MIDTERM 1 REVIEW

1. Consider the integral $\int_0^2 x \cos x dx$

   (a) Estimate the integral using $n = 4$ subintervals using rectangles (right endpoints). DRAW A PICTURE.

   (b) Estimate the integral using $n = 4$ subintervals using trapezoids. DRAW A PICTURE.

   (c) Estimate the integral using Simpson's rule with $n = 4$. DRAW A PICTURE.

   (d) Compute the exact answer. Which of the three numerical approaches (rectangles, trapezoids, Simpson's rule, which uses parabolas) was most accurate? Why would or would you not expect this approach to be the most accurate?

2. Find the following indefinite integrals

   (a) $\int x \sin(x^2 - 1) dx$

   (b) $\int \left(x^2 - 1\right) \sin x dx$

   (c) $\int \sin^2 x dx$

3. Evaluate the following definite integrals

   (a) $\int_0^2 \sqrt{4 - x^2} dx$

   (b) $\int_1^2 \frac{4}{2x^2 + x} dx$

4. Evaluate the improper integrals

   (a) $\int_1^\infty x^{-2/3} dx$

   (b) $\int_{-\infty}^0 \frac{2x}{x^2 + 1} dx$

(c) $\int_0^4 \frac{1}{x\sqrt{x}}dx$. Why is this integral improper?

5. Sketch the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$. Then find the area of the region.

6. (a) Let R be the region bounded by the curve $y = \sqrt{x}$, the x-axis, and the vertical lines $x = 0$ and $x = 4$. Find the volume of the solid generated when R is revolved about the x-axis.

   (b) Let D be the region bounded by the curve $y = \sqrt{x}$, the y-axis, and the horizontal lines $y = 0$ and $y = 2$. Find the volume of the solid generated when D is revolved about the y-axis.

7. I also strongly suggest you look at the following problems:

   (a) Problems from previous quizzes

   (b) Homework problems (assigned and suggested)

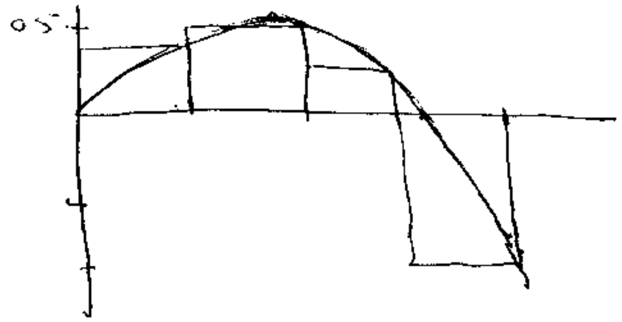
   (c) Examples done in the lecture

   Practicing is by far your best strategy for succeeding on the first midterm!

Midterm Review

1. $\int_0^2 x\cos x \, dx$

a) $n=4$ rectangles (right endpts)



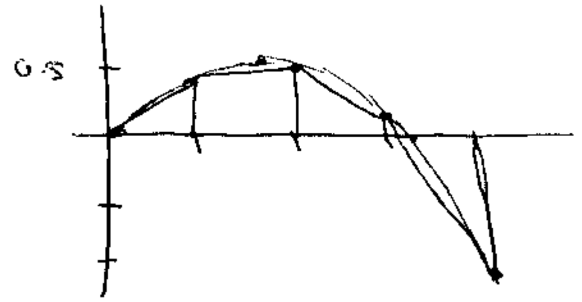
$$\int_0^2 x\cos x\,dx \sim \frac{1}{2}\left[f(0.5) + f(1) + f(1.5) + f(2)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2}\cos\left(\frac{1}{2}\right) + 1\cos(1) + \frac{3}{2}\cos\frac{3}{2} + 2\cos 2\right]$$

$$= \frac{1}{2}\left[0.43879 + 0.54030 + 0.10611 + 0.83229\right]$$

$$= 0.126455$$

b) $n=4$ trapezoids



$$\int_0^2 x\cos x\,dx \sim \frac{1}{2}\cdot\frac{1}{2}\left[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2)\right]$$

$$= \frac{1}{4}\left[0 + \cos\left(\frac{1}{2}\right) + 2\cos(1) + 3\cos\left(\frac{3}{2}\right) + 2\cos 2\right]$$

$$= \frac{1}{4}\left[0.87758 + 1.08060 + 0.21221 + 0.83229\right]$$

$$= 0.33453$$

c) $n=4$ Simpson's

$$\int_0^2 x\cos x\,dx \sim \frac{1}{3}\cdot\frac{1}{2}\left[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\right]$$

$$= \frac{1}{6}\left[0 + 2\cos\left(\frac{1}{2}\right) + 2\cos(1) + 6\cos(1.5) + 2\cos 2\right]$$

$$= \frac{1}{6}\left[1.75517 + 1.08060 + 0.42442 - 0.83229\right]$$

$$= 0.40465$$

d) Best for Simpson's as would be expected. Error for rectangles + trapezoids decreases as $\frac{1}{n^2}$, but decreases as $\frac{1}{n^4}$

2 a) $\int x\sin(x^2-1)dx$

$u = x^2-1$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int \sin u\,du$

$= -\frac{1}{2}\cos u + C$

$= -\frac{1}{2}\cos(x^2-1)+C$

b) $\int (x^2-1)\sin x\,dx$

$u = x^2-1 \qquad dv = \sin x\,dx$

$du = 2x\,dx \qquad v = -\cos x$

$= -(x^2-1)\cos x + 2\int x\cos x\,dx$

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$= -(x^2-1)\cos x + 2[x\sin x - \int \sin x]$

$= -(x^2-1)\cos x + 2x\sin x + 2\cos x + C$

c) $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)dx$

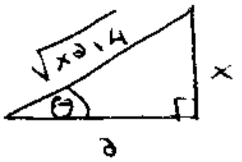
$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right] + C$

3 a) $\displaystyle\int_2^3 \frac{1}{x\sqrt{4+x^2}}\,dx$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$\tan\theta = \frac{x}{2} = \frac{opp}{adj}$



$\csc\theta = \frac{1}{\sin\theta} = \frac{hyp}{opp} = \frac{\sqrt{x^2+4}}{x}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{2}{x}$

$$= \int \frac{2\sec^2\theta\, d\theta}{2\tan\theta\sqrt{4+4\tan^2\theta}}$$

$$= \int \frac{\sec^2\theta}{\tan\theta\sqrt{4(1+\tan^2\theta)}}\,d\theta$$

$$= \int \frac{\sec^2\theta}{\tan\theta\sqrt{4\sec^2\theta}}\,d\theta$$

$$= \int \frac{\sec^2\theta}{2\tan\theta\sec\theta}\,d\theta$$

$$= \frac{1}{2}\int \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{2}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}\,d\theta$$

$$= \frac{1}{2}\int \frac{1}{\sin\theta}\,d\theta$$

$$= \frac{1}{2}\int \csc\theta\, d\theta$$

$$= -\frac{1}{2}\ln\left|\csc\theta + \cot\theta\right|$$

$$= -\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}+2}{x}\right|\Bigg|_2^3$$

$$= -\frac{1}{2}\left[\ln\left|\frac{\sqrt{13}+2}{3}\right| - \ln\left|\frac{\sqrt{8}+2}{2}\right|\right]$$

$$= -\frac{1}{2}(+0.62515 - 0.88137) = 0.12811$$

3

b) $\int_1^2 \frac{4}{x(2x+1)}\,dx$

$$\frac{4}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1}$$

$$= \frac{A(2x+1)+Bx}{x(2x+1)}$$

$4 = (2A+B)x + A$

So,

$A = 4$

$2A + B = 0 \Rightarrow 8 + B = 0$

$B = -8$

$$\int_1^2 \frac{4}{x(2x+1)}\,dx = 4\int_1^2 \frac{1}{x}\,dx - 8\int_1^2 \frac{1}{2x+1}\,dx$$

$$= 4\ln x\Big|_1^2 - \frac{8}{2}\ln(2x+1)\Big|_1^2$$

$$= 4(\ln 2 - \ln 1) - 4(\ln 5 - \ln 3)$$

$$= 4\ln 2 - 4\ln 5 + 4\ln 3$$

$$= 4(\ln 2 + \ln 3 - \ln 5)$$

$$= 4\ln\left(\frac{2\times 3}{5}\right)$$

$$= 4\ln\left(\frac{6}{5}\right)$$

$$\simeq 0.72929$$

4 a) $\int_1^\infty x^{-2/3}\,dx$

$= \lim_{N\to\infty} \int_1^N x^{-2/3}\,dx$

$= \lim_{N\to\infty} \left[3x^{1/3}\right]\Big|_1^N$

$= \lim_{N\to\infty} (3N^{1/3} - 1)$

$= \infty - 1$

$= \infty$ (DIVERGES)

b) $\int_{-\infty}^0 \frac{2x}{x^2+1}\,dx$

$= \lim_{N\to-\infty} \int_N^0 \frac{2x}{x^2+1}\,dx$

$u = x^2+1 \quad du = 2x\,dx$

$= \lim_{N\to-\infty} \int \frac{1}{u}\,du$

$= \lim_{N\to-\infty} \ln|u|$

$= \lim_{N\to-\infty} \ln|x^2+1|\Big|_N^0$

$= \lim_{N\to-\infty} (\ln 1 - \ln|N^2+1|)$

$= 0 - \lim_{N\to-\infty} \ln(N^2+1)$

$= -\infty$ (DIVERGES)

5

c) Improper b/c discontinuous at $x=0$

$$\int_0^4 \frac{1}{x^{3/2}}\,dx = \lim_{N\to 0^+}\int_N^4 x^{-3/2}\,dx$$

$$= \lim_{N\to 0^+}\left[-2x^{-1/2}\right]_N^4$$

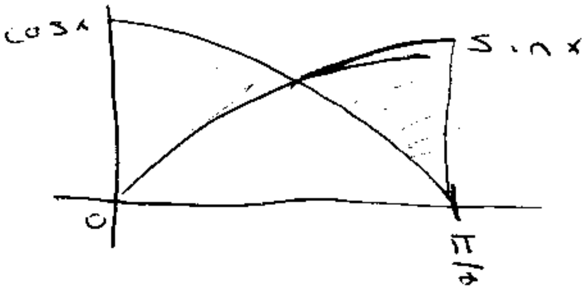
$$= 2\lim_{N\to 0^+}\left(\frac{1}{2} - \frac{1}{\sqrt{x}}\right)$$

$$= -2\left(\frac{1}{2} - \frac{1}{0}\right)$$

$$= -2\left(\frac{1}{2} - \infty\right)$$

Diverges

5. $y=\sin x,\ y=\cos x,\ x=0,\ x=\frac{\pi}{2}$



From $0 \to \frac{\pi}{4}$: $\cos x$ on top

From $\frac{\pi}{4} \to \frac{\pi}{2}$: $\sin x$ on top

$$A = \int_0^{\pi/4}(\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2}(\sin x - \cos x)\,dx$$

$$= \sin x + \cos x\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \left[\left(\sin\tfrac{\pi}{4} + \cos\tfrac{\pi}{4}\right) - \left(\sin 0 + \cos\theta\right)\right] + \left[\left(-\cos\tfrac{\pi}{2} - \sin\tfrac{\pi}{2}\right) - \left(-\cos\tfrac{\pi}{4} - \sin\tfrac{\pi}{4}\right)\right]$$

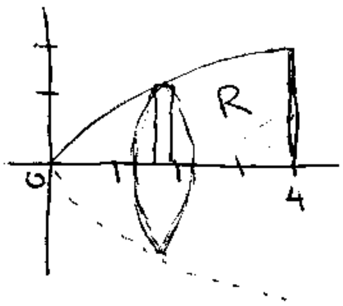
$$= \left[\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0+1)\right] + \left[(0-1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right)\right]$$

$$= \sqrt{2} - 1 - 1 + \sqrt{2}$$

$$= 2\sqrt{2} - 2 = 0.82843$$

6 Rotate $y = \sqrt{x}$ bounded by $y = 0$, $x = 4$ about y-axis

a)



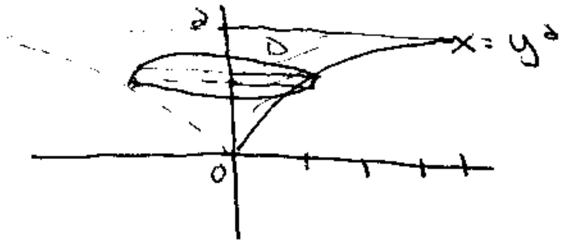
Cross-section is a circle with radius $\sqrt{x}$. So

$$A(x) = \pi(\sqrt{x})^2 = \pi x$$

$$V = \int_0^4 A(x)\,dx = \int_0^4 \pi x\,dx$$

$$= \frac{\pi}{2} x^2 \Big|_0^4 = 8\pi \approx 25.1327$$

b) about x-axis



Cross-section is a circle with radius y^2 ($\perp$ to y, so write as function of y)

So,

$$A(y) = \pi(y^2) = \pi y^4$$

$$V = \int_0^2 A(y)\,dy = \pi \int_0^2 y^4\,dy$$

$$= \frac{\pi}{5} y^5 \Big|_0^2 = \frac{32\pi}{5} \approx 20.106$$