

SOME REMARKS ON CONNECTION RICCI FLOW

JEFFREY STREETS

ABSTRACT. We study a generalization of Ricci flow to connections with torsion. This system of equations in fact arises naturally in the study of nonlinear sigma models [4]. The picture we give allows one to easily extend the proofs of derivative estimates and compactness of solutions to the case of a connection with torsion. We also examine gradient properties of this flow. Indeed it was shown in [4] that the monotonicity of Perelman's \mathcal{F} -functional extends to the case of a connection with torsion. We show that the expander entropy of Feldman, Ilmanen, and Ni [2] also extends to the connection Ricci flow.

1. INTRODUCTION

The recent advances in Ricci flow made by Perelman [5] have had an important impact in physics. In particular, the physical interpretation of the monotonicity of Perelman's \mathcal{F} functional is that the pure-gravity 1-loop renormalization group flow is a gradient flow. This result has important physical consequences, in particular ruling out nontrivial closed orbits, and was extended to the case where a B -field is included [4]. In particular, given a metric g and a closed three-form H the pertinent renormalization group flow is diffeomorphism equivalent to

$$(1.1) \quad \begin{aligned} \frac{\partial}{\partial t} g &= -2 \operatorname{Rc} + \frac{1}{2} \mathcal{H} \\ \frac{\partial}{\partial t} H &= \Delta_{LB} H \end{aligned}$$

where $\mathcal{H}_{ij} := g^{mn} g^{pq} H_{imp} H_{jnq}$ and Δ_{LB} is the usual Laplace-Beltrami operator on forms associated to $g(t)$. Since it plays no significant role we will leave the physics of B -fields out of our discussion. Please see [4] for the physical origin of these equations. We also would like to point out that the idea of interpreting the field strength of a B -field as the torsion of a Riemannian connection has been considered before, see for instance [6].

In section 2 we will discuss some basic properties of Riemannian connections with torsion, and use them to motivate the definition of connection Ricci flow. This perspective facilitates computation to allow an easy extension of the derivative estimates and compactness results to the case of connection Ricci flow which we carry out in section 3. In section 4 we show that the expander entropy of Feldman, Ilmanen, and Ni extends to connection Ricci flow.

2. GEOMETRY OF RIEMANNIAN CONNECTIONS WITH TORSION

In this section we will state and prove certain curvature formulas for Riemannian connections with torsion. The proofs are all simple modifications of the usual proofs for the Levi-Civita connection. Let (M, g) be a Riemannian manifold. Recall

that by the fundamental theorem of Riemannian geometry a metric-compatible connection ∇ is determined by its torsion tensor, τ .

Definition: 2.1. Given a Riemannian manifold (M, g) and a metric-compatible connection ∇ , the torsion of ∇ , denoted τ , is defined by

$$(2.1) \quad \tau(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Definition: 2.2. Given a Riemannian manifold (M, g) and a metric-compatible connection ∇ with torsion τ , the curvature endomorphism is

$$(2.2) \quad \text{Rm}(X, Y)Z := (\nabla^2 Z)(X, Y) - (\nabla^2 Z)(Y, X) + (\nabla Z)(\tau(X, Y))$$

where ∇^2 denotes the second total covariant derivative.

Lemma: 2.3. *The curvature endomorphism is a tensor of type (3, 1).*

Proof. The proof is elementary, and follows exactly as in the case of the usual curvature tensor, using (2.2). \square

Before continuing we need to make an important assumption. It turns out that the curvature tensor simplifies greatly if τ has a special form.

Definition: 2.4. A torsion tensor τ is *geometric* if $g_{kl}\tau_{ij}^l \in \wedge^3 T^*M$ and moreover $d\tau = 0$.

From now on we will always assume that we have a connection or family of connections with geometric torsion. The fact that ∇ has torsion means that all of the basic formulas one is used to from the curvature of the Levi-Civita connection are different. In particular, the Ricci tensor of ∇ , defined as the natural contraction on the first and last indices, is no longer symmetric.

Proposition: 2.5. *Given a Riemannian manifold (M, g) and connection ∇ with geometric torsion τ , let Rc^\wedge denote the skew-symmetric projection of the Ricci tensor of ∇ and likewise let Rc^\otimes denote the symmetric projection of the Ricci tensor of ∇ . Then we have*

$$(2.3) \quad \begin{aligned} \text{Rc}^\wedge &= -\frac{1}{2}d^*\tau \\ \text{Rc}^\otimes &= \text{Rc} - \frac{1}{4}\mathcal{H} \end{aligned}$$

where Rc denotes the Ricci tensor of g , and

$$(2.4) \quad \mathcal{H}_{jk} = g^{pq}g^{rs}\tau_{jpr}\tau_{kqs}$$

and

$$(2.5) \quad (d^*\tau)_{jk} = -{}^g\nabla^i\tau_{ijk}$$

where ${}^g\nabla$ is the Levi-Civita connection of g .

Proof. This is a straightforward calculation using the definition of the curvature tensor. \square

We now record three important properties of the curvature tensor associated to a connection with geometric torsion. These are generalizations of the usual Bianchi identities for the Levi-Civita connection. In all cases the condition that the torsion be geometric is essential. We omit the proofs, which are lengthy but straightforward calculations.

Proposition: 2.6. *Given a Riemannian manifold (M, g) and a metric-compatible connection ∇ with geometric torsion τ , we have*

$$(2.6) \quad \begin{aligned} R_{ijkl} + R_{iklj} + R_{iljk} &= \nabla_i \tau_{jkl} - \tau_{ij}^m \tau_{mkl} - \tau_{ik}^m \tau_{jml} - \tau_{il}^m \tau_{jkm} \\ &= {}^g \nabla_i \tau_{jkl} + \tau^{*2} \end{aligned}$$

Proposition: 2.7. *Given a Riemannian manifold (M, g) and a metric-compatible connection ∇ with geometric torsion τ , we have*

$$(2.7) \quad R_{ijkl} - R_{klij} = \frac{1}{2} (\nabla_i \tau_{jkl} - \nabla_j \tau_{ikl} - \nabla_k \tau_{lij} + \nabla_l \tau_{kij})$$

Proposition: 2.8. *Given a Riemannian manifold (M, g) and a metric-compatible connection ∇ with torsion τ , we have*

$$(2.8) \quad \nabla_m R_{ijk}^l + \nabla_i R_{jmk}^l + \nabla_j R_{mik}^l = -\tau_{mi}^p R_{pjk}^l - \tau_{mj}^p R_{ipk}^l - \tau_{ij}^p R_{pmk}^l$$

3. CONNECTION RICCI FLOW

Definition: 3.1. Given (M, g) a time-varying Riemannian manifold and a connection ∇ which is compatible with g with torsion τ , a family of connections

$$(3.1) \quad \nabla(t) = {}^g \nabla(t) + \tau(t)$$

is a solution to *connection Ricci flow* (CRF) with initial value g_0, τ_0 if $g(t)$ and $\tau(t)$ solve

$$(3.2) \quad \begin{aligned} \frac{\partial}{\partial t} g &= -2 \text{Rc}^\otimes \\ \frac{\partial}{\partial t} \tau &= 2d \text{Rc}^\wedge \\ g(0) &= g_0 \\ \tau(0) &= \tau_0 \end{aligned}$$

where d is the usual exterior derivative acting on the two-form Rc^\wedge . We note that using Proposition 2.5 this flow corresponds to the system of equations

$$(3.3) \quad \begin{aligned} \dot{g} &= -2 \text{Rc} + \frac{1}{2} \mathcal{H} \\ \dot{\tau} &= \Delta_{LB} \tau \end{aligned}$$

which is exactly the renormalization group flow considered in the introduction, with τ replacing the field strength H . Our definition for the evolution of τ may seem ad-hoc, but it clearly gives a very natural equation in the case of geometric torsion. We note in passing that in the physical picture one thinks of the fundamental field as $g + B$ where B is a locally defined two-form. From this perspective, it is natural to flow this tensor by the Ricci tensor of connection with torsion given by the field strength of B , which induces the equations of this definition. It may be interesting to make this rigorous.

Lemma: 3.2. *Given a solution $\nabla(t)$ to connection Ricci flow such that τ_0 is geometric, then $\tau(t)$ is geometric for all time. In fact, $[\tau(t)] = [\tau_0]$ for all time, where $[\tau]$ denotes the cohomology class of τ .*

Proof. Since $\dot{\tau} = 2d\text{Rc}^\wedge \in \wedge^3 T^*M$, it is clear that the property $\tau \in \wedge^3 T^*M$ is preserved by 3.2. We compute that

$$\begin{aligned} \frac{\partial}{\partial t} d\tau &= d\left(\frac{\partial}{\partial t}\tau\right) \\ &= 2d(d\text{Rc}^\wedge) \\ &= 0 \end{aligned}$$

Since $d^2 = 0$. Thus τ remains closed, and the cohomology class of τ is preserved. \square

Proposition: 3.3. *Given a Riemannian manifold (M, g) and a metric-compatible connection ∇ with geometric torsion, there is a unique solution to CRF with initial condition g, τ on $[0, \epsilon)$ for some $\epsilon > 0$.*

Proof. Recall from (2.3) that

$$(3.4) \quad \begin{aligned} \dot{\tau} &= \Delta_d \tau \\ &= \Delta \tau + \mathcal{R}(\tau) \end{aligned}$$

where in the last line we have applied the Bochner formula for a three form. Thus τ obeys a strictly parabolic system. It is clear that the evolution equation for g is just the usual Ricci flow modified by a term of lower order in g , in particular zeroth order. It is straightforward now to introduce a diffeomorphism parameter to adapt the usual proof of existence and uniqueness for Ricci flow, see for instance [1]. \square

We also record the basic evolution equations for the curvature tensor of ∇ . The geometric assumption on τ is essential in this computation, and the Bianchi identities of the previous section play a key role.

Proposition: 3.4. *A geometric solution to CRF satisfies*

$$(3.5) \quad \frac{\partial}{\partial t} \text{Rm} = \Delta \text{Rm} + \tau * \nabla \text{Rm} + \text{Rm}^{*2}$$

Proposition: 3.5. *A geometric solution to CRF satisfies*

$$(3.6) \quad \begin{aligned} \frac{\partial}{\partial t} \nabla^k \text{Rm} &= \Delta \nabla^k \text{Rm} + \tau * \nabla^{k+1} \text{Rm} \\ &+ \sum_{j=1}^k \nabla^j \tau * \nabla^{k+1-j} \text{Rm} + \sum_{j=0}^k \nabla^j \text{Rm} * \nabla^{k-j} \text{Rm} \\ &+ \sum_{j=0}^{k-1} \sum_{l=0}^j \nabla^l \tau * \nabla^{j-l} \text{Rm} * \nabla^{k-1-j} \text{Rm} \end{aligned}$$

4. DERIVATIVE ESTIMATES

In this section we will prove derivative estimates for geometric solutions of CRF. These estimates are generalizations of the Bernstein-Shi estimates for Ricci flow. Our proof follows that in [1].

Theorem: 4.1. *Let $(M, g(t), \tau(t))$ be a solution to CRF for which the maximum principle holds. Then for each $\alpha > 0$ and every $m \in \mathbb{N}$ there exists a constant C_m*

depending only on m, n and $\max\{\alpha, 1\}$ such that if

$$(4.1) \quad \begin{aligned} |\mathrm{Rm}|_{C^0(M_t)} &\leq K, \\ |\tau|_{C^0(M_t)}^2 &\leq K \end{aligned}$$

for all $x \in M$ and $t \in [0, \frac{\alpha}{K}]$, then

$$(4.2) \quad |\nabla^m \mathrm{Rm}|_{C^0(M_t)} \leq \frac{C_m K}{t^{m/2}}$$

for all $x \in M$ and $t \in (0, \frac{\alpha}{K}]$.

Proof. Our proof is by induction on m . First consider $m = 1$. We have the following evolution equation for $|\mathrm{Rm}|^2$:

$$(4.3) \quad \frac{\partial}{\partial t} |\mathrm{Rm}|^2 = \Delta |\mathrm{Rm}|^2 - 2 |\nabla \mathrm{Rm}|^2 + \tau * \nabla \mathrm{Rm} * \mathrm{Rm} + \mathrm{Rm}^{*3}.$$

Also, an immediate corollary of Proposition 3.5 is the evolution equation

$$(4.4) \quad \begin{aligned} \frac{\partial}{\partial t} |\nabla \mathrm{Rm}|^2 &= \Delta |\nabla \mathrm{Rm}|^2 - 2 |\nabla^2 \mathrm{Rm}|^2 + \mathrm{Rm} * \nabla \mathrm{Rm}^{*2} \\ &\quad + \tau * \nabla^2 \mathrm{Rm} * \nabla \mathrm{Rm} + \tau * \mathrm{Rm}^{*2} * \nabla \mathrm{Rm}. \end{aligned}$$

Now, we aim to use the term $-2 |\nabla \mathrm{Rm}|^2$ in the evolution of $|\mathrm{Rm}|^2$ to control the evolution of $|\nabla \mathrm{Rm}|^2$. Consider the function

$$F(x, t) := t |\nabla \mathrm{Rm}|^2 + \beta |\mathrm{Rm}|^2$$

where β is a constant to be chosen below. Putting together (4.3) and (4.4) gives

$$(4.5) \quad \begin{aligned} \frac{\partial}{\partial t} F &\leq \Delta F - 2t |\nabla^2 \mathrm{Rm}|^2 + (1 + c_1 t |\mathrm{Rm}| - 2\beta) |\nabla \mathrm{Rm}|^2 \\ &\quad + t\tau * \nabla^2 \mathrm{Rm} * \nabla \mathrm{Rm} + t\tau * \mathrm{Rm}^{*2} * \nabla \mathrm{Rm} \\ &\quad + c_2 \beta |\mathrm{Rm}|^3 + \tau * \nabla \mathrm{Rm} * \mathrm{Rm} \end{aligned}$$

where all the c_i are universal constants depending only on dimension. We must estimate the different terms in 4.5. First of all we use the Cauchy-Schwarz inequality and the assumption (4.1) to conclude

$$(4.6) \quad \begin{aligned} t\tau * \nabla^2 \mathrm{Rm} * \nabla \mathrm{Rm} &\leq tc_3 (|\nabla^2 \mathrm{Rm}|) (|\tau| |\nabla \mathrm{Rm}|) \\ &\leq tc_3 \left(\frac{|\nabla^2 \mathrm{Rm}|^2}{2c_3} + \frac{c_3 |\tau|^2 |\nabla \mathrm{Rm}|^2}{2} \right) \\ &\leq \frac{t}{2} |\nabla^2 \mathrm{Rm}|^2 + c_4 t K |\nabla \mathrm{Rm}|^2. \end{aligned}$$

Similarly we simplify

$$(4.7) \quad \begin{aligned} t\tau * \mathrm{Rm}^{*2} * \nabla \mathrm{Rm} &\leq tc_5 (|\mathrm{Rm}|^2) (|\tau| |\nabla \mathrm{Rm}|) \\ &\leq c_6 t K^4 + c_6 t K |\nabla \mathrm{Rm}|^2 \end{aligned}$$

and finally

$$(4.8) \quad \tau * \mathrm{Rm} * \nabla \mathrm{Rm} \leq \frac{c_7}{\beta} K^3 + \beta |\nabla \mathrm{Rm}|^2.$$

So, plugging (4.6)-(4.8) into (4.5) gives that for $t \in [0, \frac{\alpha}{K}]$,

$$\frac{\partial}{\partial t} F \leq \Delta F + (1 + c_8 \alpha - \beta) |\nabla \text{Rm}|^2 + c_9 (\alpha + \beta) K^3.$$

Choose $\beta \geq \frac{1+c_8\alpha}{2}$ and note that β depends only on the dimension and $\max\{\alpha, 1\}$. Then we have that for $t \in [0, \frac{\alpha}{K}]$,

$$\frac{\partial}{\partial t} F \leq c_{10} \beta K^3$$

Using that $F(0) \leq \beta K^2$ and applying the maximum principle gives

$$\sup_{x \in M} F(x, t) \leq \beta K^2 + c_{10} \beta K^3 t \leq (1 + c_{10} \alpha) \beta K^2 \leq C_1^2 K^2$$

where again C_1 depends only on n and $\max\{\alpha, 1\}$. Thus

$$|\nabla \text{Rm}| \leq \sqrt{\frac{F}{t}} \leq \frac{C_1 K}{t^{1/2}}$$

for all $x \in M$ and $t \in (0, \frac{\alpha}{K}]$. This completes the case $m = 1$. For the induction step we first conclude from Proposition 3.5 the evolution equation

$$\begin{aligned} \frac{\partial}{\partial t} |\nabla^k \text{Rm}|^2 &= \Delta |\nabla^k \text{Rm}|^2 - 2 |\nabla^{k+1} \text{Rm}|^2 + \tau * \nabla^{k+1} \text{Rm} * \nabla^k \text{Rm} \\ &\quad + \sum_{j=1}^k \nabla^j \tau * \nabla^{k+1-j} \text{Rm} * \nabla^k \text{Rm} \\ &\quad + \sum_{j=0}^k \nabla^j \text{Rm} * \nabla^{k-j} \text{Rm} * \nabla^k \text{Rm} \\ &\quad + \sum_{j=0}^{k-1} \sum_{l=0}^j \nabla^l \tau * \nabla^{j-l} \text{Rm} * \nabla^{k-1-j} \text{Rm} * \nabla^k \text{Rm} \end{aligned}$$

We first make the bound

$$\begin{aligned} \tau * \nabla^{k+1} \text{Rm} * \nabla^k \text{Rm} &\leq c |\tau| |\nabla^{k+1} \text{Rm}| |\nabla^k \text{Rm}| \\ (4.9) \quad &\leq \epsilon |\nabla^{k+1} \text{Rm}|^2 + C(\epsilon) |\tau|^2 |\nabla^k \text{Rm}|^2 \\ &\leq \epsilon |\nabla^{k+1} \text{Rm}|^2 + C(\epsilon) K |\nabla^k \text{Rm}|^2. \end{aligned}$$

Similarly using the inductive hypothesis we can bound for $j > 0$

$$\begin{aligned} \nabla^j \tau * \nabla^{k+1-j} \text{Rm} * \nabla^k \text{Rm} &\leq c |\nabla^j \tau| |\nabla^{k+1-j} \text{Rm}| |\nabla^k \text{Rm}| \\ (4.10) \quad &\leq c |\nabla^{j-1} \text{Rm}| |\nabla^{k+1-j} \text{Rm}| |\nabla^k \text{Rm}| \\ &\leq c \frac{K}{t^{(j-1)/2}} \frac{K}{t^{(k+1-j)/2}} |\nabla^k \text{Rm}| \\ &\leq c \frac{K^2}{t^{k/2}} |\nabla^k \text{Rm}| \\ &\leq c |K| |\nabla^k \text{Rm}|^2 + c \frac{K^3}{t^k} \end{aligned}$$

A similar calculation yields a bound

$$\begin{aligned}
(4.11) \quad \nabla^j \text{Rm} * \nabla^{k-j} \text{Rm} * \nabla^k \text{Rm} &\leq c |\nabla^j \text{Rm}| |\nabla^{k-j} \text{Rm}| |\nabla^k \text{Rm}| \\
&\leq c \frac{K}{t^{j/2}} \frac{K}{t^{(k-j)/2}} |\nabla^k \text{Rm}| \\
&\leq cK |\nabla^k \text{Rm}|^2 + c \frac{K^3}{t^k}.
\end{aligned}$$

Finally, we bound the last terms using the inequality $K \leq \frac{C}{t}$

$$\begin{aligned}
(4.12) \quad \nabla^l \tau * \nabla^{j-l} \text{Rm} * \nabla^{k-1-j} \text{Rm} * \nabla^k \text{Rm} &\leq c |\nabla^l \tau| |\nabla^{j-l} \text{Rm}| |\nabla^{k-1-j} \text{Rm}| |\nabla^k \text{Rm}| \\
&\leq c |\nabla^{l-1} \text{Rm}| |\nabla^{j-l} \text{Rm}| |\nabla^{k-1-j} \text{Rm}| |\nabla^k \text{Rm}| \\
&\leq c \frac{K}{t^{(l-1)/2}} \frac{K}{t^{(j-l)/2}} \frac{K}{t^{(k-1-j)/2}} |\nabla^k \text{Rm}| \\
&\leq c \frac{K^3}{t^{(k-2)/2}} |\nabla^k \text{Rm}| \\
&\leq c \frac{K^2}{t^{k/2}} |\nabla^k \text{Rm}| \\
&\leq cK |\nabla^k \text{Rm}|^2 + c \frac{K^3}{t^k}.
\end{aligned}$$

Using (4.9) - (4.12) we conclude

$$(4.13) \quad \frac{\partial}{\partial t} |\nabla^k \text{Rm}|^2 \leq \Delta |\nabla^k \text{Rm}|^2 - |\nabla^{k+1} \text{Rm}|^2 + CK \left(|\nabla^k \text{Rm}|^2 + \frac{K^2}{t^k} \right)$$

This bound is sufficient to carry out the inductive step analogously to the step $k = 1$. The details of this construction are found in [1] page 229-230. \square

5. COMPACTNESS RESULT FOR CONNECTION RICCI FLOW

In this section we will prove a compactness result for solutions to connection Ricci flow which generalizes the corresponding result for Ricci flow. Our proof is an easy modification of the case of regular Ricci flow. We use our Bernstein-Shi type estimates from theorem 4.1 to reduce to the case where one has a uniform bound on curvatures and their derivatives, and then apply a convergence result from [3]. We start with a definition which generalizes the notion of convergence of a sequence of evolving Riemannian manifolds with metric-compatible connection.

Definition: 5.1. We say that a sequence $M_i = \{M_i, g_i(t), \tau(t), p_i, F_i\}$ of evolving complete marked Riemannian manifolds converges to the evolving complete marked Riemannian manifold $M = \{M, g(t), \tau(t), p, F\}$ if there exists a sequence of open sets U_i in M containing p and a sequence of diffeomorphisms ϕ_i mapping U_i to $V_i \subset M_i$ and mapping p to p_i and F to F_i such that any compact set in M eventually lies in all U_i and the pullbacks $\phi_i^*(g_i)$ and $\phi_i^*(\tau_i)$ converge to g and τ uniformly on compact sets in $M \times (\alpha, \omega)$ together with all of their derivatives.

Theorem: 5.2. Compactness Theorem *Let*

$$\{M_i, g_i(t), \tau(t), p_i, F_i : i \in \mathbb{N}\}$$

be a sequence of complete geometric solutions to CRF existing for $t \in (\alpha, \omega)$ where $-\infty \leq \alpha < 0 < \omega \leq \infty$. Each solution has a fixed origin $p_i \in M_i$ and an frame F_i

at p_i which is orthonormal with respect to $g_i(0)$. Suppose there exists $K < \infty$ such that

$$\sup_{M_i \times (\alpha, \omega)} |\text{Rm}| \leq K$$

and

$$\sup_{M_i \times (\alpha, \omega)} |\tau|^2 \leq K$$

and $\delta > 0$ such that:

$$\text{inj}_{g_i(0)}(M_i) \geq \delta \text{ for all } i \in \mathbb{N}$$

Then there exists a subsequence which converges in the pointed category to a complete solution

$$\{M_\infty, g_\infty(t), \tau_\infty(t), p_\infty, F_\infty\}$$

of CRF on (α, ω) with the same bounds on curvature and injectivity radius.

Our proof will make use of the following fundamental theorem in [3].

Theorem: 5.3. *Given any sequence $M_i = (M_i, g_i, p_i, F_i)$ of complete Riemannian manifolds as above such that there exist constants $C_m, m \geq 0$ independent of i so that*

$$|g_i \nabla^m \text{rm}(g_i)| \leq C_m$$

and $\delta > 0$ independent of i so that

$$\text{inj}_{g_i(t)}(p_i) \geq \delta$$

there exists a convergent subsequence.

Corollary: 5.4. *Let $M_i = (M_i, g_i, \tau_i, p_i, F_i)$ be a sequence of complete marked Riemannian manifolds with connections ∇_i with geometric torsion tensors τ_i . Suppose that for all $m \geq 0$ there exists C_m independent of i so that*

$$(5.1) \quad |\nabla^m \text{Rm}_i| \leq C_m$$

and there exists C independent of i so that

$$(5.2) \quad |\tau_i|^2 \leq C$$

and $\delta > 0$ independent of i so that

$$(5.3) \quad \text{inj}_{g_i(t)}(p_i) \geq \delta$$

then there exists a convergent subsequence.

Proof. Using Proposition 2.6 it is clear that the bound on τ and the bound on Rm then gives a bound on $\nabla\tau$. One can differentiate the expression in Proposition 2.6 to then inductively get uniform bounds on all of the higher derivatives of τ in terms of the bounds on $\nabla^m \text{Rm}$. Given this and the formula for the total curvature tensor, it is clear that one now has uniform bounds on the curvature of g_i and all of its derivatives, given in terms of the assumed uniform bounds on $\nabla^m \text{Rm}$.

Now we can apply Theorem 5.3 to the sequence of underlying Riemannian manifolds to extract a subsequence where the Riemannian data converges. Using our uniform bounds on τ and its derivatives we can extract a further subsequence where τ_i and its derivatives converge. \square

Proof of Theorem 5.2: First of all a diagonalization argument allows us to assume $\alpha > -\infty$ and $\omega < \infty$. So, for fixed $\epsilon > 0$, using our assumption of uniformly bounded curvature we have uniform C^k estimates on curvature for all k on $(\alpha + \epsilon, \omega)$. So if we can prove the theorem for sequences with uniform C^k bounds on curvature then we can do another diagonalization as $\epsilon \rightarrow 0$ to conclude the theorem. Thus we make this assumption.

So, fix a sequence (M_i, g_i, p_i, F_i) of complete marked solutions to CRF. Theorem 5.3 shows that the slices at time 0 contain a subsequence which converges to a metric g on a manifold (M, p) . Recall that this convergence means that there exist a sequence of open sets $\{U_i\}$ in M containing p and a sequence of maps $F_i : U_i \rightarrow V_i \subset M_i$ such that $\tilde{g}_i := F_i^*(g_i)$ converges uniformly in C^∞ to g . Note that these pullbacks \tilde{g}_i still have uniform bounds on all derivatives of curvatures and are defined on (α, ω) while g is not yet. We now want to show that we have uniform bounds on the covariant derivatives of \tilde{g}_i taken with respect to the metric g . We sketch the proof here for solutions to CRF, as it is identical to the proof of lemma 2.4 in [3].

Lemma: 5.5. *Let (M, g) be a Riemannian manifold, K a compact subset of M and g_i a collection of solutions to CRF defined on neighborhoods of $K \times [a, b]$ where $0 \in [a, b]$. Let ∇ denote the covariant derivative with respect to g and ∇_i denote the covariant derivative with respect to g_i .*

Suppose that the metrics g_i are uniformly equivalent to g at $t = 0$ on K and that the covariant derivatives of g_i with respect to g are uniformly bounded at $t = 0$ on K . Finally assume that the covariant derivatives of the curvatures of Rm_i with respect to g_i are uniformly bounded on $K \times [a, b]$. Then the metrics g_i are uniformly equivalent to g on $K \times [a, b]$, and the covariant derivatives of g_i with respect to g are uniformly bounded on $K \times [a, b]$.

Proof. We have assumed the bound

$$cg(V, V) \leq g_i(V, V) \leq Cg(V, V)$$

at time $t = 0$. Using the equation

$$\frac{\partial}{\partial t} g_i(V, V) = -2 \text{Rc}^\otimes$$

and the bound

$$|\text{Rc}^\otimes(V, V)| \leq C |g_i(V, V)|$$

we get

$$\left| \frac{\partial}{\partial t} \ln g_i(V, V) \right| \leq C$$

which allows us to extend the bound on $\ln g_i(V, V)$ at $t = 0$ to the finite time interval $[a, b]$. The derivative bounds are the same, simply bounding the time derivative using our assumption of uniform bounds on all derivatives of curvature. See [3] pg. 550 for more detail. \square

Applying this lemma to the sequence of metrics \tilde{g}_i above it is clear that the metric g is defined on $[a, b]$ and is a solution to CRF. \square

6. EXPANDING ENTROPY

Let R denote the scalar curvature of the underlying Riemannian metric. Consider first the following Proposition.

Proposition: 6.1. ([4] Proposition 3.1, Corollary 3.2) *CRF is the gradient flow of the lowest eigenvalue of the Schrödinger operator $-4\Delta + R - \frac{1}{12}|\tau|^2$.*

This is a generalization of the corresponding fact for Ricci flow when $\tau = 0$. It is worth noting that, due essentially to the lack of scale-invariance of the evolution of g , the obvious generalization of the Perelman shrinker entropy does not generalize to the case of nonzero torsion. However, the expanding entropy of Feldman, Ilmanen, and Ni introduces an extra term with the “right” sign and so extends to the case of nonzero torsion. In particular define

$$(6.1) \quad \begin{aligned} \mathcal{W}_+(g, \tau, u, \sigma) &:= \int_M \left[\sigma \left(\frac{|\nabla u|^2}{u} + Ru - \frac{1}{12}|\tau|^2 u \right) + u \log u \right] dV \\ &= \int_M \left[\sigma \left(|\nabla f_+|^2 + R - \frac{1}{12}|\tau|^2 \right) - f_+ + n \right] u dV \end{aligned}$$

restricted to functions u satisfying

$$\int_M u dV = 1$$

where f_+ is defined implicitly by

$$u = \frac{e^{-f_+}}{(4\pi\sigma)^{\frac{n}{2}}}.$$

Suppose u satisfies the conjugate heat equation

$$(6.2) \quad \frac{\partial u}{\partial t} = -\Delta u + Ru - \frac{1}{4}|\tau|^2 u.$$

It follows then that

$$\frac{d}{dt} \int_M u dV = 0$$

and moreover f_+ satisfies

$$(6.3) \quad \frac{\partial f_+}{\partial t} = -\Delta f_+ + |\nabla f_+|^2 - R + \frac{1}{4}|\tau|^2 - \frac{n}{2(t-T)}.$$

Theorem: 6.2. *Given (g, τ) a solution to CRF and u a solution to 6.2 define*

$$v_+ := \left[(t-T) \left(2\Delta f_+ - |\nabla f_+|^2 + R - \frac{1}{12}|\tau|^2 \right) - f_+ + n \right] u dV.$$

Then

$$(6.4) \quad \begin{aligned} &\left(\frac{\partial}{\partial t} + \Delta - R + \frac{1}{4}|\tau|^2 \right) v_+ \\ &= 2(t-T) \left(\left| \text{Rc} - \frac{1}{4}\mathcal{H} + \nabla^2 f_+ + \frac{g}{2t} \right|^2 + \frac{1}{4}|d^*\tau - \nabla f_+ \lrcorner \tau|^2 \right) u \\ &\quad + \frac{1}{6}|\tau|^2 u. \end{aligned}$$

Proof. We assume without loss of generality that $T = 0$. Let $V := 2\Delta f_+ - |\nabla f_+|^2 + R - \frac{1}{12}|\tau|^2$. We will compute $(\frac{\partial}{\partial t} + \Delta)V$ in stages. First of all, using Lemma 8.1 and (6.3) we compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) 2\Delta f_+ &= 2 \left[\left\langle 2\text{Rc} - \frac{1}{2}\mathcal{H}, \nabla^2 f_+ \right\rangle \right. \\ &\quad \left. + \Delta \left(-\Delta f_+ + |\nabla f_+|^2 - R + \frac{1}{4}|\tau|^2 - \frac{n}{2t} \right) \right. \\ &\quad \left. - \left(\frac{1}{2}\nabla^i \mathcal{H}_{ij} - \frac{1}{4}\nabla^j |\tau|^2 \right) \nabla^j f_+ + \Delta \Delta f_+ \right] \\ &= \langle 4\text{Rc} - \mathcal{H}, \nabla^2 f_+ \rangle + 2\Delta |\nabla f_+|^2 - 2\Delta R \\ &\quad + \frac{1}{2}\Delta |\tau|^2 + \left\langle \frac{1}{2}\nabla |\tau|^2 - \text{div } \mathcal{H}, \nabla f_+ \right\rangle. \end{aligned}$$

Also we can compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) (-|\nabla f_+|^2) &= - \left[2 \left\langle \nabla \left(-\Delta f_+ + |\nabla f_+|^2 - R + \frac{1}{4}|\tau|^2 - \frac{n}{2t} \right), \nabla f_+ \right\rangle \right. \\ &\quad \left. + \left\langle 2\text{Rc} - \frac{1}{2}\mathcal{H}, \nabla f_+ \otimes \nabla f_+ \right\rangle + \Delta |\nabla f_+|^2 \right] \\ &= 2 \left\langle \nabla \left(\Delta f_+ - |\nabla f_+|^2 + R - \frac{1}{4}|\tau|^2 \right), \nabla f_+ \right\rangle \\ &\quad + \left\langle \frac{1}{2}\mathcal{H} - 2\text{Rc}, \nabla f_+ \otimes \nabla f_+ \right\rangle - \Delta |\nabla f_+|^2. \end{aligned}$$

Now using Lemma 8.2 we compute

$$\left(\frac{\partial}{\partial t} + \Delta\right) R = 2\Delta R + 2|\text{Rc}|^2 - \frac{1}{2}\Delta |\tau|^2 + \frac{1}{2}\text{div div } \mathcal{H} - \frac{1}{2}\langle \text{Rc}, \mathcal{H} \rangle.$$

Then using Lemma 8.3 we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) \left(-\frac{1}{12}|\tau|^2\right) &= -\frac{1}{12} \left[\left\langle 6\text{Rc} - \frac{3}{2}\mathcal{H}, \mathcal{H} \right\rangle + 2 \left\langle \frac{\partial}{\partial t} \tau, \tau \right\rangle + \Delta |\tau|^2 \right] \\ &= \left\langle \frac{1}{8}\mathcal{H} - \frac{1}{2}\text{Rc}, \mathcal{H} \right\rangle - \frac{1}{6} \langle \Delta_{LB} \tau, \tau \rangle - \frac{1}{12} \Delta |\tau|^2. \end{aligned}$$

Combining the above four calculations gives

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta\right) V &= \Delta |\nabla f_+|^2 - \frac{1}{12}\Delta |\tau|^2 - \frac{1}{6} \langle \Delta_{LB} \tau, \tau \rangle + \frac{1}{2} \text{div div } \mathcal{H} \\ &\quad + \left\langle \frac{1}{2}\nabla |\tau|^2 - \text{div } \mathcal{H}, \nabla f_+ \right\rangle + \left\langle \frac{1}{2}\mathcal{H} - 2\text{Rc}, \nabla f_+ \otimes \nabla f_+ \right\rangle \\ (6.5) \quad &\quad + 2 \left| \text{Rc} - \frac{1}{4}\mathcal{H} + \nabla^2 f_+ \right|^2 - 2|\nabla^2 f_+|^2 \\ &\quad + 2 \left\langle \nabla \left(\Delta f_+ - |\nabla f_+|^2 + R - \frac{1}{4}|\tau|^2 \right), \nabla f_+ \right\rangle. \end{aligned}$$

Now using Lemma 8.4 we get

$$(6.6) \quad \frac{1}{2} \text{div div } \mathcal{H} - \frac{1}{6} \langle \Delta_{LB} \tau, \tau \rangle - \frac{1}{12} \Delta |\tau|^2 = \frac{1}{2} |d^* \tau|^2$$

and

$$(6.7) \quad \left\langle \frac{1}{2} \nabla |\tau|^2 - \operatorname{div} \mathcal{H}, \nabla f_+ \right\rangle = \frac{1}{3} \left\langle \nabla |\tau|^2, \nabla f_+ \right\rangle - \langle d^* \tau, \nabla f_+ \lrcorner \tau \rangle.$$

Also, a calculation shows that

$$(6.8) \quad \Delta |\nabla f_+|^2 - 2 |\nabla^2 f_+|^2 - 2 \langle \operatorname{Rc}, \nabla f_+ \otimes \nabla f_+ \rangle = 2 \langle \nabla \Delta f_+, \nabla f_+ \rangle.$$

Thus plugging (6.6) - (6.8) into (6.5) gives

$$(6.9) \quad \left(\frac{\partial}{\partial t} + \Delta \right) V = 2 \left| \operatorname{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ \right|^2 + \frac{1}{2} |d^* \tau - \nabla f_+ \lrcorner \tau|^2 + 2 \langle \nabla V, \nabla f_+ \rangle.$$

Now let $W := tV - f_+ + n$. Using (6.9) we get

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta \right) W &= V + 2t \left| \operatorname{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ \right|^2 + \frac{t}{2} |d^* \tau - \nabla f_+ \lrcorner \tau|^2 \\ &\quad + 2t \langle \nabla V, \nabla f_+ \rangle - |\nabla f_+|^2 + R - \frac{1}{4} |\tau|^2 - \frac{n}{2t} \\ &= 2\Delta f_+ + 2R - \frac{1}{3} |\tau|^2 + \frac{n}{2t} + 2 \langle \nabla W, \nabla f_+ \rangle \\ &\quad + 2t \left| \operatorname{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ \right|^2 + \frac{t}{2} |d^* \tau - \nabla f_+ \lrcorner \tau|^2. \\ &= 2t \left| \operatorname{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ + \frac{g}{2t} \right|^2 + \frac{t}{2} |d^* \tau - \nabla f_+ \lrcorner \tau|^2 \\ &\quad + \frac{1}{6} |\tau|^2 + 2 \langle \nabla W, \nabla f_+ \rangle. \end{aligned}$$

Finally we can compute

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \Delta \right) v_+ &= \left(\frac{\partial}{\partial t} + \Delta \right) (Wu) \\ &= \left(\left(\frac{\partial}{\partial t} + \Delta \right) W \right) u + W \left(\frac{\partial}{\partial t} + \Delta \right) u + 2 \langle \nabla W, \nabla u \rangle \\ &= \left(2t \left| \operatorname{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ + \frac{g}{2t} \right|^2 + \frac{t}{2} |d^* \tau - \nabla f_+ \lrcorner \tau|^2 \right. \\ &\quad \left. + \frac{1}{6} |\tau|^2 + 2 \langle \nabla W, \nabla f_+ \rangle \right) u + W \left(Ru - \frac{1}{4} |\tau|^2 u \right) - 2 \langle \nabla W, \nabla f_+ \rangle u \\ &= 2t \left| \operatorname{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ + \frac{g}{2t} \right|^2 u + \frac{t}{2} |d^* \tau - \nabla f_+ \lrcorner \tau|^2 u + \frac{1}{6} |\tau|^2 u \\ &\quad + Rv_+ - \frac{1}{4} |\tau|^2 v_+ \end{aligned}$$

and the result follows. \square

Corollary: 6.3. *Let $(g(t), \tau(t))$ be a solution to CRF and $u(t)$ a solution to (6.2). Then*

$$(6.10) \quad \frac{\partial}{\partial t} \mathcal{W}_+(g(t), \tau(t), u(t), t-T) = \int_M 2u \left[(t-T) \left| \text{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ + \frac{g}{2(t-T)} \right|^2 + \frac{1}{4} |d^* \tau - \nabla f_+ \lrcorner \tau|^2 + \frac{1}{12} |\tau|^2 \right] dV$$

We can now define certain quantities related to \mathcal{W}_+ in direct analogy with [5], [2]. In particular let

$$(6.11) \quad \mu_+(g, \tau, \sigma) := \inf_{u|_{\int_M u dV \equiv 1}} \mathcal{W}_+(g, \tau, u, \sigma)$$

and

$$(6.12) \quad \nu_+(g, \tau) := \sup_{\sigma > 0} \mu_+(g, \tau, \sigma)$$

Proposition: 6.4. *Let $(M^m, g(t), \tau(t))$ be a solution to connection Ricci flow on a compact manifold. If $\lambda(g, \tau) < 0$ then the supremum defining ν_+ is attained by a unique σ . Moreover, $\nu_+(g(t), \tau(t))$ is monotonically nondecreasing and is constant only on a Ricci expander with $\tau \equiv 0$.*

Proof. This proof is identical to that of Theorem 1.7(b) of [2]. In particular, the formula

$$\frac{d}{dt} \nu_+(g(t), \tau(t)) = \int_M 2u \left[\sigma \left| \text{Rc} - \frac{1}{4} \mathcal{H} + \nabla^2 f_+ + \frac{g}{2\sigma} \right|^2 + \frac{1}{4} \sigma |d^* \tau - \nabla f_+ \lrcorner \tau|^2 + \frac{1}{12} |\tau|^2 \right] dV$$

holds. Thus if $\nu_+(g(t), \tau(t))$ is constant then $\tau \equiv 0$ and $g(t)$ is a Ricci expander. \square

7. CONCLUSIONS

The definition we have given for connection Ricci flow certainly makes sense for a connection with non-geometric torsion. It may be interesting to find a weaker condition that still gives a parabolic equation.

Also, the obvious generalization of the Perelman shrinker entropy is not monotonic for a solution to connection Ricci flow. An important open problem is to find such a generalization, or to find a way to prove noncollapsing for solutions to connection Ricci flow.

8. APPENDIX: VARIATION FORMULAE

Lemma: 8.1. *Given $g(t)$ a family of Riemannian metrics such that $\frac{\partial}{\partial t} g = h$ and a family of smooth functions $f(x, t)$ then*

$$\begin{aligned} \frac{\partial}{\partial t} \Delta_{g(t)} f &= -\langle h, \nabla^2 f \rangle + \Delta \left(\frac{\partial}{\partial t} f \right) \\ &\quad - \frac{1}{2} g^{ij} (\nabla_i h_{jk} + \nabla_j h_{ik} - \nabla_k h_{ij}) \nabla^k f \end{aligned}$$

and

$$\frac{\partial}{\partial t} |\nabla f|^2 = -\langle h, \nabla f \otimes \nabla f \rangle + 2 \left\langle \nabla \left(\frac{\partial}{\partial t} f \right), \nabla f \right\rangle$$

Lemma: 8.2. *Given $g(t)$ a family of Riemannian metrics such that $\frac{\partial}{\partial t} g = h$ then*

$$\frac{\partial}{\partial t} R = -\Delta \operatorname{tr} h + \operatorname{div} \operatorname{div} h - \langle \operatorname{Rc}, h \rangle$$

Lemma: 8.3. *Given $g(t)$ a family of Riemannian metrics such that $\frac{\partial}{\partial t} g = h$ and $\tau(x, t)$ a family of three-forms then*

$$\frac{\partial}{\partial t} |\tau|^2 = -3 \langle h, \mathcal{H} \rangle + 2 \left\langle \frac{\partial}{\partial t} \tau, \tau \right\rangle$$

Lemma: 8.4. *Given (M^n, g) a Riemannian manifold and τ a closed three-form on M , then*

$$(\operatorname{div} \mathcal{H})_i = \frac{1}{6} \nabla_i |\tau|^2 + (d^* \tau)^{mn} \tau_{imn}$$

and

$$\operatorname{div} \operatorname{div} \mathcal{H} = \frac{1}{3} \langle \Delta_{LB} \tau, \tau \rangle + \frac{1}{6} \Delta |\tau|^2 + |d^* \tau|^2$$

Proof. We begin with the proof of the first statement. We directly compute using that τ is closed

$$\begin{aligned} (\operatorname{div} \mathcal{H})_j &= \nabla^i \mathcal{H}_{ij} \\ &= \nabla^i (\tau_{imn} \tau_{jmn}) \\ &= (d^* \tau)^{mn} \tau_{jmn} + \tau_{imn} \nabla^i \tau_{jmn} \\ &= (d^* \tau)^{mn} \tau_{jmn} + \tau_{imn} (\nabla_j \tau_{imn} - \nabla_m \tau_{ijn} + \nabla_n \tau_{ijm}) \\ &= (d^* \tau)^{mn} \tau_{jmn} + \frac{1}{2} \nabla_j |\tau|^2 - 2 \tau_{imn} \nabla^i \tau_{jmn} \end{aligned}$$

where in the last line we rearranged indices. The first result follows from this calculation. For the second we take the divergence of the first equation and get

$$\begin{aligned} \operatorname{div} \operatorname{div} \mathcal{H} &= \nabla^i \left((d^* \tau)^{mn} \tau_{imn} + \frac{1}{6} \nabla_i |\tau|^2 \right) \\ &= \langle \nabla_i (d^* \tau)_{mn}, \tau_{imn} \rangle + |d^* \tau|^2 + \frac{1}{6} \Delta |\tau|^2 \\ &= \frac{1}{3} \langle \Delta_{LB} \tau, \tau \rangle + \frac{1}{6} \Delta |\tau|^2 + |d^* \tau|^2 \end{aligned}$$

□

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FINE HALL, PRINCETON UNIVERSITY, PRINCETON, NJ 08544
E-mail address: jstreets@math.princeton.edu