to the initial Sobolev constant and the H^1 norm of the gradient vector then the flow exists for all time and converges to a flat metric. We also improve the regularity result for the flow proved in [13] on four-manifolds.

1. Introduction

In [13] we studied the gradient flow of the L^2 norm of the Riemannian curvature tensor. Specifically let

$$\mathcal{F}(g) := \int_M \left|\mathrm{Rm}_g\right|_g^2 dV_g.$$

In what follows we will often drop the explicit reference to g, as all objects in sight will be referencing a given time-dependent metric. A basic calculation [1] shows that

$$\operatorname{grad} \mathcal{F} = \delta d \operatorname{Rc} - \check{R} + \frac{1}{4} \left|\mathrm{Rm}\right|^2 g. \tag{1}$$

where d is the acting as the exterior derivative on the Ricci tensor treated as a one-form with values in the tangent bundle, and δ is the adjoint of d. Moreover,

$$\check{R}_{ij} = R_{ipqr} R_j^{pqr}.$$

We will say that metrics satisfying $\operatorname{grad} \mathcal{F} \equiv 0$ are *critical*. The negative gradient flow equation for $\mathcal{F}$ is then

$$\begin{aligned} \frac{\partial}{\partial t} g =& \ - \operatorname{grad} \mathcal{F} \\ g(0) =& \ g_0. \end{aligned} \tag{2}$$

This is a nonlinear fourth order degenerate parabolic equation. Since the equation is fourth order maximum principle techniques are not readily available, and thus we tend to rely on integral estimates. In [13] we showed

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short-time existence of the initial value problem as well as derivative estimates and a long-time existence obstruction. The results of [13] are summed up in the next theorem, which we state after a definition.

Definition 1. Let (M^n, g) be a compact Riemannian manifold. The *Sobolev constant* of g is the smallest constant $C_S(g)$ such that

$$\left|\left|f\right|\right|_{L^{\frac{2n}{n-2}}} \leq C_S \left(\left|\left|\nabla f\right|\right|_{L^2} + \mathrm{Vol}(g)^{-\frac{1}{n}} \left|\left|f\right|\right|_{L^2} \right)$$

for any function $f \in C^{0,1}(M)$.

Theorem 2. [13] *Let (M^n, g) be a smooth Riemannian manifold. There exists a maximal constant $T \in \mathbb{R}_{>0} \cup \infty$ such that the solution to (2) with initial condition g exists and is unique on the time interval $[0, T)$. Moreover, if $T < \infty$ then either*

$$\lim_{t \to T} \left| \mathrm{Rm}(g_t) \right|_{C^0} = \infty$$

or

$$\lim_{t \to T} C_S(g_t) = \infty.$$

In this paper we aim to investigate the low-energy behavior of (2) on four manifolds. We restrict to four dimensions because it is here that the low-energy hypothesis has real significance, due to the fact that $\mathcal{F}$ is scale invariant in this dimension. Indeed examples due to Gromov ([1] pg. 136) show that all manifolds of dimension 5 and higher admit metrics with unit volume and arbitrarily small value for $\mathcal{F}$. First note that an "analytic" stability theorem for solutions to (2) should certainly hold, by which we mean convergence to a flat metric assuming C^k-closeness to a flat metric for some large k. However, one expects a small enough initial energy bound to suffice for convergence to a flat metric. Indeed in the work of Gao (e.g. [5]) the topological conclusion that a four-manifold admits a flat metric if the L^2 norm of curvature is small enough with respect to bounds on the injectivity radius and curvature was obtained. Moreover, studying the "energy concentration" behavior of (2) is natural as well. For these reasons we restrict to studying the behavior of solutions to (2) under small local or global energy bounds.

Our main result is long-time existence and convergence assuming small enough initial energy. In particular we have

Theorem 3. *Let (M^4, g) be a smooth Riemannian four-manifold. Suppose $\mathrm{Vol}(g) = 1$, $C_S(g) < C$, and $\left|\left|\mathrm{grad}\, \mathcal{F}\right|\right|_{H^1} < K$. There exists a constant $\epsilon_0 = \epsilon_0(C, K)$ such that if*

$$\int_M \left|\mathrm{Rm}\right|^2 \leq \epsilon_0,$$

Then the solution to (2) with initial condition g exists for all time and converges exponentially to a flat metric.

One should compare Theorem 3 to other recent low-energy convergence results for fourth-order flows. In particular a result of Kuwert-Schazle [8] gives convergence of the Willmore flow where the initial energy is less than a small universal constant. The proof of this fact relies on integral estimates related to those we employ here together with a blow-up argument. In later, much deeper, work the same result was shown where the initial energy is only assumed less than 8π [9]. This result is quite strong since the energy bound is explicitly given and is moreover sharp. The argument employs the Michael-Simon inequality which is a Sobolev-type inequality which holds for Euclidean submanifolds with a uniform constant at the expense of a mild nonlinearity in the lower-order term. The overall argument combines integral estimates of the form we employ here together with a detailed blow-up analysis.

Another analogous result is the work of Chen-He [4] on the convergence of Calabi flow on certain low-energy Kähler surfaces. Using an argument of Tian [15] the low-energy hypothesis together with certain symmetry hypotheses implies a uniform Sobolev constant bound for the solution to the flow. The argument again uses integral estimates together with a blow-up analysis of potential singularities. Arguments from algebraic geometry are employed to classify the singularity models.

In light of these two results, the weakness of our result is apparent: one would prefer to remove the dependence on K, and ideally C as well. The reason for our extra hypotheses is the need to bound the Sobolev constant along the flow. Indeed, as we noted, both approaches above exploit arguments quite specific to the individual flows to yield a uniform Sobolev constant bound along the flow. We are unable to find any geometrically natural situations where such a bound holds for solutions to (2), but such a bound may be possible. One idea might be to embed the family of metrics given by the solution to (2) in an ambient Riemannian space, use the Michael-Simon Sobolev inequality, and attempt to bound the mean curvature of the embedding as the metric changes. However we were unable to gain anything by this approach.

The proof of Theorem 3 has two main stages. For the first stage we rely on integral estimates, and use of the Sobolev inequality is essential to this approach. We bound the time derivative of the L^2 norm of $\operatorname{grad} \mathcal{F}$ using the low-energy hypothesis. Essential to this approach is a coercivity estimate for $\operatorname{grad} \mathcal{F}$. In particular the L^2 norm of $\operatorname{grad} \mathcal{F}$ bounds the H^2 norm of curvature up to a lower-order term which is bounded in our setting. Thus having bounded the H^2 norm of curvature we bound the variation of the H^1 norm of $\operatorname{grad} \mathcal{F}$, which implies a bound on the H^3 norm of curvature. We can now employ the Sobolev constant to derive a curvature bound. This suffices to obtain regularity and arbitrarily long existence along the flow, but we are unable to derive any convergence behavior of the flow using such estimates. For this reason we will always eventually lose control of the

Sobolev constant. In the process of these estimates we derive two theorems of independent interest.

Theorem 4. *Let* (M^4, g) *be a smooth Riemannian four-manifold with unit volume. Given* $T > 0$ *there exists* $\epsilon_0 = \epsilon_0(C_S(g), \left|\left|\operatorname{grad}\mathcal{F}\right|\right|_{H^1}, T)$ *such that if*

$$\int_M \left|\mathrm{Rm}\right|^2 \leq \epsilon_0$$

then the solution to (2) with initial condition g_0 *exists on* $[0, T]$*. Moreover, given* $k \in \mathbb{N}$ *and* $\epsilon > 0$*, there is a (smaller), and ineffective constant* $\epsilon_0' = \epsilon_0'(C_S(g), \left|\left|\operatorname{grad}\mathcal{F}\right|\right|_{H^1}, T, k)$ *such that if*

$$\int_M \left|\mathrm{Rm}\right|^2 \leq \epsilon_0'$$

then M^4 *admits a flat metric which is* ϵ*-close to* $g(T)$ *in the* C^k *topology.*

In the process of proving Theorem 4 we will derive an improvement of Theorem 2.

Theorem 5. *Let* (M^4, g) *be a smooth Riemannian four-manifold. Suppose the maximal interval of existence of the solution to (2) with initial condition* g *is* $[0, T)$*, with* $T < \infty$*. Then*

$$\limsup_{t \to T} C_S(g_t) = \infty$$

or

$$\limsup_{t \to T} \left|\left|\mathrm{Rm}\right|\right|_{L^p} = \infty$$

for every $p > 2$*.*

In an appendix we give the second stage of the proof of Theorem 3, which is an analytic stability theorem. In particular, assuming one is sufficiently C^k-close to a flat metric, we prove by a spectral analysis that the solution to (2) exists for all time and converges. Arguments of this kind are becoming quite standard and common, but we include a proof for convenience. The main theorem clearly follows by combining Theorem 4 with Theorem 6 below.

Theorem 6. *Let* (M^4, g_0) *be a flat Riemannian manifold. There exists a constant* k *and* $\epsilon > 0$ *so that if* g *is another metric on* M *such that*

$$\left|g - g_0\right|_{C^k} < \epsilon$$

then the solution to (2) with initial condition g *exists for all time and converges exponentially to a flat metric.*

We now outline the remainder of the paper. In section 2 we record some basic properties of $\operatorname{grad}\mathcal{F}$ and moreover prove a coercivity-type estimate which is at the heart of our estimates. Section 3 records some basic properties of equation (2), including evolution equations for $\operatorname{grad}\mathcal{F}$ and $\nabla \operatorname{grad}\mathcal{F}$

and L^2 derivative estimates for curvature. In section 4 we give an a-priori estimate for general solutions to (2) on time intervals where the change in energy is small. This estimate is quite general, and we use it to give the proof of Theorem 5.

2. PROPERTIES OF $\operatorname{grad} \mathcal{F}$

In this section we recall some basic properties of $\mathcal{F}$ and its gradient. We also show a coercive estimate for $\operatorname{grad} \mathcal{F}$ which holds under a low-energy hypothesis. This bound is essential to the estimates of (2) which follow in the next section.

Lemma 7. ***Divergence-Free condition*** *Let (M^n, g) be a Riemannian manifold. Then*

$$\operatorname{div} \operatorname{grad} \mathcal{F} = 0. \tag{3}$$

Proof. This follows from general principles because $\mathcal{F}$ is a diffeomorphism invariant functional, therefore its gradient is L^2-orthogonal to the infinitesimal action of the diffeomorphism group, which are given by gradients of vector fields. The adjoint of that action is to pair the vector field with $\operatorname{div} \operatorname{grad} \mathcal{F}$, which therefore must vanish. This of course can also be verified by direct calculation. □

Lemma 8. ***Coercive Estimate*** *There exists a universal $C > 0$ so that*

$$\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 \leq C \int_M \left|\operatorname{grad} \mathcal{F}\right|^2 + \left|\nabla \mathrm{Rm}\right|^2 \left|\mathrm{Rm}\right|.$$

Proof. First we observe that a formula exists expressing $\Delta \mathrm{Rm}$ in terms of second covariant derivatives of the Ricci tensor. In particular,

$$\Delta \mathrm{Rm} = \mathcal{L}(\nabla^2 \mathrm{Rc})$$

where $\mathcal{L}$ denotes a universal linear expression. This implies that

$$\left|\Delta \mathrm{Rm}\right| \leq C \left|\nabla^2 \mathrm{Rc}\right|.$$

Thus integrating by parts we see that

$$\begin{aligned} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 &= \int_M \nabla_i \nabla_j R_{klmn} \nabla^i \nabla^j R^{klmn} \\ &= -\int_M \nabla_j R_{klmn} \nabla_i \nabla^i \nabla^j R^{klmn} \\ &= \int_M \nabla_j R_{klmn} \nabla^j \Delta R^{klmn} + \nabla \mathrm{Rm}^{*2} * \mathrm{Rm} \\ &= \int_M \left|\Delta \mathrm{Rm}\right|^2 + \nabla \mathrm{Rm}^{*2} * \mathrm{Rm} \\ &\leq C \int_M \left|\nabla^2 \mathrm{Rc}\right|^2 + \left|\nabla \mathrm{Rm}\right|^2 \left|\mathrm{Rm}\right|. \end{aligned} \tag{4}$$

A similar integration by parts identity shows

$$\begin{aligned}\int_M |\Delta \operatorname{Rc}|^2 &= \int_M \nabla_i \nabla_i \operatorname{Rc}_{jk} \nabla_l \nabla_l \operatorname{Rc}_{jk}\\ &= \int_M \left|\nabla^2 \operatorname{Rc}\right|^2 + \operatorname{Rm} * \nabla \operatorname{Rm}^{*2} \qquad (5)\\ &\geq \int_M \left|\nabla^2 \operatorname{Rc}\right|^2 - C \int_M |\operatorname{Rm}| \left|\nabla \operatorname{Rm}\right|^2.\end{aligned}$$

Next we need a Kato-type inequality for the derivatives of the Ricci curvature achieved by a representation-theoretic decomposition of $\nabla \operatorname{Rc}$. Let E denote the traceless component of the covariant derivative of the Ricci tensor. In particular one has for a general Riemannian metric

$$E_{ijk} = \nabla_i \operatorname{Rc}_{jk} - \frac{n-2}{2(n+2)(n-1)} \left(\nabla_k s g_{ij} + \nabla_j s g_{ik}\right) - \frac{n}{(n+2)(n-1)} \nabla_i s g_{jk}.$$

Using this decomposition we can further write

$$\begin{aligned}\nabla_i \nabla_j R_{kl} = \nabla_i E_{jkl} &+ \frac{n-2}{2(n+2)(n-1)} \left(\nabla_i \nabla_l s g_{jk} + \nabla_l \nabla_k s g_{jl}\right)\\ &+ \frac{n}{(n+2)(n-1)} \nabla_i \nabla_j s g_{kl}\end{aligned}$$

and moreover ∇E is orthogonal to the other terms in the above formula. Therefore taking the norm squared of the above formula when $n = 4$ yields

$$\left|\nabla^2 \operatorname{Rc}\right|^2 \geq \frac{90}{324} \left|\nabla^2 s\right|^2 > \frac{1}{4} \left|\nabla^2 s\right|^2.$$

Finally we observe the identity

$$\begin{aligned}\int_M \left\langle \Delta \operatorname{Rc}, \nabla^2 s\right\rangle dV &= \int_M \nabla^i \nabla_i \operatorname{Rc}_{kl} \nabla^k \nabla^l s\\ &= -\int_M \nabla^k \nabla^i \nabla_i \operatorname{Rc}_{kl} \nabla_l s\\ &= -\int_M \Delta \nabla^k R_{kl} \nabla_l s + \nabla \operatorname{Rm}^{*2} * \operatorname{Rm}\\ &= \int_M \frac{1}{2} \nabla_i \nabla_i \nabla_l s \nabla_l s + \nabla \operatorname{Rm}^{*2} * \operatorname{Rm}\\ &= \int_M \frac{1}{2} \left|\nabla^2 s\right|^2 + \nabla \operatorname{Rm}^{*2} * \operatorname{Rm}\end{aligned}$$

which follows from the Bianchi identity and integrating by parts. This implies

$$\begin{aligned}
\int_M |\operatorname{grad}\mathcal{F}|^2 &= \int_M \left|2\Delta\,\mathrm{Rc} - \nabla^2 s + \mathrm{Rm}^{*2}\right|^2 dV \\
&\geq \int_M 4\,|\Delta\,\mathrm{Rc}|^2 - 4\left\langle \Delta\,\mathrm{Rc}, \nabla^2 s\right\rangle + \left|\nabla^2 s\right|^2 + \nabla^2\,\mathrm{Rm} * \mathrm{Rm}^{*2} \\
&\geq \int_M 4\,|\Delta\,\mathrm{Rc}|^2 - \left|\nabla^2 s\right|^2 + \mathrm{Rm} * \nabla\,\mathrm{Rm}^{*2} \\
&\geq \frac{1}{100}\int_M |\Delta\,\mathrm{Rc}|^2 - C\int_M |\nabla\,\mathrm{Rm}|^2\,|\mathrm{Rm}|\,.
\end{aligned} \tag{6}$$

Combining inequalities (4), (5) and (6) yields the result. □

Corollary 9. *Let (M^4, g) be a Riemannian four-manifold. There exists $\epsilon_0 = \epsilon_0(C_S(g))$ depending only on the Sobolev constant of g such that if*

$$\int_M |\mathrm{Rm}|^2 \leq \epsilon_0$$

then

$$\int_M \left|\nabla^2\,\mathrm{Rm}\right|^2 \leq C\int_M |\operatorname{grad}\mathcal{F}|^2 + C(C_S)\epsilon_0$$

where C is a universal constant.

Proof. From the previous lemma we have

$$\int_M \left|\nabla^2\,\mathrm{Rm}\right|^2 \leq C\int_M |\operatorname{grad}\mathcal{F}|^2 + |\nabla\,\mathrm{Rm}|^2\,|\mathrm{Rm}|\,.$$

We now estimate using Hölders inequality, the Sobolev inequality and interpolation

$$\begin{aligned}
\int_M |\nabla\,\mathrm{Rm}|^2\,|\mathrm{Rm}| \leq{}& \left(\int_M |\nabla\,\mathrm{Rm}|^4\right)^{\frac{1}{2}}\left(\int_M |\mathrm{Rm}|^2\right)^{\frac{1}{2}} \\
&\leq C_S\epsilon_0^{\frac{1}{2}}\left(\int_M \left|\nabla^2\,\mathrm{Rm}\right|^2 + \int_M |\nabla\,\mathrm{Rm}|^2\right) \\
&\leq CC_S\epsilon_0^{\frac{1}{2}}\left(\int_M \left|\nabla^2\,\mathrm{Rm}\right|^2 + \epsilon_0\right).
\end{aligned}$$

Thus for ϵ_0 chosen small with respect to the Sobolev constant, we can absorb the term $C\int_M |\nabla\,\mathrm{Rm}|^2\,|\mathrm{Rm}|$ and the result follows. □

3. Basic Properties of Equation (2)

In this section we recall some basic properties of solutions to (2). In the introduction we stated the general short-time existence of solutions together with a long-time existence obstruction. Here we note that the volume stays fixed along a solution to (2), recall the H_k-estimates of curvature, and further derive evolution equations for curvature quantities for a solution to (2).

Lemma 10. *Let* $\left(M^4, g(t)\right)$ *be a solution to (2). Then* $\mathrm{Vol}(g(t)) = \mathrm{Vol}(g(0))$.

Proof. A basic computation shows that when $n = 4$ one has $\operatorname{tr}_g \operatorname{grad} \mathcal{F} = 2\Delta R$. Therefore

$$\frac{\partial}{\partial t} \mathrm{Vol}(g(t)) = \frac{1}{2} \int_M \operatorname{tr}_g \operatorname{grad} \mathcal{F} dV = \int_M \Delta R dV = 0.$$

The result follows. $\square$

Lemma 11. [13] *Let* $(M^n, g(t))$ *be a solution to (2). Then*

$$\frac{\partial}{\partial t} \nabla^k \mathrm{Rm} = -\Delta^2 \nabla^k \mathrm{Rm} + P_2^{k+2}(\mathrm{Rm}) + P_3^k(\mathrm{Rm})$$

where $P_m^k(\mathrm{Rm})$ *refers to a product of* m *terms in the curvature and its derivatives where the sum of the total number of derivatives appearing is* k.

A consequence of these evolution equations is an L^2 heat kernel estimate for solutions to (2).

Theorem 12. [13] *Let* $(M^n, g(t))$ *be a solution to equation (2) with* M *compact. Then for every* $\alpha > 0$ *and* $m \in \mathbb{N}$, *there is a constant* C_m *depending only on* m, n *and* $\max\{\alpha, 1\}$ *such that if*

$$|\mathrm{Rm}|_{C^0(M_t)} \leq K \qquad \text{for } t \in \left[0, \frac{\alpha}{K^2}\right]$$

then

$$||\nabla^m \mathrm{Rm}||_{L^2(M_t)} \leq \frac{C_m || \mathrm{Rm} ||_{L^2(M_0)}}{t^{m/4}} \qquad \text{for } t \in (0, \frac{\alpha}{K^2}].$$

Lemma 13. *Let* $(M^n, g(t))$ *be a solution to (2). Then*

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{grad} \mathcal{F} = & -\Delta^2 \operatorname{grad} \mathcal{F} + \nabla^2 \operatorname{grad} \mathcal{F} * \mathrm{Rm} + \nabla \operatorname{grad} \mathcal{F} * \nabla \mathrm{Rm} \\ & + \operatorname{grad} \mathcal{F} * \mathrm{Rm}^{*2} + \operatorname{grad} \mathcal{F} * \nabla^2 \mathrm{Rm}. \end{aligned}$$

Proof. It is well known that for a general family of metrics $g(s)$ with $\frac{\partial}{\partial s} g(s) = h$ one has the formulas

$$\begin{aligned} \frac{\partial}{\partial s} \mathrm{Rm} = & \nabla^2 h + h * \mathrm{Rm}, \\ \frac{\partial}{\partial s} \mathrm{Rc}_{ij} = & -\frac{1}{2} \left(\Delta_L h_{ij} + \nabla_i \nabla_j \operatorname{tr} h - \nabla_i \operatorname{div}_j h - \nabla_j \operatorname{div}_j h \right), \\ \frac{\partial}{\partial s} R = & -\Delta \operatorname{tr} h + \operatorname{div} \operatorname{div} h - \langle h, \mathrm{Rc} \rangle. \end{aligned}$$

In the second line Δ_L is the Lichnerowicz Laplacian. From the first formula we derive for a solution to (2)

$$\frac{\partial}{\partial t} \mathrm{Rm}^{*2} = \nabla^2 \operatorname{grad} \mathcal{F} * \mathrm{Rm} + \operatorname{grad} \mathcal{F} * \mathrm{Rm}^{*2}.$$

Consider next

$$\Delta \operatorname{Rc}_{ij} = g * (\partial + \Gamma)(\partial + \Gamma) \operatorname{Rc}_{ij} .$$

Together with the formula for the variation of the Ricci tensor and (3), we see that for a a solution to (2),

$$\begin{aligned}\frac{\partial}{\partial t}(-2\Delta \operatorname{Rc}) = & -\Delta\Delta \operatorname{grad}\mathcal{F} - \Delta\nabla_i\nabla_j \operatorname{tr}\operatorname{grad}\mathcal{F} \\ & + \nabla^2 \operatorname{grad}\mathcal{F} * \operatorname{Rm} + \nabla \operatorname{grad}\mathcal{F} * \nabla \operatorname{Rm} + \operatorname{grad}\mathcal{F} * \operatorname{Rm}^{*2} + \operatorname{grad}\mathcal{F} * \nabla^2 \operatorname{Rm} .\end{aligned}$$

Likewise we have

$$\begin{aligned}\frac{\partial}{\partial t}\nabla_i\nabla_j s = {} & \nabla_i\nabla_j\Delta \operatorname{tr}\operatorname{grad}\mathcal{F} + \nabla^2 \operatorname{grad}\mathcal{F} * \operatorname{Rm} + \nabla \operatorname{grad}\mathcal{F} * \nabla \operatorname{Rm} \\ & + \operatorname{grad}\mathcal{F} * \operatorname{Rm}^{*2} + \operatorname{grad}\mathcal{F} * \nabla^2 \operatorname{Rm} .\end{aligned}$$

Finally we note that one can commute derivatives to yield

$$\begin{aligned}\nabla_i\nabla_j\Delta \operatorname{tr}\operatorname{grad}\mathcal{F} = {} & \Delta\nabla_i\nabla_j \operatorname{grad}\mathcal{F} + \nabla^2 \operatorname{grad}\mathcal{F} * \operatorname{Rm} + \nabla \operatorname{grad}\mathcal{F} * \nabla \operatorname{Rm} \\ & + \operatorname{grad}\mathcal{F} * \nabla^2 \operatorname{Rm} .\end{aligned}$$

Combining these calculations yields the result. □

Lemma 14. *Let $(M^n, g(t))$ be a solution to (2). Then*

$$\begin{aligned}\frac{\partial}{\partial t}\nabla \operatorname{grad}\mathcal{F} = & -\Delta^2\nabla \operatorname{grad}\mathcal{F} + \sum_{j=0}^{3} \nabla^j \operatorname{grad}\mathcal{F} * \nabla^{3-j} \operatorname{Rm} \\ & + \nabla \operatorname{grad}\mathcal{F} * \operatorname{Rm}^{*2} + \operatorname{grad}\mathcal{F} * \nabla \operatorname{Rm} * \operatorname{Rm} .\end{aligned}$$

Proof. Using the result of the previous lemma we compute

$$\begin{aligned}\frac{\partial}{\partial t}\nabla \operatorname{grad}\mathcal{F} &= \frac{\partial}{\partial t}(\partial + \Gamma)\operatorname{grad}\mathcal{F} \\ &= \nabla\left(\frac{\partial}{\partial t}\operatorname{grad}\mathcal{F}\right) + \nabla \operatorname{grad}\mathcal{F} * \operatorname{grad}\mathcal{F} \\ &= \nabla\big(-\Delta^2 \operatorname{grad}\mathcal{F} + \nabla^2 \operatorname{grad}\mathcal{F} * \operatorname{Rm} + \nabla \operatorname{grad}\mathcal{F} * \nabla \operatorname{Rm} \\ &\quad + \operatorname{grad}\mathcal{F} * \operatorname{Rm}^{*2} + \operatorname{grad}\mathcal{F} * \nabla^2 \operatorname{Rm}\big) + \nabla \operatorname{grad}\mathcal{F} * \operatorname{grad}\mathcal{F} \\ &= -\Delta^2\nabla \operatorname{grad}\mathcal{F} + \sum_{j=0}^{3} \nabla^j \operatorname{grad}\mathcal{F} * \nabla^{3-j} \operatorname{Rm} \\ &\quad + \nabla \operatorname{grad}\mathcal{F} * \operatorname{Rm}^{*2} + \operatorname{grad}\mathcal{F} * \nabla \operatorname{Rm} * \operatorname{Rm}\end{aligned}$$

□

4. An a-priori estimate for $\operatorname{grad} \mathcal{F}$

In this section we give a proof of Theorem 5. Most of the section will be consumed with deriving an a priori estimate for the L^2 norm of $\operatorname{grad} \mathcal{F}$, and the proof will be given at the end of the section using these estimates. To simplify notation we will set

$$E := \operatorname{grad} \mathcal{F}.$$

We fix a certain setting to derive estimates. Let (M^4, g) be a Riemannian manifold. Let ϵ_0 be a small constant which will be fixed later and fix a time interval $[t_0, t_1]$ such that the solution to (2) with initial condition g satisfies

$$\int_{t_0}^{t_1} \int_M |E|^2 \, dV dt \leq \epsilon_0 \leq 1.$$

Note that this condition is satisfied for arbitrary time intervals if the initial condition satisfies

$$\int_M |\mathrm{Rm}|^2 \, dV \leq \epsilon_0.$$

Furthermore assume that for any $t \in [t_0, t_1]$ one has

$$C_S(g_t) \leq A.$$

Without loss of generality we assume $A \geq 1$. Our goal is to derive further estimates on the time-dependent metric given these hypotheses. In what follows C will denote different universal constants. All dependencies on ϵ_0 and A will be made explicit. We will now derive an estimate for the L^2 norm of E. An immediate corollary of Lemma 13 is the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} ||E||_{L^2}^2 = & - ||\Delta E||_{L^2}^2 + \int_M E * \nabla^2 E * \mathrm{Rm} \\ & + \int_M E * \nabla E * \nabla \mathrm{Rm} + E^{*2} * \mathrm{Rm}^{*2} + E^{*2} * \nabla^2 \mathrm{Rm}. \end{aligned} \tag{7}$$

Integrating by parts and commuting derivatives yields

$$\begin{aligned} ||\Delta E||_{L^2}^2 = & \int_M \nabla_i \nabla_i E_{jk} \nabla_l \nabla_l E_{jk} \\ = & \int_M \left|\nabla^2 E\right|^2 + \mathrm{Rm} * \nabla E^{*2} \\ \geq & \int_M \left|\nabla^2 E\right|^2 - C \int_M |E| \left|\nabla^2 E\right| |\mathrm{Rm}| \\ & - C \int_M |E| \, |\nabla E| \, |\nabla \mathrm{Rm}|. \end{aligned}$$

Combining this with (7) and integrating over the time interval $[t_0, t_1]$ yields

$$\begin{aligned}
&||E||^2_{L^2(g_{t_1})} + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \\
&\qquad \leq \; ||E||^2_{L^2(g_{t_0})} + C \int_{t_0}^{t_1} \int_M \Big[|E| \left|\nabla^2 E\right| |\mathrm{Rm}| \\
&\qquad\qquad + |E| \, |\nabla E| \, |\nabla \mathrm{Rm}| + |E|^2 |\mathrm{Rm}|^2 + |E|^2 \left|\nabla^2 \mathrm{Rm}\right|\Big] .
\end{aligned} \tag{8}$$

We now proceed to bound the terms on the right hand side of the above inequality in a series of lemmas.

Lemma 15. *Given $(M^4, g(t))$ a solution to (2) with the above hypotheses satisfied, one has*

$$\int_{t_0}^{t_1} \int_M |E|^2 |\mathrm{Rm}|^2 \leq CA^2 \epsilon_0^{\frac{1}{2}} \left[1 + \sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2\right].$$

Proof. First we apply Hölder's inequality and the Sobolev inequality to yield

(9)

$$\begin{aligned}
\int_M |E|^2 |\mathrm{Rm}|^2 &\leq \left(\int_M |E|^4\right)^{\frac{1}{2}} \left(\int_M |\mathrm{Rm}|^4\right)^{\frac{1}{4}} \\
&\leq CA^2 \left(\int_M |\nabla E|^2 + \int_M |E|^2\right) \left(\int_M |\nabla \mathrm{Rm}|^2 + \int_M |\mathrm{Rm}|^2\right) \\
&= I + II + III + IV,
\end{aligned}$$

where the Roman numerals refer to the four different terms in the expanded product. First we bound the main term, integrating by parts

$$\begin{aligned}
I &= CA^2 \int_M |\nabla E|^2 \int_M |\nabla \mathrm{Rm}|^2 \\
&= CA^2 \left(\int_M \langle E, \Delta E\rangle\right) \left(\int_M \langle \mathrm{Rm}, \Delta \mathrm{Rm}\rangle\right) \\
&\leq CA^2 \left(\int_M |E|^2\right)^{\frac{1}{2}} \left(\int_M |\Delta E|^2\right)^{\frac{1}{2}} \left(\int_M |\mathrm{Rm}|^2\right)^{\frac{1}{2}} \left(\int_M |\Delta \mathrm{Rm}|^2\right)^{\frac{1}{2}} \\
&\leq CA^2 \left(\int_M |E|^2\right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 E\right|^2\right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}} .
\end{aligned}$$

Integrating this bound in time and applying Hölders inequality to the time integral yields

(10)

$$\begin{aligned}
\int_{t_0}^{t_1} I &\leq CA^2 \sup_{t_0 \leq t \leq t_1} \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} \int_M |E|^2\right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2\right)^{\frac{1}{2}} \\
&\leq CA^2 \epsilon_0^{\frac{1}{2}} \left[\sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2\right].
\end{aligned}$$

The lower order terms are easier to bound. We bound by interpolation

$$\begin{aligned}\int_{t_0}^{t_1} II &= \int_{t_0}^{t_1} \int_M |\nabla E|^2 \int_M |\mathrm{Rm}|^2\\ &\leq C\epsilon_0 \int_{t_0}^{t_1} \left(\int_M |E|^2\right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\\ &\leq C\epsilon_0 \left(\int_{t_0}^{t_1}\int_M |E|^2\right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\\ &\leq C\epsilon_0^{\frac{3}{2}} \left[1 + \int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right|^2\right].\end{aligned}$$

For the third term we again interpolate

$$\begin{aligned}\int_{t_0}^{t_1} III &= C\int_{t_0}^{t_1} \int_M |E|^2 \int_M |\nabla \mathrm{Rm}|^2\\ &\leq C\int_{t_0}^{t_1} \int_M |E|^2 \left(\int_M |\mathrm{Rm}|^2\right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}}\\ &\leq C\epsilon_0^{\frac{1}{2}} \sup_{t_0 \leq t \leq t_1} \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}} \int_{t_0}^{t_1}\int_M |E|^2\\ &\leq C\epsilon_0^{\frac{3}{2}} \left[1 + \sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right].\end{aligned}$$

Finally we make the bound

$$\begin{aligned}\int_{t_0}^{t_1} IV &= C\int_{t_0}^{t_1} \int_M |E|^2 \int_M |\mathrm{Rm}|^2\\ &\leq C\epsilon_0 \int_{t_0}^{t_1} \int_M |E|^2\\ &\leq C\epsilon_0^{\frac{3}{2}}.\end{aligned}$$

Combining these bounds gives the result. □

Lemma 16. *Given* $(M^4, g(t))$ *a solution to (2) with the above hypotheses satisfied, one has*

(11)
$$\int_{t_0}^{t_1} \int_M |E|^2 \left|\nabla^2 \mathrm{Rm}\right| \leq CA^2\epsilon_0^{\frac{1}{2}} \left[1 + \sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right|^2\right].$$

Proof. First we apply Hölder's inequality and the Sobolev inequality to bound

$$\begin{aligned}\int_M |E|^2 \left|\nabla^2 \mathrm{Rm}\right| \leq & \int_M |E|^2 \left|\nabla^2 \mathrm{Rm}\right| \\ \leq & \left(\int_M |E|^4\right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}} \\ \leq & A\left(\int_M |\nabla E|^2 + \int_M |E|^2\right)\left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}} \\ \leq & CA\left[\left(\int_M |E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^2 E\right|^2\right)^{\frac{1}{2}} + \int_M |E|^2\right]\left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

In the last line we applied interpolation to the integral $\int_M |\nabla E|^2$. The second term above may be integrated in time to yield

$$\begin{aligned}CA\int_{t_0}^{t_1}\int_M |E|^2 \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}} \leq & CA\left(\sup_{t_0\leq t\leq t_1}\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right)^{\frac{1}{2}}\epsilon_0 \\ \leq & CA^2\epsilon_0\left[1+\sup_{t_0\leq t\leq t_1}\int_M \left|\nabla^2 \mathrm{Rm}\right|^2\right].\end{aligned}$$

The first term above is integrated in time and bounded as in line (10), yielding the result. □

Lemma 17. *Given* $(M^4, g(t))$ *a solution to (2) with the above hypotheses satisfied, one has*

$$\int_{t_0}^{t_1}\int_M |E|\,|\nabla E|\,|\nabla \mathrm{Rm}| \leq CA^2\epsilon_0^{\frac{1}{2}}\left[1+\sup_{t_0\leq t\leq t_1}\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right|^2\right].$$

Proof. We apply Hölder's inequality, the Sobolev inequality and interpolation to bound

$$\begin{aligned}\int_M |E|\,|\nabla E|\,|\nabla \mathrm{Rm}| \leq & \left(\int_M |E|^2\right)^{\frac{1}{2}}\left(\int_M |\nabla E|^4\right)^{\frac{1}{4}}\left(\int_M |\nabla \mathrm{Rm}|^4\right)^{\frac{1}{4}} \\ \leq & CA^2\left(\int_M |E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^2 E\right|^2 + \int_M |\nabla E|^2\right)^{\frac{1}{2}} \cdot \\ & \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_M |\nabla \mathrm{Rm}|^2\right)^{\frac{1}{2}} \\ \leq & CA^2\left(\int_M |E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^2 E\right|^2 + \int_M |E|^2\right)^{\frac{1}{2}} \cdot \\ & \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_M |\mathrm{Rm}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

The time integral of each of the terms above has been bounded in the previous two lemmas, and so the result follows. □

Lemma 18. *Given* $(M^4, g(t))$ *a solution to (2) with the above hypotheses satisfied, one has*

$$\int_{t_0}^{t_1}\int_M |E|\left|\nabla^2 E\right||\mathrm{Rm}| \leq CA^2\epsilon_0^{\frac{1}{4}}\left[1+\sup_{t_0\leq t\leq t_1}\int_M\left|\nabla^2\mathrm{Rm}\right|^2+\int_{t_0}^{t_1}\int_M\left|\nabla^2 E\right|^2\right].$$

Proof. We start by applying Hölder's inequality and the Sobolev inequality to bound

$$\begin{aligned}\int_M |E|\left|\nabla^2 E\right||\mathrm{Rm}| \leq{}& \left(\int_M |E|^4\right)^{\frac{1}{4}}\left(\int_M\left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\left(\int_M|\mathrm{Rm}|^4\right)^{\frac{1}{4}}\\ \leq{}& A^2\left(\int_M|\nabla E|^2+\int_M|E|^2\right)^{\frac{1}{2}}\left(\int_M\left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\cdot\\ &\left(\int_M|\nabla\mathrm{Rm}|^2+\int_M|\mathrm{Rm}|^2\right)^{\frac{1}{2}}\\ ={}& A^2\left(I+II+III+IV\right).\end{aligned}$$

where the Roman numerals denote the four terms in the expanded product above after applying the inequality $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$. First we bound the highest order term

$$\begin{aligned}\int_{t_0}^{t_1} I ={}& \int_{t_0}^{t_1}\left(\int_M|\nabla E|^2\right)^{\frac{1}{2}}\left(\int_M\left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\left(\int_M|\nabla\mathrm{Rm}|^2\right)^{\frac{1}{2}}\\ \leq{}& \int_{t_0}^{t_1}\left(\int_M|E|^2\right)^{\frac{1}{4}}\left(\int_M\left|\nabla^2 E\right|^2\right)^{\frac{3}{4}}\left(\int_M|\mathrm{Rm}|^2\right)^{\frac{1}{4}}\left(\int_M\left|\nabla^2\mathrm{Rm}\right|^2\right)^{\frac{1}{4}}\\ \leq{}& C\sup_{t_0\leq t\leq t_1}\left(\int_M\left|\nabla^2\mathrm{Rm}\right|^2\right)^{\frac{1}{4}}\left(\int_{t_0}^{t_1}\int_M|E|^2\right)^{\frac{1}{4}}\left(\int_{t_0}^{t_1}\int_M\left|\nabla^2 E\right|^2\right)^{\frac{3}{4}}\\ \leq{}& C\epsilon_0^{\frac{1}{4}}\left[\sup_{t_0\leq t\leq t_1}\int_M\left|\nabla^2\mathrm{Rm}\right|^2+\int_{t_0}^{t_1}\int_M\left|\nabla^2 E\right|^2\right].\end{aligned}$$

Next we bound

$$\begin{aligned}\int_{t_0}^{t_1} II ={}& \int_{t_0}^{t_1}\left(\int_M|\nabla E|^2\right)^{\frac{1}{2}}\left(\int_M|\mathrm{Rm}|^2\right)^{\frac{1}{2}}\left(\int_M\left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\\ \leq{}& C\int_{t_0}^{t_1}\left(\int_M|E|^2\right)^{\frac{1}{4}}\left(\int_M\left|\nabla^2 E\right|^2\right)^{\frac{3}{4}}\\ \leq{}& C\left(\int_{t_0}^{t_1}\int_M|E|^2\right)^{\frac{1}{4}}\left(\int_{t_0}^{t_1}\int_M\left|\nabla^2 E\right|^2\right)^{\frac{3}{4}}\\ \leq{}& C\epsilon_0^{\frac{1}{4}}\left[1+\int_M\left|\nabla^2 E\right|^2\right].\end{aligned}$$

For the third term we bound

$$\begin{aligned}
\int_{t_0}^{t_1} III &= \int_{t_0}^{t_1} \left(\int_M |E|^2 \right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 E\right|^2 \right)^{\frac{1}{2}} \left(\int_M |\nabla \mathrm{Rm}|^2 \right)^{\frac{1}{2}} \\
&\leq \int_{t_0}^{t_1} \left(\int_M |E|^2 \right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 E\right|^2 \right)^{\frac{1}{2}} \left(\int_M |\nabla^2 \mathrm{Rm}|^2 \right)^{\frac{1}{4}} \left(\int_M |\mathrm{Rm}|^2 \right)^{\frac{1}{4}} \\
&\leq C \left(\sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 \right)^{\frac{1}{4}} \left(\int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \right)^{\frac{1}{2}} \\
&\leq C \epsilon_0^{\frac{1}{2}} \left[1 + \sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \right].
\end{aligned}$$

Finally we estimate

$$\begin{aligned}
\int_{t_0}^{t_1} IV &= \int_{t_0}^{t_1} \left(\int_M |E|^2 \right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 E\right|^2 \right)^{\frac{1}{2}} \left(\int_M |\mathrm{Rm}|^2 \right)^{\frac{1}{2}} \\
&\leq C \left(\int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \right)^{\frac{1}{2}} \\
&\leq C \epsilon_0^{\frac{1}{2}} \left[1 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \right].
\end{aligned}$$

Combining these four estimates and using that $\epsilon_0 \leq 1$ gives the result. □

Proposition 19. *Given $(M^4, g(t))$ a solution to (2) with the above hypotheses satisfied, for ϵ_0 chosen small with respect to A one has*

$$\sup_{t_0 \leq t \leq t_1} ||E||_{L^2}^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \leq 2 \, ||E||_{L^2(g_{t_0})}^2 + C A^2 \epsilon_0^{\frac{1}{4}} \left[1 + \sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 \right].$$

Proof. Combining Lemmas 15 - 18 and plugging into (7) yields

$$\begin{aligned}
&\sup_{t_0 \leq t \leq t_1} ||E||_{L^2}^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \\
&\qquad \leq \ ||E||_{L^2(g_{t_0})}^2 + C A^2 \epsilon_0^{\frac{1}{4}} \left[1 + \sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \right]
\end{aligned}$$

Therefore for ϵ_0 chosen small enough with respect to universal constants and A we conclude the result. □

Proposition 20. *Given $(M^4, g(t))$ a solution to (2) with*

$$\int_M |\mathrm{Rm}|^2 \leq \epsilon_0.$$

Suppose the solution exists on $[0, T)$. Then for ϵ_0 chosen small with respect to A one has for any $t_0 \leq t_1 \leq T$.

$$\sup_{t_0 \leq t \leq t_1} ||E||_{L^2}^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^2 E\right|^2 \leq 2 \, ||E||_{L^2(g_{t_0})}^2 + C A^2 \epsilon_0^{\frac{1}{4}}$$

Proof. As noted above, the hypothesis $\int_M |\mathrm{Rm}|^2 \leq \epsilon_0$ implies $\int_{t_0}^{t_1} \int_M |E|^2 \leq \epsilon_0$ for any $t_0 \leq t_1$ using the gradient property. Therefore the result of Proposition 19 applies. We now apply the coercive estimate to absorb the term $\sup_{t_0 \leq t \leq t_1} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2$ into the left hand side for ϵ_0 small. First we note the bound

$$\begin{aligned}\int_M |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}| \leq & \left(\int_M |\nabla \mathrm{Rm}|^4 \right)^{\frac{1}{2}} \left(\int_M |\mathrm{Rm}|^2 \right)^{\frac{1}{2}} \\ & \leq \epsilon_0^{\frac{1}{2}} A \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + \int_M |\nabla \mathrm{Rm}|^2 \right)\end{aligned}$$

Moreover we interpolate to estimate

$$\begin{aligned}\int_M |\nabla \mathrm{Rm}|^2 = & -\int_M \langle \mathrm{Rm}, \Delta \mathrm{Rm} \rangle \\ & \leq C \left(\int_M |\mathrm{Rm}|^2 \right)^{\frac{1}{2}} \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 \right)^{\frac{1}{2}} \\ & \leq C \epsilon_0^{\frac{1}{2}} + \epsilon_0^{\frac{1}{2}} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 .\end{aligned}$$

Therefore by the coercive estimate of Lemma 8, for a Riemannian metric g satisfying our hypotheses,

$$\begin{aligned}\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 \leq & C \int_M |E|^2 + |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}| \\ & \leq C \int_M |E|^2 + 2 \epsilon_0^{\frac{1}{2}} A \left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + C \epsilon_0^{\frac{1}{2}} \right) .\end{aligned}$$

This implies that if we choose $\epsilon_0^{\frac{1}{2}} \leq \frac{1}{4CA}$ we obtain

$$\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 \leq C \int_M |E|^2 + C A \epsilon_0^{\frac{1}{2}} . \tag{12}$$

The result follows. $\square$

Proof of Theorem 5: Fix (M^4, g) a smooth Riemannian four manifold. Suppose that the maximal interval of existence of the solution to (2) with initial condition g is $[0, T)$ with $T < \infty$. Moreover suppose that

$$\limsup_{t \to T} C_S(g_t) = A < \infty$$

and

$$\limsup_{t \to T} ||\mathrm{Rm}||_{L^p} = Q < \infty$$

for some $p > 2$. We may rescale our solution in space and time to yield a new solution which, relabelling the new time interval to $[0, T]$, satisfies

$$\limsup_{t \to T} ||\mathrm{Rm}||_{L^p} = \delta$$

where δ is chosen small with respect to A, which doesn't change under rescalings, and universal constants.

Since the functional $\mathcal{F}$ is monotonically decreasing and bounded below, given any $\epsilon_0 > 0$ we may cover $[0,T]$ by finitely many intervals $\{[t_i, t_{i+1}]\}$ of a fixed length such that for all $i \geq 0$ one has

$$\mathcal{F}(g_{t_i}) - \mathcal{F}(g_{t_{i+1}}) = \int_{t_i}^{t_{i+1}} \int_M |E|^2 \leq \epsilon_0. \tag{13}$$

We are also able to exploit the coercive estimate with our given bounds. In particular, for our given p, choose q such that $L^{\frac{2p}{p-1}}$ embeds in H_1^q, and note that $q < 2$. Thus we can estimate

$$\begin{aligned}\int_M |\nabla \mathrm{Rm}|^2 |\mathrm{Rm}| &\leq \left(\int_M |\nabla \mathrm{Rm}|^{\frac{2p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_M |\mathrm{Rm}|^p\right)^{\frac{1}{p}} \\ &\leq \delta A \left[\left(\int_M \left|\nabla^2 \mathrm{Rm}\right|^q\right)^{2/q} + \left(\int_M |\nabla \mathrm{Rm}|^q\right)^{\frac{2}{q}}\right] \\ &\leq \frac{\delta}{2} \int_M \left|\nabla^2 \mathrm{Rm}\right|^2 + C(A, \mathrm{Vol}(g)).\end{aligned}$$

Plugging this estimate into Lemma 8 yields

$$\int_M \left|\nabla^2 \mathrm{Rm}\right|^2 \leq C(A, \mathrm{Vol}(g)) \left[1 + ||E||_{L^2}^2\right]. \tag{14}$$

Using (13) and (14) in Proposition 19 yields that

$$\sup_{t_i \leq t \leq t_{i+1}} ||E||_{L^2}^2 \leq 2\, ||E||_{L^2(g_{t_i})}^2 + C(\delta, A, \mathrm{Vol}(g)).$$

Therefore $||E||_{L^2}^2$ grows at worst linearly over the interval $[t_i, t_{i+1}]$. It follows that $||E||_{L^2}^2$ is uniformly bounded on $[0,T]$ and moreover $\int_M \left|\nabla^2 \mathrm{Rm}\right|^2$ is bounded. By interpolation we have that $\int_M |\nabla \mathrm{Rm}|^2$ is bounded as well. Since the Sobolev constant is bounded, we may know by Theorem 2 that

$$\lim_{t \to T} |\mathrm{Rm}|_{C^0(g_t)} = \infty.$$

Fix a sequence of times $t_i \to T$ and let $\lambda_i := |\mathrm{Rm}|_{C^0(g_{t_i})}$. Consider the sequence of solutions

$$g_i(t) = \lambda_i g\left(\frac{t}{\lambda_i^2} + t_i\right).$$

These solutions have uniformly bounded curvature and Sobolev constant up to time 0, and therefore uniformly bounded injectivity radii. It follows from the compactness result ([13] Theorem 7.1) that the $g_i(t)$ converge to a smooth nonflat solution $(M_\infty, g_\infty(t))$ of (2) on $(-\infty, 0]$. Moreover, since the volume is fixed along a compact solution to (2) one has that $\mathrm{Vol}(g_\infty(0)) = \infty$. Also, since bound on $\int_M |\nabla \mathrm{Rm}|^2$ is uniformly bounded in the given flow and has scaling weight -1, it follows from Fatou's Lemma that $\int_{M_\infty} |\nabla \mathrm{Rm}|_{g_\infty}^2 =$

0. It also follows from Fatou's Lemma that $\int_{M_\infty} |\mathrm{Rm}|^2_{g_\infty} \leq C$. It follows that (M_∞, g_∞) is a nonflat symmetric space with infinite volume and finite L^2 norm of curvature, a contradiction. Therefore we conclude

$$\lim_{t \to T} |\mathrm{Rm}|_{C^0(g_t)} < \infty,$$

and so existence follows from Theorem 2. □

5. Proof of Theorem 4

In this section we give a proof of Theorem 4. Our main goal is to derive estimates of the L^2 norm of ∇E. We fix a certain setting to derive these estimates. Let (M^4, g) be a Riemannian manifold of unit volume and let g_t be the corresponding solution to (2). Let ϵ_0 be a small constant which will be fixed later and suppose

$$\int_M |\mathrm{Rm}|^2_{g_0} \, dV \leq \epsilon_0.$$

Furthermore assume that for any $t \in [t_0, t_1]$ one has

$$C_S(g_t) \leq A.$$

Without loss of generality we assume $A \geq 1$. It follows from Proposition 20 that this implies a bound on $||E||^2_{L^2}$ depending on A, as well as a bound on $\int_M \left|\nabla^2 \mathrm{Rm}\right|^2$ as we argued in the proof of Theorem 5. So, in what follows below, C will denote a generic constant depending on A and $\int_M \left|\nabla^2 \mathrm{Rm}\right|^2$, which is bounded by constants depending on $||E||^2_{L^2(g_0)}$ and A.

We derive the evolution of the L^2 norm of ∇E. As a consequence of Lemma 14 we get

$$\begin{aligned}\frac{\partial}{\partial t} ||\nabla E||^2_{L^2} = &- ||\Delta \nabla E||^2_{L^2} + \int_M \sum_{j=0}^{3} \nabla^j E * \nabla^{3-j} \mathrm{Rm} * \nabla E \\ &+ \int_M \nabla E^{*2} * \mathrm{Rm}^{*2} + E * \nabla \mathrm{Rm} * \mathrm{Rm} * \nabla E.\end{aligned}$$

First we integrate by parts to see

$$\begin{aligned}||\Delta \nabla E||^2_{L^2} =& \int_M \nabla_i \nabla^i \nabla E \nabla^j \nabla_j \nabla E \\ =& -\int_M \nabla^i \nabla E \nabla_i \nabla^j \nabla_j \nabla E \\ =& -\int_M \nabla^i \nabla E \nabla^j \nabla_i \nabla_j \nabla E + \mathrm{Rm} * \nabla^2 E^{*2} \\ =& \left|\left|\nabla^3 E\right|\right|^2_{L^2} + \int_M \nabla E * \mathrm{Rm} * \nabla^3 E + \nabla E * \nabla \mathrm{Rm} * \nabla^2 E.\end{aligned}$$

Therefore, integrating in time we conclude

$$\begin{aligned}
&\sup_{t_0\le t\le t_1} ||\nabla E||_{L^2}^2 + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2 \\
&\quad \le ||\nabla E||^2_{L^2(g_{t_0})} + \int_{t_0}^{t_1}\int_M \sum_{j=0}^{3} \nabla^j E * \nabla^{3-j}\mathrm{Rm} * \nabla E \qquad (15)\\
&\qquad + \int_{t_0}^{t_1}\int_M \nabla E^{*2} * \mathrm{Rm}^{*2} + E * \nabla\mathrm{Rm} * \mathrm{Rm} * \nabla E.
\end{aligned}$$

Lemma 21. *Given $(M^4, g(t))$ a solution to (2) with the above hypotheses satisfied, one has*

$$\int_{t_0}^{t_1}\int_M |\nabla E|^2 |\mathrm{Rm}|^2 \le C\epsilon_0^{\frac{1}{2}} \left[1 + \sup_{t_0\le t\le t_1} ||\nabla E||^2_{L^2} + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right].$$

where $C = C(A, ||E||_{L^2(0)})$.

Proof. We estimate using Hölder's inequality and the Sobolev inequality

$$\begin{aligned}
\int_{t_0}^{t_1}\int_M |\nabla E|^2 |\mathrm{Rm}|^2 &\le \int_{t_0}^{t_1}\left(\int_M |\nabla E|^4\right)^{\frac{1}{2}}\left(\int_M |\mathrm{Rm}|^4\right)^{\frac{1}{2}}\\
&\le C\int_{t_0}^{t_1}\left(\int_M \left|\nabla^2 E\right|^2 + \int_M |\nabla E|^2\right)\cdot\\
&\qquad \left(\int_M |\nabla\mathrm{Rm}|^2 + \int_M |\mathrm{Rm}|^2\right)\\
&= C\int_{t_0}^{t_1} I + II
\end{aligned}$$

where the terms in Roman numerals are the two terms in the expanded product. We bound them each. First using interpolation we see

$$\begin{aligned}
\int_{t_0}^{t_1} I &= \int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right|^2 \left(\int_M |\nabla\mathrm{Rm}|^2 + \int_M |\mathrm{Rm}|^2\right)\\
&\le \int_{t_0}^{t_1}\left(\int_M |\nabla E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^3 E\right|^2\right)^{\frac{1}{2}}\cdot\\
&\qquad \left(\left(\int_M |\mathrm{Rm}|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^2\mathrm{Rm}\right|^2\right)^{\frac{1}{2}} + \int_M |\mathrm{Rm}|^2\right)\\
&\le \left(C\epsilon_0^{\frac{1}{2}} + \epsilon_0\right)\sup_{t_0\le t\le t_1} ||\nabla E||_{L^2}\left(\int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right)^{\frac{1}{2}}\\
&\le C\epsilon_0^{\frac{1}{2}}\left[\sup_{t_0\le t\le t_1} ||\nabla E||^2_{L^2} + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right].
\end{aligned}$$

Next we bound the lower order term

$$\begin{aligned}\int_{t_0}^{t_1} II &= \int_{t_0}^{t_1}\int_M |\nabla E|^2 \left(\int_M |\nabla \mathrm{Rm}|^2 + \int_M |\mathrm{Rm}|^2\right)\\ &\leq \left(C\epsilon_0^{\frac{1}{2}} + \epsilon_0\right)\int_{t_0}^{t_1}\left(\int_M |E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\\ &\leq C\epsilon_0^{\frac{1}{2}}\int_{t_0}^{t_1}\left(\int_M |\nabla E|^2\right)^{\frac{1}{4}}\left(\int_M \left|\nabla^3 E\right|\right)^{\frac{1}{4}}\\ &\leq C\epsilon_0^{\frac{1}{2}} \sup_{t_0\leq t\leq t_1} ||\nabla E||_{L^2}^{\frac{1}{2}}\left(\int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right)^{\frac{1}{4}}\\ &\leq C\epsilon_0^{\frac{1}{2}}\left[1 + \sup_{t_0\leq t\leq t_1} ||\nabla E||_{L^2}^2 + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right].\end{aligned}$$

The result follows. □

Lemma 22. *Given $(M^4, g(t))$ a solution to (2) with the above hypotheses satisfied, one has*

$$\int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right| |\nabla \mathrm{Rm}| |\nabla E| \leq C\epsilon_0^{\frac{1}{2}}\left[1 + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right]$$

and

$$\int_{t_0}^{t_1}\int_M \nabla E^{*2} * \nabla^2 \mathrm{Rm} \leq C\epsilon_0^{\frac{1}{2}}\left[1 + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right]$$

where $C = C(A, ||E||_{L^2(0)})$.

Proof. We estimate using Hölder's inequality, the Sobolev inequality and interpolation

$$\begin{aligned}\int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right| |\nabla \mathrm{Rm}| |\nabla E| &\leq \int_{t_0}^{t_1}\left(\int_M \left|\nabla^2 E\right|^4\right)^{\frac{1}{4}}\left(\int_M |\nabla \mathrm{Rm}|^2\right)^{\frac{1}{2}}\left(\int_M |\nabla E|^4\right)^{\frac{1}{4}}\\ &\leq C\epsilon_0^{\frac{1}{4}}\int_{t_0}^{t_1}\left(\int_M \left|\nabla^3 E\right|^2 + \int_M \left|\nabla^2 E\right|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^2 E\right|^2 + |\nabla E|^2\right)^{\frac{1}{2}}\\ &\leq C\epsilon_0^{\frac{1}{4}}\int_{t_0}^{t_1}\left(\int_M \left|\nabla^3 E\right|^2 + \int_M |E|^2\right)\\ &\leq C\epsilon_0^{\frac{1}{4}}\left[1 + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right].\end{aligned}$$

The first formula follows. For the second we simply integrate by parts to see

$$\begin{aligned}\int_M \nabla E^{*2} * \nabla^2 \mathrm{Rm} &= \int_M \nabla^2 E * \nabla E * \nabla \mathrm{Rm}\\ &\leq C\int_M \left|\nabla^2 E\right| |\nabla E| |\nabla \mathrm{Rm}|\end{aligned}$$

and the second result follows. ☐

Lemma 23. *Given $(M^4, g(t))$ a solution to (2) with the above hypotheses satisfied, one has*

$$\int_{t_0}^{t_1}\int_M \nabla^3 E * \mathrm{Rm} * \nabla E \leq C\epsilon_0^{\frac{1}{2}}\left[1+\int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right].$$

where $C = C(A, ||E||_{L^2(0)})$.

Proof. First we integrate by parts to see

$$\begin{aligned}\int_{t_0}^{t_1}\int_M \nabla^3 E * \mathrm{Rm} * \nabla E &= \int_{t_0}^{t_1}\int_M \nabla^2 E^{*2} * \mathrm{Rm} + \nabla^2 E * \nabla \mathrm{Rm} * \nabla E\\ &\leq C\int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right|^2 |\mathrm{Rm}| + \left|\nabla^2 E\right| |\nabla \mathrm{Rm}| |\nabla E|.\end{aligned}$$

The second term is bounded using Lemma 22. For the first we estimate using Hölder's inequality and the Sobolev inequality

$$\begin{aligned}\int_{t_0}^{t_1}\int_M \left|\nabla^2 E\right|^2 |\mathrm{Rm}| &\leq \int_{t_0}^{t_1}\left(\int_M \left|\nabla^2 E\right|^4\right)^{\frac{1}{2}}\left(\int_M |\mathrm{Rm}|^2\right)^{\frac{1}{2}}\\ &\leq C\epsilon_0^{\frac{1}{2}}\int_{t_0}^{t_1}\left(\int_M \left|\nabla^3 E\right|^2 + \int_M \left|\nabla^2 E\right|^2\right)\\ &\leq C\epsilon_0^{\frac{1}{2}}\int_{t_0}^{t_1}\left(\int_M \left|\nabla^3 E\right|^2 + \int_M |E|^2\right)\\ &\leq C\epsilon_0^{\frac{1}{2}}\left[1+\int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right].\end{aligned}$$

The result follows. ☐

Lemma 24. *Given $(M^4, g(t))$ a solution to (2) with the above hypotheses satisfied, one has*

$$\int_{t_0}^{t_1}\int_M |E| |\nabla E| |\nabla \mathrm{Rm}| |\mathrm{Rm}| \leq C\epsilon_0^{\frac{1}{2}}\left[1+\sup_{t_0\leq t\leq t_1} ||\nabla E||_{L^2}^2 + \int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right]$$

where $C = C(A, ||E||_{L^2(0)})$.

Proof. We estimate using Hölder's inequality and the Sobolev inequality

$$\begin{aligned}
\int_{t_0}^{t_1}\int_M &\left|E\right|\left|\nabla E\right|\left|\nabla \operatorname{Rm}\right|\left|\operatorname{Rm}\right| \\
&\leq \int_{t_0}^{t_1}\left(\int_M |E|^4\right)^{\frac{1}{4}}\left(\int_M |\nabla E|^4\right)^{\frac{1}{4}}\left(\int_M |\nabla \operatorname{Rm}|^4\right)^{\frac{1}{4}}\left(\int_M |\operatorname{Rm}|^4\right)^{\frac{1}{4}} \\
&\leq C\epsilon_0^{\frac{1}{4}}\int_{t_0}^{t_1}\left(\int_M |\nabla E|^2+\int_M |E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^2 E\right|^2+\int_M |E|^2\right)^{\frac{1}{2}} \\
&\leq C\epsilon_0^{\frac{1}{4}}\int_{t_0}^{t_1}\left(\int_M \left|\nabla^2 E\right|^2+\int_M |E|^2\right) \\
&\leq C\epsilon_0^{\frac{1}{4}}\left[1+\int_{t_0}^{t_1}\left(\int_M |\nabla E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^3 E\right|^2\right)^{\frac{1}{2}}\right] \\
&\leq C\epsilon_0^{\frac{1}{4}}\left[1+\sup_{t_0\leq t\leq t_1}||\nabla E||_{L^2}^2+\int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right].
\end{aligned}$$

The result follows. ☐

Lemma 25. *Given* $(M^4, g(t))$ *a solution to (2) with the above hypotheses satisfied, for arbitrary* $\delta > 0$ *one has*

$$\int_{t_0}^{t_1} E * \nabla^3 \operatorname{Rm} * \nabla E \leq C\epsilon_0^{\frac{1}{4}}\left[1+\sup_{t_0\leq t\leq t_1}||\nabla E||_{L^2}^2+\int_{t_0}^{t_1}\int_M \left|\nabla^3 E\right|^2\right]$$

where $C = C(A, ||E||_{L^2(0)})$.

Proof. First we integrate by parts to see

$$\begin{aligned}
\int_{t_0}^{t_1}\int_M E * \nabla^3 \operatorname{Rm} * \nabla E &= \int_{t_0}^{t_1}\int_M \nabla E^{*2} * \nabla^2 \operatorname{Rm} + E * \nabla^2 \operatorname{Rm} * \nabla E \\
&\leq C\int_{t_0}^{t_1}\int_M \nabla E^{*2} * \nabla^2 \operatorname{Rm} + |E|\left|\nabla^2 \operatorname{Rm}\right|\left|\nabla E\right|.
\end{aligned}$$

The first term is bounded using Lemma 23. For the second we estimate using Hölder's inequality and the Sobolev inequality

$$\begin{aligned}
\int_{t_0}^{t_1}\int_M &|E|\left|\nabla^2 E\right|\left|\nabla^2 \operatorname{Rm}\right| \\
&\leq \int_{t_0}^{t_1}\left(\int_M |E|^4\right)^{\frac{1}{4}}\left(\int_M \left|\nabla^2 E\right|^4\right)^{\frac{1}{4}}\left(\int_M \left|\nabla^2 \operatorname{Rm}\right|^2\right)^{\frac{1}{2}} \\
&\leq C\int_{t_0}^{t_1}\left(\int_M |\nabla E|^2+\int_M |E|^2\right)^{\frac{1}{2}}\left(\int_M \left|\nabla^3 E\right|^2+\int_M |E|^2\right)^{\frac{1}{2}} \\
&\leq C\int_{t_0}^{t_1} I + II + III + IV.
\end{aligned}$$

where the Roman numerals refer to the terms in the expanded product after applying the inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. In particular we can bound

$$\begin{aligned}
\int_{t_0}^{t_1} I &= \int_{t_0}^{t_1} \left(\int_M |\nabla E|^2 \right)^{\frac{1}{2}} \left(\int_M \left|\nabla^3 E\right|^2 \right)^{\frac{1}{2}} \\
&\leq \int_{t_0}^{t_1} \left(\int_M |E|^2 \right)^{\frac{1}{4}} \left(\int_M \left|\nabla^2 E\right|^2 \right)^{\frac{1}{4}} \left(\int_M \left|\nabla^3 E\right|^2 \right)^{\frac{1}{2}} \\
&\leq \int_{t_0}^{t_1} \left(\int_M |E|^2 \right)^{\frac{1}{4}} \left(\int_M |\nabla E|^2 \right)^{\frac{1}{8}} \left(\int_M \left|\nabla^3 E\right|^2 \right)^{\frac{5}{8}} \\
&\leq \sup_{t_0 \leq t \leq t_1} ||\nabla E||_{L^2}^{\frac{1}{4}} \left(\int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{4}} \left(\int_{t_0}^{t_1} \left(\int_M \left|\nabla^3 E\right|^2 \right)^{\frac{5}{6}} \right)^{\frac{3}{4}} \\
&\leq \epsilon_0^{\frac{1}{4}} \left[1 + \sup_{t_0 \leq t \leq t_1} ||\nabla E||_{L^2}^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^3 E\right|^2 \right].
\end{aligned}$$

The lower order terms are much easier. First we have

$$\begin{aligned}
\int_{t_0}^{t_1} II &= \int_{t_0}^{t_1} \left(\int_M |\nabla E|^2 \right)^{\frac{1}{2}} \left(\int_M |E|^2 \right)^{\frac{1}{2}} \\
&\leq \sup_{t_0 \leq t \leq t_1} ||\nabla E||_{L^2} \left(\int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{2}} \\
&\leq \epsilon_0^{\frac{1}{2}} \left[1 + \sup_{t_0 \leq t \leq t_1} ||\nabla E||_{L^2}^2 \right].
\end{aligned}$$

Next we see

$$\begin{aligned}
\int_{t_0}^{t_1} \left(\int_M |E|^2 \right)^{\frac{1}{2}} \left(\int_M \left|\nabla^3 E\right|^2 \right)^{\frac{1}{2}} &\leq \left(\int_{t_0}^{t_1} \int_M |E|^2 \right)^{\frac{1}{2}} \left(\int_{t_0}^{t_1} \int_M \left|\nabla^3 E\right|^2 \right)^{\frac{1}{2}} \\
&\leq \epsilon_0^{\frac{1}{2}} \left[1 + \int_{t_0}^{t_1} \int_M \left|\nabla^3 E\right|^2 \right].
\end{aligned}$$

Finally we bound

$$\int_{t_0}^{t_1} IV = \int_{t_0}^{t_1} \int_M |E|^2 \leq \epsilon_0.$$

The result follows. □

Proposition 26. *Given $(M^4, g(t))$ a solution to (2) with the above hypotheses satisfied, there exists $\epsilon_0 = \epsilon_0(A, ||E||_{L^2(0)})$ such that if $\mathcal{F}(g_0) \leq \epsilon_0$ then*

(16)

$$\sup_{t_0 \leq t \leq t_1} ||\nabla E||_{L^2}^2 + \int_{t_0}^{t_1} \int_M \left|\nabla^3 E\right|^2 \leq 2\, ||\nabla E||_{L^2(g_{t_0})}^2 + C(A, ||E||_{L^2(0)}) \epsilon_0^{\frac{1}{4}}.$$

Proof. This follows immediately applying Lemmas 21 - 25 in (15). □

Proof of Theorem 4: Fix (M^4, g) a smooth Riemannian four-manifold with unit volume, and fix $T > 0$. We aim to show that for ϵ_0 chosen small with respect to $||E(g)||_{H_1}$, $C_S(g)$ and T the flow exists on $[0, T]$. Let $A = 4C_S(g_0)$. We want to show that the bound $C_S(g_t) < A$ is preserved on the time interval $[0, T]$. First choose ϵ_0 small with respect to A and $||E||_{L^2(0)}$ such that the results of Propositions 20 and 26 apply. We apply a multiplicative Sobolev inequality to estimate the C^0 norm of E. In particular, we apply Theorem 34 with $m = 2, p = 8$ and $\alpha = \frac{4}{5}$ to conclude

$$||E||_{C^0} \leq C(A) \, ||E||_{L^2}^{\frac{1}{5}} \left[||\nabla E||_{L^8} + ||E||_{L^8} \right]^{\frac{4}{5}}.$$

Applying the Sobolev inequality and interpolation inequalities one has

$$\begin{aligned}
||\nabla E||_{L^8} &= \left|\left| |\nabla E|^2 \right|\right|_{L^4}^{\frac{1}{2}} \\
&\leq C(A) \left[\left|\left| \nabla |\nabla E|^2 \right|\right|_{L^2}^{\frac{1}{2}} + \left|\left| |\nabla E|^2 \right|\right|_{L^2}^{\frac{1}{2}} \right] \\
&\leq C(A) \left[\left(\int_M \left|\nabla^2 E\right|^2 |\nabla E|^2 \right)^{\frac{1}{4}} + \left(\int_M |\nabla E|^4 \right)^{\frac{1}{4}} \right] \\
&\leq C(A) \left[\left|\left|\nabla^2 E\right|\right|_{L^4} + ||\nabla E||_{L^4} \right] \\
&\leq C(A) \left[\left|\left|\nabla^3 E\right|\right|_{L^2} + \left|\left|\nabla^2 E\right|\right|_{L^2} + ||\nabla E||_{L^2} \right] \\
&\leq C(A) \left[\left|\left|\nabla^3 E\right|\right|_{L^2} + ||E||_{L^2} \right].
\end{aligned}$$

Likewise $||E||_{L^8}$ can be bounded to yield

$$||\nabla E||_{L^8} + ||E||_{L^8} \leq C(A) \left(\left|\left|\nabla^3 E\right|\right|_{L^2} + ||E||_{L^2} \right).$$

Thus plugging this into the above equation and integrating in time we yield

$$\begin{aligned}
\int_0^T ||E||_{C^0} &\leq C(A) \int_0^T ||E||_{L^2}^{\frac{1}{5}} \left[\left|\left|\nabla^3 E\right|\right|_{L^2} + ||E||_{L^2} \right]^{\frac{4}{5}} \\
&\leq C(A) \left(\int_0^T ||E||_{L^2}^2 \right)^{\frac{1}{10}} \left(\int_0^T \left[\left|\left|\nabla^3 E\right|\right|_{L^2}^2 + ||E||_{L^2}^2 \right] \right)^{\frac{2}{5}} \left(\int_0^T 1 \right)^{\frac{1}{2}} \\
&\leq C(A, ||\nabla E||_{L^2}) \epsilon_0^{\frac{1}{10}} T^{\frac{1}{2}}.
\end{aligned}$$

In the last line we applied the result of Proposition 26.

We have bounded the time integral of the C^0 norm of the time derivative of g. Thus

$$e^{-C(A, ||\nabla E||_{L^2}) \epsilon_0^{\frac{1}{10}} T^{\frac{1}{2}}} g_0 \leq g_t \leq e^{C(A, ||\nabla E||_{L^2}) \epsilon_0^{\frac{1}{10}} T^{\frac{1}{2}}} g_0.$$

it follows from Lemma 33 that for ϵ_0 chosen small with respect to A and T and $||E||_{H^1}$ one has

$$C_S(g_t) \leq 2C_S(g_0) < A$$

for all $t \leq T$. Also, applying Proposition 20 and arguing as before we conclude a bound on $||\mathrm{Rm}||_{L^4}$ for all $t \leq T$. Thus Theorem 5 applies to conclude smooth existence up to time T. This proves the first statement of Theorem 4.

To show the second statement we will use a compactness argument. We have noted before that bounds on $C_S(g)$ and $||E(g)||_{H^1}$ imply a curvature bound. Also, volume and Sobolev constant bounds imply a diameter bound for g. Thus we may apply [5] Theorem 0.11 with $\Delta = 0$ to conclude that there exists a small constant ϵ so that if $\mathcal{F}(0) \leq \epsilon$ then there exists a flat metric on M. Moreover, any metric satisfying these same bounds is $C^{1,\alpha}$-close to a flat metric. Thus in particular $g(T)$ is at least $C^{1,\alpha}$-close to a flat metric once ϵ_0 is chosen small as in [5] Theorem 0.11. It remains to show bounds on the derivatives of curvature at time T. Suppose at least $T \geq 1$. Now fix some k. The L^2-derivative estimates of Theorem 12 imply that $||\nabla^m \mathrm{Rm}||_{L^2(g(T))} \leq C_m \epsilon_0^{\frac{1}{2}}$ where the constant C_m depends on m and the curvature bound, which in turn only depends on the bounds on $C_S(g)$ and $||E(g)||_{H^1}$. Choosing m large enough and applying the Sobolev inequality it follows that the C^k-norm of $g(T)$ can be made small with respect to ϵ_0, and the result follows. □

6. Appendix: Analytic Stability

In this section we give the proof of Theorem 6. As mentioned in the introduction, the proof essentially relies on a linear analysis, specifically understanding the spectrum of the linearization of $\operatorname{grad} \mathcal{F}$ at a flat metric. Results of this type are quite standard, see for instance [3] [6], [11], [12], [14]. Our argument is quite similar to that used in [11] and [14]. Since these techniques are becoming quite common, and they are not the main thrust of this paper, we will at points give only sketches of proofs. The main points are that the linearization of the gauge-fixed operator of $\operatorname{grad} \mathcal{F}$ is positive semidefinite at a flat metric, and that the kernel of said operator is realized through variations of flat metrics.

The linearization of $\operatorname{grad} \mathcal{F}$ is the second variation of $\mathcal{F}$, and as such we will denote it $\mathcal{D}^2 \mathcal{F}$. We will show that when acting on divergence-free symmetric two-tensors this operator is semi-positive. To that end, fix a flat metric g_0 and let $\delta^*_{g_0}$ denote the L^2-adjoint of the divergence operator associated to g_0, and let $\Pi_{\mathrm{Im}\,\delta^* g_0}$ be L^2 projection onto its image. Define

$$P_{g_0}(h) = \mathcal{D}^2_{g_0} \mathcal{F}\left(\Pi_{\mathrm{Im}\,\delta^*} h\right) \tag{17}$$

and let

$$L_{g_0}(h) := \left(\mathcal{D}^2_{g_0} \mathcal{F} - P_{g_0}\right)(h). \tag{18}$$

Proposition 27. *Fix (M^4, g_0) a flat Riemannian manifold of unit volume, and fix $h \in \mathrm{Sym}^2 T^* M$. The second derivative of $\mathcal{F}$ at g_0 with respect to h*

is

$$\mathcal{D}^2_{g_0}\mathcal{F}(h) = \Delta^2 h + \nabla^2_{jk}\left(\operatorname{div}^2 h\right) - \nabla_k \Delta \operatorname{div} h_j - \nabla_j \Delta \operatorname{div} h_k. \tag{19}$$

Moreover, $L_{g_0} = \Delta^2 \geq 0$.

Proof. First of all it since the underlying metric is flat it is clear that

$$\left(\check{R} - \frac{1}{4}\left|\mathrm{Rm}\right|^2 g\right)' \cdot h = 0$$

since these terms are quadratic in curvature. Next we have

$$\begin{aligned}(\delta d \operatorname{Rc})' \cdot h = & -\frac{1}{2}\delta d\left(\Delta h + \nabla^2 \operatorname{tr} h - \nabla_j \operatorname{div} h_k - \nabla_k \operatorname{div} h_j\right)\\ &= \Delta\left(\Delta h_{jk} + \nabla^2_{jk} \operatorname{tr} h - \nabla_j \operatorname{div} h_k - \nabla_k \operatorname{div} h_j\right)\\ &\quad - \nabla^m \nabla_j\left(\Delta h_{mk} + \nabla^2_{mk} \operatorname{tr} h - \nabla_m \operatorname{div} h_k - \nabla_k \operatorname{div} h_m\right)\\ &= \Delta^2 h + \nabla^2_{jk}\left(\operatorname{div}^2 h\right) - \nabla_k \Delta \operatorname{div} h_j - \nabla_j \Delta \operatorname{div} h_k.\end{aligned}$$

Thus we have shown (19). The operator L_{g_0} is given by the action of $\mathcal{D}^2_{g_0}\mathcal{F}$ on divergence-free tensors, thus $L_{g_0} = \Delta^2 \geq 0$. ☐

Note that because of the diffeomorphism invariance of $\mathcal{F}$, we will always have some zero eigenvalues of L coming from the action of the diffeomorphism group. However, there are further degeneracies coming from the nontrivial moduli of flat metrics near g_0. For this reason, we do not necessarily expect that the flow should converge to the given flat metric g_0, but perhaps to one close by. An important property of flat metrics is that the space of nearby flat metrics has a manifold structure. In general this property is often referred to as *integrability* of the metric g_0. Specifically, for a flat metric g_0, the operator $L = \Delta^2$ simply acts as the usual bilaplacian on the coordinate entries of a tensor h, i.e. $(\Delta^2 h)_{ij} = \Delta^2(h_{ij})$. Since the bilaplacian on functions has kernel given only by constant functions, we have that the kernel of L is made up of tensors h with constant entries. The metrics $g(t) = g + th$ are all flat, with first variation h. This is the condition we will exploit in the proof below.

Proof of Theorem 6: Fix h a symmetric two tensor such that $|h_0|_{C^k} < \epsilon' < \epsilon$ where k, ϵ' and ϵ are to be chosen later. In fact all we will need is $k > 6$. We want to show that solution to the equation

$$\begin{aligned}\frac{\partial}{\partial t} g = & -\operatorname{grad} \mathcal{F}\\ g(0) = & \ g_0 + h_0\end{aligned} \tag{20}$$

exists for all time and converges for ϵ' chosen small enough. We will want to use standard parabolic regularity theory, so we in fact look at the gauge-fixed flow

$$\begin{aligned} \frac{\partial}{\partial t} g = & -\operatorname{grad}_g \mathcal{F} + P_{g_0}(g) \\ g(0) = & \, g_0 + h_0 \end{aligned} \tag{21}$$

where P is defined as above. We showed in [13] that the operator P_{g_0} can be accounted for by diffeomorphism flow. Indeed this is how short-time existence of solutions to equation (20) was proved. In particular there exists a one-parameter family of diffeomorphisms ϕ_t such that if $g(t)$ is a solution to (20) then $\phi_t^*(g(t))$ solves (21). Note that g_0 is stationary for (21) since it is critical and $P_{g_0}(g_0) = 0$. Let $h(t) = g(t) - g_0$ where $g(t)$ is the solution to (21). We start with the basic calculation

$$\begin{aligned} \frac{\partial}{\partial t} h = & -\operatorname{grad}_g \mathcal{F} + P_{g_0}(g) \\ = & -\big(\operatorname{grad}_{g_0} \mathcal{F} + \mathcal{D}^2_{g_0}\mathcal{F}(h) + A(g_0, h)\big) + P_{g_0}(g) \\ = & -\big(\mathcal{D}^2_{g_0}\mathcal{F}(h) - P_{g_0}(h)\big) + A(g_0, h) \\ = & -L(h) + A(g_0, h) \end{aligned} \tag{22}$$

where A represents the higher order terms in the approximation of $\operatorname{grad}_g \mathcal{F}$ by $\mathcal{D}^2_{g_0}\mathcal{F}(h)$. Specifically we have the bounds

$$|A(g_0, h)|_{C^k} \leq C \sum_{j=0}^{4} \left|\nabla^j h\right|_{C^k} \left|\nabla^{4-j} h\right|_{C^k} \tag{23}$$

where the constant C depends on bounds on the C^0 geometry of $g(t)$, which we are assuming is staying bounded along the flow anyways since $|g(t) - g_0|_{C^k} < \epsilon$. So, fix $T > 0$ and a small $\epsilon > 0$. We would like to show that for ϵ' small enough as above our solution exists on $[0, T)$ and $|h(t)|_{C^k} < \epsilon$ on this interval. We start with an L^2 growth estimate. In all of the calculations to follow, unless explicit reference is given the metric being used is the background flat metric g_0.

Lemma 28. *There exists a uniform (independent of ϵ, ϵ', T) constant C so that if $|h|_{C^k} < \epsilon$ for all $t \in [0, T)$, we have*

$$\int_M |h(t)|^2 \, dV_{g_0} \leq e^{C\epsilon t} \int_M |h_0|^2 \, .dV_{g_0}$$

Proof. Multiplying the final equation in (22) by h and integrating over M gives

$$\frac{\partial}{\partial t} \int_M |h(t)|^2 \, dV_{g_0} \leq \int_M (A * h) \, dV_{g_0} \tag{24}$$

using that L is positive semidefinite. By straightforward bounds using integration by parts and the assumed C^k bound on h for $k > 4$ we are able to get the bound

$$\left|\int_M (A * h)\, dV_{g_0}\right| \leq C\epsilon \int_M |h|^2\, dV_{g_0}$$

where C depends only on g_0. The result follows immediately ☐

Lemma 29. *There exists $\epsilon' = \epsilon'(T, \epsilon) < \epsilon$ such that if $|h_0|_{C^k} < \epsilon'$, then the solution $g(t)$ exists on $[0, T)$ with $|h(t)|_{C^k} < \epsilon$ for all $t \in [0, T)$.*

Proof. We plan to use the standard parabolic regularity theory. First we rewrite the evolution equation for h as

$$\frac{\partial}{\partial t} h = -\Delta^2 h + A(g_0, h). \tag{25}$$

Fix a time $\tau < T$. We first will get an estimate for $\int_0^\tau \int_M \left|\nabla^2 h\right|^2 dV_{g_0} dt$. Take the inner product of (25) with h and integrate over M to get

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t}\int_M |h|^2 =& -\int_M |\Delta h|^2 + \int_M A(g_0, h) * h \\ \leq& -\int_M |\Delta h|^2 + C\epsilon \int_M |h|^2 \end{aligned} \tag{26}$$

Now we integrate by parts, commute derivatives and use the flatness of g_0 to conclude

$$\int_M \left|\nabla^2 h\right|^2 = \int |\Delta h|^2 . \tag{27}$$

Using this and integrating over time we conclude

$$\int_0^\tau \int_M \left|\nabla^2 h\right|^2 \leq \int_M |h_0|^2 + C\epsilon\tau \sup_{[0,\tau)} \int_M |h(t)|^2 .$$

So, using Lemma 28 we see that $\int_0^\tau \int_M \left|\nabla^2 h\right|^2$ can be made small comparable to ϵ'. Continuing in this fashion we can induct to get a bound of the above form for all covariant derivatives of h. Indeed, following the usual proof of parabolic regularity we can in fact get bounds of the form

$$\int_0^\tau \int_M \left|\frac{\partial^p}{\partial t^p}\nabla^q h\right|^2 \leq C(p, q)\epsilon'$$

for all $p, q > 0$. One can now apply the Sobolev inequality (with respect to g_0) to conclude C^k bounds on h in terms of ϵ'. These bounds will hold over any time interval where the L^2 norm of h is still small. This time can be made arbitrarily large with small choice of ϵ' by Lemma 28. At this point standard parabolic regularity results apply to conclude smooth existence of the flow up to this time, and the result follows. ☐

Say T is a maximal time such that $|h|_{C^k} < \epsilon$ on $[0, T)$. Divide the interval $[0, T)$ into intervals of length τ and let N be the integer so that $N\tau < T < (N+1)\tau$. Let $I_j = [j\tau, (j+1)\tau]$. On $M_j := M \times I_j$ define the inner product

$$||f||^2_{M_j} := \int_{j\tau}^{(j+1)\tau} ||f(t)||^2_{L^2(g_0)} dt \tag{28}$$

Let π^j denote the orthogonal projection onto $\ker\left(\frac{\partial}{\partial t} + L\right)$ with respect to $||f||_{M_j}$. Furthermore, let $\pi^j_0(h)$ denote the temporally constant component, i.e. the kernel of L. This component is potentially troublesome, as it would prevent concluding exponential decay of h along M_j. We will show using the smooth manifold structure of nearby flat metrics that there exists a flat metric g_j on M_j close to g_0 such that $\pi^j_0(g(t) - g_j)$ is small on M_j. This will allow us to conclude L^2 decay of h and then convergence.

Lemma 30. *Fix an interval* $I = [t_0, t_0 + \tau]$. *Given* $\alpha > 0$ *there exists* $\delta = \delta(n, \tau)$ *such that if* $\sup_I |h(t)|_{C^k} < \delta$ *then there exists a flat metric* g_1 *such that*

$$\left|\pi^j_0(g - g_1)\right|_{C^k} \leq \alpha(t - t_0) \left|g - g_1\right|_{C^k} \tag{29}$$

and

$$\left|g_1 - g_0\right|_{C^k} \leq C \sup_I \left|g - g_0\right|_{C^k}. \tag{30}$$

Proof. Recall that we have shown that the space of flat metrics near g_0 up to diffeomorphism equivalence, call it $\mathcal{U}$, has a natural smooth manifold structure. The tangent space to $\mathcal{U}$ is given by the kernel of L, call it $\mathcal{K}$, which is finite dimensional since L is elliptic. Let $\{B_i\}$ be a basis for $\mathcal{K}$ orthonormal with respect to the L^2 norm induced by g_0. Also using ellipticity, we get a system of eigenvectors $\{E_\lambda\}$ for L orthonormal with respect to the L^2 inner product above. We see that there exist constants r_λ such that $C_\lambda = r_\lambda E_\lambda e^{-\lambda t}$ is a basis for $\ker\left(\frac{\partial}{\partial t} + L\right)$ which is orthonormal with respect to the inner product in (28).

Define the map $\Psi : \mathcal{U} \to \mathcal{K}$ by $\Psi(g) = \sum_i \langle g, B_i \rangle B_i$. A simple calculation using the bases described above shows that for $g_1 \in \mathcal{U}$, $\Psi(g_1) = \Psi(\pi^j(g_1)) = \pi^j_0(g_1)$. Also it is easy to see that the differential of Ψ at g_0 is the identity map, so we can apply the inverse function theorem. If $|g(t) - g_0|_{C^k}$ is small enough, then in particular $\pi^j_0(g(t_0) - g_0) = \pi^j(g(t_0))$ can be made small, so that by the argument above there exists $g_1 \in \mathcal{U}$ such that

$$\Psi(g_1) = \pi^j_0(g(t_0)).$$

Thus in particular using the above equalities we have $\pi^j_0(g_1 - g(t_0)) = 0$. Using the evolution equations satisfied by g and g_1 and the bound on h it is clear that one has estimate (29). Also note that we have $g_1 = \Psi^{-1}((\pi^j g)_0)$

and $g_0 = \Psi^{-1}((\pi^j g_0)_0)$ thus using our bound from the inverse function theorem we get

$$||g_1 - g_0||_{M_j} \leq C||\pi^j(g - g_0)||_{M_j}$$

and again using that these are all solutions of the same parabolic equation, we can get the bound

$$|g_1 - g_0|_{C^k} \leq C \sup_{I_j} |g - g_0|_{C^k}.$$

□

Lemma 31. *Let $I = [\tau_0, \tau_0 + \tau]$ and choose g_1 as in Lemma 30. Then there exists $\epsilon = \epsilon(\tau) > 0$ such that if $|g - g_1(0)|_{C^k} < \epsilon$ then*

$$\sup_{\left[\tau_0 + \frac{\tau}{2}, \tau_0 + \tau\right]} \int_M |g - g_1|^2 \, dV_{g_0} \leq e^{-\frac{\tau\lambda}{2}} \sup_{\left[\tau_0, \tau_0 + \frac{\tau}{2}\right]} \int_M |g - g_1|^2 \, dV_{g_0} \tag{31}$$

where $\lambda = \min\{\lambda_i : \lambda_i$ is an eigenvalue of $L_{g_0}, \lambda_i \neq 0\} > 0$.

Proof. Let $h_1(t) = g(t) - g_1$. Note that $|h_1|_{C^k} < \epsilon$ using estimate (30). We will linearize $\operatorname{grad} \mathcal{F}$ at g_1 and emulate the calculation of Lemma 28. Since g_1 is close to g_0 in C^k for $k > 4$, the lowest nonzero eigenvalue of $\Delta^2_{g_1}$ is within $\frac{1}{10}$ of that for g_0. Thus using the estimates (29) and (30) we yield

$$\begin{aligned}
\frac{d}{dt} \int_M |h_1|^2 =& \int_M \langle 2L_{g_1} h_1, h_1 \rangle \, dV_{g_0} + \int_M A(h_1, g_0) * h_1 dV_{g_0} \\
\leq& -\frac{5}{4}\lambda \int_M \left| h_1 - \pi_0^j(h_1) \right|^2 dV_{g_0} + C\epsilon \int_M |h_1|^2 \, dV_{g_0} \\
\leq& \left(-\frac{3}{2}\lambda + C\epsilon\,(1 + \tau) \right) \int_M |h_1|^2 \\
\leq& -\frac{1}{2}\lambda \int_M |h_1|^2
\end{aligned}$$

as long as $\epsilon < \frac{\lambda}{2C(1+\tau)}$. Thus $\int_M |h_1(t)|^2 \, dV_{g_0} \leq e^{-\lambda(t-\tau)} \int_M |h_1(\tau)|^2 \, dV_{g_0}$ from which the claim follows immediately. □

We will need one more lemma, which roughly says that if a solution to the (21) is decaying at a certain rate at a particular time then it decayed at that rate earlier in time. This lemma is inspired by Lemma 5.31 in [2], and the proof is the same.

Lemma 32. *There exists a constant $\nu(n, \tau) > 0$ with the following property. Let k be a symmetric two-tensor satisfying the equation*

$$\frac{\partial}{\partial t} k = -Lk + A(g_0, k),$$

for $t \in [t_0, t_0 + \tau]$ and

$$\sup_{[t_0, t_0 + \tau]} |k|_{C^k} < \nu$$

and

$$\left|\pi_0(k)\right|_{C^k} \leq \alpha(t-t_0)\left|k\right|_{C^k}$$

where here we mean projection onto the kernel of $\left(\frac{\partial}{\partial t}+L\right)$ *restricted to the interval* $\left[\tau_0-\frac{\tau}{2},\tau\right]$. *Then if*

$$\sup_{\left[\tau_0+\frac{\tau}{2},\tau\right]}\int_M |k|^2\, dV_{g_0} \leq e^{-\frac{\tau\lambda}{2}}\sup_{\left[\tau_0,\tau_0+\frac{\tau}{2}\right]}\int_M |k|^2\, dV_{g_0} \tag{32}$$

then

$$\sup_{\left[\tau_0,\tau_0+\frac{\tau}{2}\right]}\int_M |k|^2\, dV_{g_0} \leq e^{-\frac{\tau\lambda}{2}}\sup_{\left[\tau_0-\frac{\tau}{2},\tau_0\right]}\int_M |k|^2\, dV_{g_0} \tag{33}$$

Proof. First note that the the analogous claim where k satisfies the linear equation $\left(\frac{\partial}{\partial t}+L\right)k=0$ is obvious since L is positive semidefinite and by definition $\pi_0(k(t_0))=0$. So, if the claim were false, then for a sequence $\nu_i \to 0$ we would have k_i satisfying the above properties with the bound $\left|k_i\right|_{C^k}<\nu_i$. By standard compactness arguments we can extract a subsequence converging to k_∞, which satisfies the initial decay hypothesis but not the conclusion. Moreover, given that A is quadratic in k, it is clear that this k_∞ satisfies the linear equation $\left(\frac{\partial}{\partial t}+L\right)k=0$, contradicting the above. □

We now proceed with the main proof. Recall that T is our maximal time of existence. Suppose $T<\infty$, and subdivide into N intervals of length τ labelled I_j as above. For fixed j, let g_j be the metric such that $\pi^j(g(t)-g_j)_0=0$ on I_j given by Lemma 30. Define $h_j:=g(t)-g_j$. By Lemma 31 and parabolic regularity we have that

$$\sup_{\left[(j+\frac{1}{2})\tau,(j+1)\tau\right]}\left|h_j\right|_{C^k} \leq Ce^{-\frac{\tau\lambda}{2}}\sup_{\left[j\tau,(j+\frac{1}{2})\tau\right]}\left|h_j\right|_{C^k} \tag{34}$$

And we can apply Lemma 32 inductively to conclude

$$\sup_{\left[j\tau,(j+1)\tau\right]}\left|h_j\right|_{C^k} \leq e^{-\lambda\tau(j-1)}\sup_{\left[0,\frac{\tau}{2}\right]}\left|h_j\right|_{C^k}$$

This allows us to conclude that on I_j we have

$$\begin{aligned}\left|\frac{\partial}{\partial t}g\right|_{C^{k-4}} &= \left|\frac{\partial}{\partial t}\left(g-g_j\right)\right|_{C^{k-4}}\\ &\leq C\sup_{I_j}\left|h_j\right|_{C^k}\\ &\leq C\epsilon e^{-\lambda\tau(j-1)} = C\epsilon p^{j-1}.\end{aligned}$$

Now note that simply integrating over time we see that

$$\sup_{I_j}\left|g-g_0\right|_{C^{k-4}} \leq 2\tau\sup_{I_j\cup I_{j-1}}\left|\frac{\partial}{\partial t}g\right|_{C^{k-4}}+\sup_{I_{j-1}}\left|g-g_0\right|_{C^{k-4}}.$$

Applying this estimate inductively and using parabolic regularity we see that

$$\begin{aligned}\sup_{I_j} |g - g_0|_{C^k} &\leq C \sup_{I_j} |g - g_0|_{C^{k-4}} \\ &\leq 2C\tau \sum_{k=1}^{N} \sup_{I_k \cup \cdots \cup I_N} \left| \frac{\partial}{\partial t} g \right|_{C^{k-4}} + \sup_{I_0} |g - g_0|_{C^{k-4}} \\ &\leq \sum_{k=1}^{\infty} \frac{2\tau C \epsilon}{p^{k-1}} + \sup_{I_0} |h_0|_{C^k} \\ &\leq \frac{2\tau C\epsilon}{p-1} + \sup_{I_0} |h_0|_{C^k}\end{aligned} \tag{35}$$

Now we want to choose our constants ϵ, ϵ' and τ to derive a contradiction from this equation. So, choose τ initially so large that

$$\frac{1}{Ce^{\tau\lambda}} + \frac{2C\tau}{e^{\tau\lambda} - 1} < \frac{1}{C} e^{-\frac{\tau\lambda}{4}} \tag{36}$$

where C is a constant depending only on g_0.

Now let $\epsilon = \min\{\delta(n,\tau), \nu(n,\tau), \frac{\lambda}{C_0}\}$ where $\delta(n,\tau)$ is as in Lemma 30, $\nu(n,\tau)$ is as in Lemma 32, and C_0 is a constant depending only on g_0 and the dimension which we now make explicit. By Lemma 28 we can bound the growth of L^2 derivatives of h, and then by Sobolev embeddings we can bound C^k norms. Specifically there exists a constant depending only on g_0 so that

$$|h(t)|_{C^k} < Ce^{C\epsilon\tau} |h(0)|_{C^k}$$

Then let $C_0 := 12C$. Note that if $\epsilon < \frac{\lambda}{C_0}$ and we start our flow with some $k(t_0)$ satisfying $|k(t_0)|_{C^k} < \frac{\epsilon}{C} e^{-\tau\lambda/4}$, the solution exists at least on $[t_0, t_0 + 3\tau)$ and moreover $\sup_{[t_0, t_0+3\tau)} |k(t)|_{C^k} < \epsilon$.

Now again using Lemma 28 we see that we may choose ϵ' so that the solution exists on $[0, 3\tau]$ and further

$$\sup_{[0,3\tau]} |h(t)|_{C^k} < \frac{\epsilon}{C} e^{-\tau\lambda}$$

Since $\epsilon < \min\{\delta(n,\tau), \nu(n,\tau)\}$ we can apply line (35) to get that

$$\begin{aligned}\sup_{I_j} |g - g_0|_{C^k} &\leq \epsilon \left(\frac{1}{Ce^{\tau\lambda}} + \frac{2C\tau}{e^{\tau\lambda} - 1} \right) \\ &\leq \frac{\epsilon}{C} e^{-\frac{\tau\lambda}{4}}.\end{aligned}$$

Thus by the above statement the solution may extended on an interval of length 3τ with $|h|_{C^k} < \epsilon$, contradicting the maximality of T. Thus the solution exists for all time and $|g(t) - g_0|_{C^k} < \epsilon$ for all time. Indeed we have decay

$$|g(t) - g_j|_{C^k} \leq Ce^{-\lambda t}$$

for all $t \in [0, j\tau)$ and for all j. Since $\{g_j\}$ is a sequence of flat metrics with uniform C^k bounds, we get a convergent subsequence $g_j \to g_\infty$ a flat metric, with exponential convergence $g(t) \to g_\infty$. This shows that equation (21) exists for all time and converges exponentially to g_∞. To show that (2) also exists for all time and converges, we simply note that since $\operatorname{grad}\mathcal{F}$ is diffeomorphism invariant if we let $g(t)$ now denote the solution to (2) and we let ϕ be the diffeomorphisms as noted at the start of the proof then

$$\begin{aligned}|\operatorname{grad}\mathcal{F}(g(t))|_{C^{k-4}} &= |\operatorname{grad}\mathcal{F}(\phi(t)^*(g(t)))|_{C^{k-4}} \\ &= |\operatorname{grad}\mathcal{F}(\phi^*(t)g(t)) - \operatorname{grad}\mathcal{F}(g_\infty)|_{C^{k-4}} \\ &\leq C\,|\phi^*(g(t)) - g_\infty|_{C^k} \\ &\leq Ce^{-t\lambda}.\end{aligned}$$

Given this decay one gets the existence of a metric $\bar{g}_\infty$ which $g(t)$ is converging to in any C^k norm. Since $\mathcal{F}$ is diffeomorphism invariant and going to zero for $\phi(t)^*g$, $\bar{g}_\infty$ is flat. This finishes the proof. ☐

7. Appendix: Sobolev Inequalities

Lemma 33. *Let (M^4, g) be a Riemannian manifold and suppose $\widetilde{g}$ is another metric on M satisfying*

$$\frac{1}{C}g \leq \widetilde{g} \leq Cg.$$

Then

$$C_S(\widetilde{g}) \leq C^4 C_S(g).$$

Proof. Fix a function $f \in C^1(M)$, and compute

$$\left(\int_M |f|^4 \, dV_{\widetilde{g}}\right)^{\frac{1}{2}} \leq C\left(\int_M |f|^4 \, dV\right).$$

Also,

$$\int_M |df|^2_{\widetilde{g}} \, dV_{\widetilde{g}} \geq \frac{1}{C^3}\int_M |df|^2_g \, dV_g$$

and

$$\operatorname{Vol}(\widetilde{g})^{-\frac{1}{2}} \int_M |f|^2 \, dV_{\widetilde{g}} \geq \frac{1}{C^3}\operatorname{Vol}(g)^{-\frac{1}{2}}\int_M |f|^2 \, dV.$$

Combining these yields

$$\begin{aligned}\frac{\left(\int_M |f|^4 \, dV_{\widetilde{g}}\right)^{\frac{1}{2}}}{\int_M |df|^2_{\widetilde{g}} + \operatorname{Vol}(\widetilde{g})^{-\frac{1}{2}}\int_M |f|^2 \, dV_{\widetilde{g}}} &\leq C^4 \frac{\left(\int_M |f|^4 \, dV_g\right)^{\frac{1}{2}}}{\int_M |df|^2_g + \operatorname{Vol}(g)^{-\frac{1}{2}}\int_M |f|^2 \, dV_g} \\ &\leq C^4 C_S(g)\end{aligned}$$

☐

Theorem 34. *Let (M^n, g) be a Riemannian manifold. For $u \in C^1_0(M)$, $n < p \leq \infty, 0 \leq m \leq \infty$ we have*

$$||u||_\infty \leq C_S \cdot C(n,m,p) ||u||_m^{1-\alpha} \left(||\nabla u||_p + ||u||_p \right)^\alpha \tag{37}$$

where $0 < \alpha \leq 1$ satisfies $\frac{1}{\alpha} = \left(\frac{1}{n} - \frac{1}{p} \right) m + 1$

Proof. See [7] for the argument on a surface. Their argument follows the more general argument in [10]. □

References

[1] Besse, A. *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), vol. 10, Springer-Verlag, Berlin, 1987.

[2] Cheeger, Jeff; Tian, G. *On the cone structure at infinity of Ricci flat manifolds with Euclidean volume growth and quadratic curvature decay*, Invent. Math. 118, 493-571 (1994).

[3] Chen, X.X. He, W.Y *On the Calabi flow*, arXiv:0603523

[4] Chen, X.X. He, W.Y *The Calabi flow on Kähler surfaces with bounded Sobolev constant*, arXiv:0710.5159.

[5] Gao, L. *Convergence of Riemannian manifolds; Ricci and $L^{\frac{n}{2}}$-curvature pinching*, J. Diff. Geom. 32 (1990) 349-381.

[6] Gunther, C. Isenberg, J. Knopf, D. *Stability of the Ricci flow at Ricci-flat metrics*, Comm. Anal. Geom. 10 (2002), no. 4, 741-777.

[7] Kuwert, E. Schätzle, R. *Gradient flow for the Willmore functional*, Comm. Anal. Geom. **10** (2002), no. 2, 307-339.

[8] Kuwert, E. Schätzle, R. *The Willmore flow with small initial energy*, J. Diff. Geom. 57 (2001), no. 3, 409-441.

[9] Kuwert E. Schätzle, R. *Removability of point singularities of Willmore surfaces*, Ann. of Math. (2) 160 (2004), no. 1, 315-357.

[10] Ladyzhenskaya, O. Uraltzeva, N. *On linear and quasi-linear equations and systems of elliptic and parabolic types*, Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963), Acad. Sci. USSR Siberian Branch, Moscow, 1963, 146-150.

[11] Sesum, N. *Linear and dynamical stability of Ricci-flat metrics*, Duke Math. J. 133 (2006), no. 1, 1-26.

[12] Simonett, G. *The Willmore flow near spheres*, Differential Integral Equations 14 (2001), no. 8, 1005-1014.

[13] Streets, J. *The gradient flow of $\int_M |\mathrm{Rm}|^2$*, J. Geom. Anal. 18 (2008), no. 1, 249-271.

[14] Streets, J. Tian, G. *Hermitian curvature flow*, arXiv:0804.4109

[15] Tian, G. unpublished.

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