Final Review

Midterm 1

1a) $\int \frac{2x+3}{x-1}dx$

$$\begin{array}{r} 2 \\ x-1 \overline{)\,2x+3} \\ -(2x-2) \\ \hline 5 \end{array}$$

So,

$$\int \frac{2x+3}{x-1}dx = \int\left(2+\frac{5}{x-1}\right)dx$$

$$= 2x + 5\ln|x-1| + c$$

b) $\int_0^2 \frac{2x+3}{x-1}dx = \int_0^2 \left(2+\frac{5}{x-1}\right)dx$ (from (a))

$$= \int_0^2 2\,dx + \int_0^2 \frac{5}{x-1}dx$$

$$= 2x\Big|_0^2 + \int_0^1 \frac{5}{x-1}dx + \int_1^2 \frac{5}{x-1}dx$$

$$= (4-0) + \lim_{N\to1^-}\int_0^N \frac{5}{x-1}dx + \lim_{N\to1^+}\int_N^2 \frac{5}{x-1}dx$$

$$= 4 + \lim_{N\to1^-} 5\ln|x-1|\Big|_0^N + 5\lim_{N\to1^+}\ln|x-1|\Big|_N^2$$

$$= 4 + 5\lim_{N\to1^-}(\ln|N-1| - 0) + 5\lim_{N\to1^+}(0-\ln|N-1|)$$

$$= 4 + \underbrace{5(-\infty - 0)}_{\text{diverge}} + \underbrace{5(0-(-\infty))}_{\text{diverge}}$$

Since one of the integrals diverge, the original integral

$$\int_0^2 \frac{2x+3}{x-1}\,dx$$

also diverges.

c) $\int xe^{x^2}dx$

$u = x^2$

$du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$= \frac{1}{2}\int e^u du$

$= \frac{1}{2}e^u + C$

$= \frac{1}{2}e^{x^2} + C$

d) $\int x^2 e^x dx$

$u = x^2 \qquad dv = e^x dx$

$du = 2x\,dx \qquad v = e^x$

$= x^2e^x - 2\int xe^x dx$

$u = x \qquad dv = e^x dx$

$du = dx \qquad v = e^x$

$= x^2e^x - 2[xe^x - \int e^x]$

$= x^2e^x - 2xe^x + 2e^x + C$

e) $\int \sin^2 x \cos^3 x \, dx$

"Peel off" from $\cos^3 x$:

$$\int \sin^2 x \cos^2 x \cdot \cos x \, dx$$

Use identity

$$\sin^2 x + \cos^2 x = 1 \Rightarrow$$
$$\cos^2 x = 1 - \sin^2 x$$

$$\int \sin^2 x (1 - \sin^2 x) \cos x \, dx$$

$u = \sin x$
$du = \cos x \, dx$

$$= \int u^2 (1 - u^2) \, du$$

$$= \int (u^2 - u^4) \, du$$

$$= \frac{1}{3} u^3 - \frac{1}{5} u^5 + c$$

$$= \frac{1}{3} \sin^3 x - \frac{1}{5} \cos^5 x + c$$

f) $\int \frac{1}{x^2 - 4x - 12} dx = \int \frac{1}{(x-6)(x+2)} dx$

Use partial fractions:

$$\frac{1}{(x-6)(x+2)} = \frac{A}{x-6} + \frac{B}{x+2}$$

$$= \frac{A(x+2) + B(x-6)}{(x-6)(x+2)}$$

So,

$$1 = (A+B)x + (2A - 6B)$$

$$A + B = 0 \Rightarrow B = -A$$

$$2A - 6B = 1 \Rightarrow 2A - 6(-A) = 1$$

$$2A + 6A = 1$$

$$8A = 1$$

$$A = \frac{1}{8} \Rightarrow B = -\frac{1}{8}$$

So,

$$\int \frac{1}{(x-6)(x+2)}\,dx = \frac{1}{8}\int \frac{1}{x-6}\,dx - \frac{1}{8}\int \frac{1}{x+2}\,dx$$

$$= \frac{1}{8}\ln|x-6| \quad \frac{1}{8}\ln|x+2| + c$$

9) $\int \frac{1}{x^2\sqrt{1+x^2}}\,dx$: Use trig substitution

$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$\int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\,d\theta$$

Since $1 + \tan^2\theta = \sec^2\theta$

$$= \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta = \int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

$$= \int \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta}{\cancel{\cos^2\theta}}} \cdot \frac{\cos^2\theta}{\sin^2\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

Now we need substitution

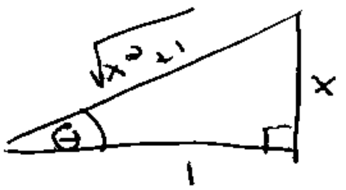
$u = \sin\theta \quad du = \cos\theta\,d\theta$

$$= \int \frac{1}{u^2}\,du = \int u^{-2}\,du = -u^{-1} + C$$

$$= -\frac{1}{\sin\theta} + C$$

Go back to x:

$$\tan\theta = \frac{opp}{adj} = \frac{x}{1}$$



$$\sin\theta = \frac{x}{\sqrt{x^2+1}}$$

So,

$$\int \frac{1}{x^2\sqrt{1+x^2}}\,dx = \frac{-1}{\left(\frac{x}{\sqrt{x^2+1}}\right)} + C$$

$$= -\frac{\sqrt{x^2+1}}{x} + C$$

h) $\displaystyle\int_2^\infty \frac{1}{(x+2)^3}\,dx$

Improper integral

$$\lim_{N\to\infty} \int_2^N (x+2)^{-3}\,dx$$

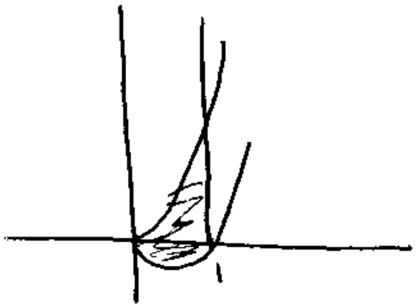
$$u = x+2$$
$$du = dx$$

$$\lim_{N\to\infty} \int u^{-3}\,du = \lim_{N\to\infty} -\frac{1}{2}u^{-2}$$

$$-\frac{1}{2}\lim_{N\to\infty} \frac{1}{(x+2)^2}\Bigg|_2^N = -\frac{1}{2}\lim_{N\to\infty}\left[\frac{1}{(N+2)^2} - \frac{1}{4^2}\right]$$

$$= -\frac{1}{2}\left(0 - \frac{1}{16}\right) = \frac{1}{32}$$

2. a) $y = e^x - 1$, $y = x^2 - x$, $x = 1$



Top: $y = e^x - 1$
Bottom: $x^2 - x$

$$A = \int_0^1 (e^x - 1) - (x^2 - x)\,dx$$

$$= \int_0^1 (e^x - 1 - x^2 + x)\,dx$$

$$= e^x - x - \frac{1}{3}x^3 + \frac{1}{2}x^2 \Big|_0^1$$

$$= \left(e^1 - 1 - \frac{1}{3} + \frac{1}{2}\right) - \left(e^0 - 0 - 0 + 0\right)$$

$$= \left(e - \frac{5}{6}\right) - 1$$

$$= e - \frac{11}{6} \sim 0.8849$$

b) $x = 2y - 7$, $x = y^2 - 6y$

To graph, let's realize the first eqn is

$$2y = x + 7 \Rightarrow y = \frac{1}{2}x + \frac{7}{2}$$

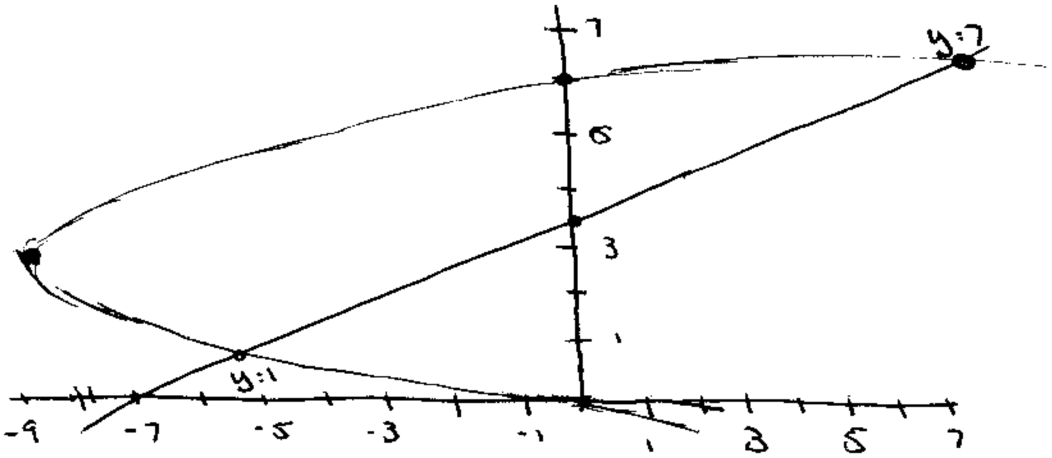
Also,

$x = y(y-6)$ is a parabola with

$x = 0$ when $y = 0, 6$

when $y = 3$, $x = 3(-3) = -9$

Plot



Intersection points:

$2y - 7 = y^2 - 6y \Rightarrow$

$y^2 - 8y + 7 = 0$

$(y-7)(y-1) = 0$

$y = 7, 1$

$x = 2y - 7$ is "ahead of" $x = y^2 - 6y$

$$A = \int_1^7 (2y-7) - (y^2 - 6y)\,dy$$

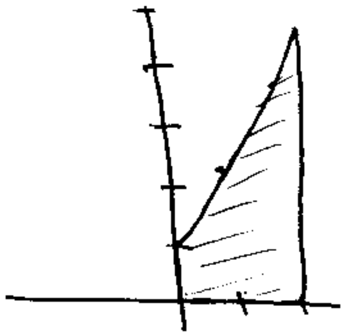
$$= \int_1^7 (-y^2 + 8y - 7)\,dy$$

$$= -\frac{1}{3}y^3 + 4y^2 - 7y \Big|_1^7$$

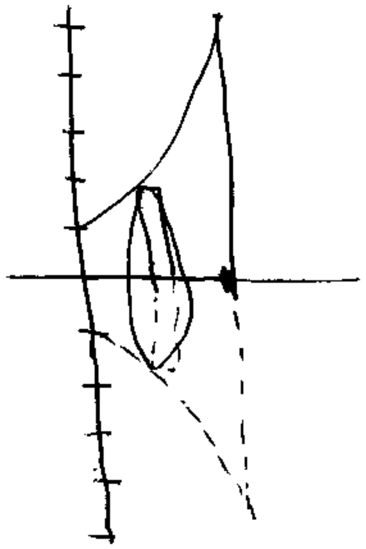
$$= \left(-\frac{7^3}{3} + 4(49) - 49\right) - \left(-\frac{1}{3} + 4 - 7\right)$$

$= 36$

3. $y = e^{\frac{3}{4}x}$, $x = 0$, $x = 2$, $y = 0$



a) About x-axis



Disk with radius

$$r = e^{\frac{3}{4}x}$$

$$V = \int_0^2 \pi \left(e^{\frac{3}{4}x}\right)^2 dx$$

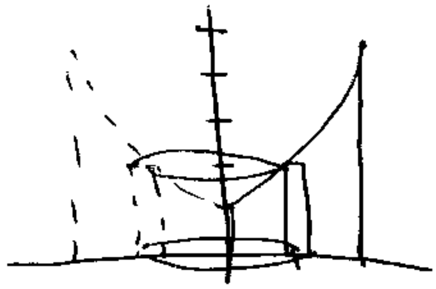
$$= \pi \int_0^2 e^{\frac{3}{2}x}\, dx$$

$$= \frac{2\pi}{3} e^{\frac{3}{2}x} \Big|_0^2$$

$$= \frac{2\pi}{3}\left(e^3 - e^0\right)$$

$$= \frac{2\pi}{3}\left(e^3 - 1\right) \sim 39.97$$

b). About y-axis



Shells

$$V = \int_0^2 2\pi x f(x)\, dx$$

$$= 2\pi \int_0^2 x e^{\frac{3}{4}x}\, dx$$

Use integration by parts

$u = x \qquad dv = e^{\frac{3}{4}x}\,dx$

$du = dx \qquad v = \frac{4}{3}e^{\frac{3}{4}x}$

$$= 2\pi\left[\frac{4}{3}xe^{\frac{3}{4}x}\Big|_0^2 - \frac{4}{3}\int_0^2 e^{\frac{3}{4}x}\,dx\right]$$

$$= 2\pi\left[\left(\frac{8}{3}e^{\frac{3}{2}} - 0\right) - \frac{4}{3}\cdot\frac{4}{3}e^{\frac{3}{4}x}\Big|_0^2\right]$$

$$= 2\pi\left[\frac{8}{3}e^{\frac{3}{2}} - \frac{16}{9}\left(e^{\frac{3}{2}} - e^0\right)\right]$$

$$= 2\pi\left[\frac{24}{9}e^{\frac{3}{2}} - \frac{16}{9}e^{\frac{3}{2}} + \frac{16}{9}\right]$$

$$= 2\pi\left(\frac{8}{9}e^{\frac{3}{2}} + \frac{16}{9}\right) \sim 36.2$$

Midterm 2

1 a) $a_k = 2 + \cos(k\pi)$

$a_1 = 2 + \cos\pi = 2 - 1 = 1$ All odd terms $= 1$

$a_2 = 2 + \cos(2\pi) = 2 + 1 = 3$ All even terms $= 3$

$a_3 = 2 + \cos(3\pi) = 2 - 1 = 1$

$a_4 = 2 + \cos(4\pi) = 2 + 1 = 3$

Sequence doesn't approach one number, so the sequence $a_k = 2 + \cos(k\pi)$ diverges.

Thus, $\sum a_k = \sum(2 + \cos(k\pi))$ diverges by the Divergence Test, since

$$\lim_{k\to\infty}(2 + \cos(k\pi)) \text{ diverges}$$

b) $a_k : \frac{1-k}{k^3+4}$

$$\lim_{k\to\infty} \frac{1-k}{k^3+4} \cdot \frac{\frac{1}{k^3}}{\frac{1}{k^3}} = \lim_{k\to\infty} \frac{\frac{1}{k^3} - \frac{1}{k^2}}{1 + \frac{4}{k^3}} = 0$$

So, the sequence converges since its limit is NOT $\pm\infty$. We cannot say anything about the series $\sum \frac{1-k}{k^3+4}$ by looking at the sequence b/c when

$$\lim_{k\to\infty} \frac{1-k}{k^3+4} = 0,$$

the Divergence Test gives us no info.

2 a) $\sum_{k=1}^{\infty} \frac{k}{(k+1)2^k}$

Try Generalized Ratio Test

$$\lim_{k\to\infty} \left| \frac{k+1}{(k+2)2^{k+1}} \cdot \frac{(k+1)2^k}{k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)^2\, \cancel{2^k}}{k(k+2)\cancel{2^k}\, 2^1} \right|$$

$$= \frac{1}{2} \lim_{k\to\infty} \left| \frac{k^2+2k+1}{k^2+2k} \right| \cdot \frac{\frac{1}{k^2}}{\frac{1}{k^2}}$$

$$= \frac{1}{2} \lim_{k\to\infty} \left| \frac{1+\frac{2}{k}+\frac{1}{k^2}}{1+\frac{2}{k}} \right| : \quad \frac{1}{2} \times 1 = \frac{1}{2} < 1, \text{ converges}$$

b) $$\sum_{k=2}^{\infty} \frac{2^{2k+3}}{3^{k-1}} : \sum_{k=2}^{\infty} \frac{2^{2k}2^3}{3^k 3^{-1}} : \sum_{k=2}^{\infty} \frac{8}{3^{-1}} \cdot \left(\frac{2^2}{3}\right)^k$$

$$: \sum_{k=2}^{\infty} 24\left(\frac{4}{3}\right)^k \quad \text{Geometric with } |r| : \frac{4}{3} > 1$$

Diverges

c) $\sum_{k=1}^{\infty} \frac{4^k + 3^k}{2^k}$

Compare to $\sum \frac{4^k}{2^k} = \sum 2^k$

Which is geometric with $|r| = 2 > 1$
so it diverges

Notice that

$$\frac{4^k + 3^k}{2^k} > \frac{4^k + 0}{2^k} = \frac{4^k}{2^k}$$

since $3^k > 0$ for all k. Since we are bigger than a divergent series,

$$\sum_{k=1}^{\infty} \frac{4^k + 3^k}{2^k}$$

diverges by Direct Comparison Test

d) $\sum_{k=}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right)$ Telescoping!

Look at first few partial sums

$$S_4 = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}}$$

$$S_5 = \left(\frac{1}{\sqrt{4}} - \cancel{\frac{1}{\sqrt{5}}} \right) + \left(\cancel{\frac{1}{\sqrt{5}}} - \frac{1}{\sqrt{6}} \right)$$

$$S_6 = \left(\frac{1}{\sqrt{4}} - \cancel{\frac{1}{\sqrt{5}}} \right) + \left(\cancel{\frac{1}{\sqrt{5}}} - \cancel{\frac{1}{\sqrt{6}}} \right) + \left(\cancel{\frac{1}{\sqrt{6}}} - \frac{1}{\sqrt{7}} \right)$$

$$\vdots$$

$$S_n = \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{n+1}}$$

So,

$$\sum_{k=4}^{\infty} \left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}} \right) = \lim_{n \to \infty} \left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{n+1}} \right) = \frac{1}{\sqrt{4}} - 0 = \frac{1}{2},$$

converges to $\frac{1}{2}$

3 a) $\sum_{k=0}^{\infty} \frac{(-5)^k x^k}{\sqrt{k+3}}$

Generalized Ratio Test

$$\lim_{k\to\infty} \left| \frac{(-5)^{k+1} x^{k+1}}{\sqrt{k+4}} \cdot \frac{\sqrt{k+3}}{(-5)^k x^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{(-5)^k} (-5)^1 \cancel{x^k} x^1 \sqrt{k+3}}{\cancel{(-5)^k} \cancel{x^k} \sqrt{k+4}} \right|$$

$$= |-5x| \lim_{k\to\infty} \left| \frac{\sqrt{k+3}}{\sqrt{k+4}} \right| = |5x| \lim_{k\to\infty} \sqrt{\frac{k+3}{k+4} \cdot \frac{\frac{1}{k}}{\frac{1}{k}}}$$

$$= |5x| \lim_{k\to\infty} \sqrt{\frac{1+3/k}{1+4/k}} = |5x| \cdot 1 = |5x|$$

Converge when $L<1$

$|5x|<1 \iff -1<5x<1$

$\iff -\frac{1}{5}<x<\frac{1}{5}$

Test endpoints

$x=-\frac{1}{5}$: $\sum_{k=0}^{\infty} \frac{(-5)^k(-\frac{1}{5})^k}{\sqrt{k+3}} = \sum_{k=0}^{\infty} \frac{(-5 \times -\frac{1}{5})^k}{\sqrt{k+3}} = \sum \frac{1}{\sqrt{k+3}}$

Compare to $\sum \frac{1}{\sqrt{k}} = \sum \frac{1}{k^{1/2}}$ which is a p-series with $p = \frac{1}{2} \leq 1$, so it diverges

Use Limit Comparison

$$\lim_{k\to\infty} \frac{1}{\sqrt{k+3}} \cdot \frac{\sqrt{k}}{1} = \lim_{k\to\infty} \sqrt{\frac{k}{k+3} \cdot \frac{\frac{1}{k}}{\frac{1}{k}}}$$

$$= \lim_{k\to\infty} \sqrt{\frac{1}{1+3/k}} = 1$$ so both diverge

$x = \frac{1}{5}$: $\sum_{k=0}^{\infty} \frac{(-5)^k(\frac{1}{5})^k}{\sqrt{k+3}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+3}}$

Alternating series

1) $\lim_{k\to\infty} \frac{1}{\sqrt{k+3}} = 0$ ✓

2) Decreasing b/c

$$\frac{1}{\sqrt{(k+1)+3}} = \frac{1}{\sqrt{k+4}} < \frac{1}{\sqrt{k+3}}$$

So this converges

Interval of convergence $-\frac{1}{5} < x \le \frac{1}{5}$

b) $\sum_{k=0}^{\infty} k!\, x^k$

Generalized Ratio Test

$$\lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{k!\, x^k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)\cancel{k!}\, \cancel{x^k} x^1}{\cancel{k!}\, \cancel{x^k}} \right|$$

$$= |x| \lim_{k\to\infty} (k+1) = \infty > 1$$

Diverges for all x except $x = 0$.

c) $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-1}}{(2k-1)!}$

Generalized Ratio Test

$$\lim_{k\to\infty} \left| \frac{(-1)^{k+1}\, x^{2(k+1)-1}}{[2(k+1)-1]!} \cdot \frac{(2k-1)!}{(-1)^k x^{2k-1}} \right|$$

$$= \lim_{k\to\infty}\left|\frac{\cancel{(-1)^k}(-1)^1 x^{2k+2-1}(2k-1)!}{[2k+2-1]!\,\cancel{(-1)^k}\,x^{2k-1}}\right|$$

$$= \lim_{k\to\infty}\left|\frac{-\cancel{x^{2k}}\,x^1 (2k-1)!}{(2k+1)!\,\cancel{x^{2k}}\,x^{-1}}\right|$$

$$= \left|-\frac{x}{x^{-1}}\right| \lim_{k\to\infty}\left|\frac{\cancel{(2k-1)!}}{(2k+1)(2k)\cancel{(2k-1)!}}\right|$$

$$= |x^2| \lim_{k\to\infty}\left|\frac{1}{4k^2+2k}\right| = 0 < 1$$

Converge for all x.

<u>New Material</u>

1 a) $f(x) = x^2 e^{-x}$

We know

$$e^u = \sum_{k=0}^{\infty}\frac{u^k}{k!}$$

Let $u = -x$:

$$e^{-x} = \sum_{k=0}^{\infty}\frac{(-x)^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!}$$

multiply by x^2:

$$x^2 e^{-x} = x^2\sum_{k=0}^{\infty}\frac{(-1)^k x^k}{k!} = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2+k}}{k!}$$

b) $f(x) = x\sin(x^4)$

We know

$$\sin u = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

So, with $u = x^4$:

$$\sin(x^4) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (x^4)^{2k+1}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{8k+4}$$

Multiply by x:

$$x\sin(x^4) = x^1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{8k+4}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{8k+5}$$

c) $\int x\sin(x^4)dx$

Use answer from (b):

$$\int x\sin(x^4)dx = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int x^{8k+5} dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \frac{1}{8k+6} x^{8k+6} + C$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(8k+6)(2k+1)!} x^{8k+6} + C$$

2 a) $f(x) = \ln x$, $c = 2$

$f(x) = \ln x \qquad f(2) = \ln 2$

$f'(x) = \frac{1}{x} = x^{-1} \qquad f'(2) = \frac{1}{2}$

$f''(x) = -x^{-2} = -\frac{1}{x^2} \qquad f''(2) = -\frac{1}{4}$

$f'''(x) = +2x^{-3} = \frac{+2}{x^3} \qquad f'''(2) = \frac{+2}{2^3} = \frac{1}{4}$

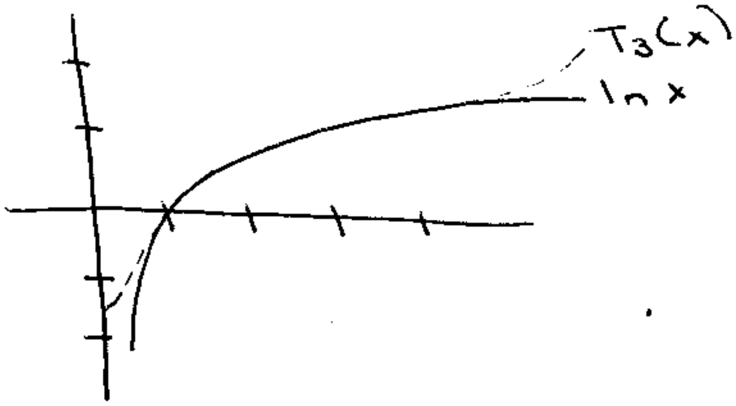
$$T_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(2)}{k!}(x-2)^k$$

$$= f(2) + \frac{f'(2)}{1!}(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \frac{f'''(2)}{3!}(x-2)^3$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{4\times 2!}(x-2)^2 + \frac{1}{4\times 3!}(x-2)^3$$

$$= \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3$$

b) Graphs



- Very similar near $x = 2$
- Look different further away from $x = 2$

c) Estimate $\ln(2.5)$

$T_3(2.5) = \ln 2 + \frac{1}{2}(2.5-2) - \frac{1}{8}(2.5-2)^2 + \frac{1}{24}(2.5-2)^3$

≈ 0.91711

Exact: $\ln 2.5 \approx 0.91629$

Very similar, as expected b/c close to 2.

3. a) $\sqrt{\dfrac{1-y^2}{x+1}} = \dfrac{dy}{dx}$

Separate:

$$\frac{\sqrt{1-y^2}}{\sqrt{x+1}} = \frac{dy}{dx}$$

$$\frac{\sqrt{1-y^2}}{\sqrt{x+1}}\,dx = dy$$

$$\frac{1}{\sqrt{x+1}}\,dx \quad \frac{1}{\sqrt{1-y^2}}\,dy$$

Integrate

$$\int (x+1)^{-1/2}\,dx = \int \frac{1}{\sqrt{1-y^2}}\,dy$$

$u = x+1$ $\qquad$ $y = \sin\theta$ (Trig subst.)

$du = dx$ $\qquad$ $dy = \cos\theta\, d\theta$

$$\int u^{-1/2}\,du = \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}}$$

$$2u^{1/2} + c_1 = \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}}$$

$$2\sqrt{x+1} + C_1 = \int d\theta$$

$$2\sqrt{x+1} + C_1 = \theta + C_2$$

Solve for θ:

$\sin^{-1} y = \theta$

So,

$$2\sqrt{x+1} = \sin^{-1} y + C_2 - C_1$$

$$2\sqrt{x+1} = \sin^{-1} y + C$$

b) $\frac{dy}{dx} = \frac{1}{e^{4x}\cos y}$

Separate

$$dy = \frac{e^{-4x}}{\cos y}\,dx$$

$$\cos y\,dy = e^{-4x}\,dx$$

Integrate

$$\sin y + C_1 = -\frac{1}{4}e^{-4x} + C_2$$

Simplify

$$\sin y + \frac{1}{4}e^{-4x} = C_2 - C_1$$

$$\sin y + \frac{1}{4}e^{-4x} = C$$

4 a) $\frac{1}{x}\frac{dy}{dx} + \frac{2}{1+x^2} = x$

1) Get in correct form

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = x^2$$

2) Find $I(x)$

$$I(x) = e^{\int \frac{2x}{1+x^2}dx}$$

$$u = 1+x^2 \qquad du = 2x\,dx$$

$$I(u) = e^{\int \frac{1}{u}du} = e^{\ln u} = u$$

$$I(x) = 1+x^2$$

3) Multiply by $I(x)$

$$(1+x^2)\frac{dy}{dx} + 2xy = (1+x^2)x^2$$

We can write the left-hand side as

$$\frac{d}{dx}\left[(1+x^2)y\right]$$

Check:

$$\frac{d}{dx}\left[(1+x^2)y\right] = (1+x^2)\frac{dy}{dx} + y(2x) \checkmark$$

4) Integrate

$$\int \frac{d}{dx}\left[(1+x^2)y\right]dx = \int (x^2+x^4)\,dx$$

$$(1+x^2)y = \frac{1}{3}x^3 + \frac{1}{5}x^5 + C$$

5) Solve for y:

$$y = \frac{\frac{1}{3}x^3 + \frac{1}{5}x^5 + C}{1+x^2}$$

b) $x\frac{dy}{dx} = -3y + 2xe^{x^4}$

1) Put in correct form:

$$\frac{dy}{dx} + \frac{3}{x}y = 2e^{x^4}$$

2) Find $I(x)$

$$I(x) = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

3) Multiply by $I(x) = x^3$:

$$\underbrace{x^3\frac{dy}{dx} + 3x^2y}_{\frac{d}{dx}(x^3y)} = 2x^3e^{x^4}$$

4) Integrate

$$\int \frac{d}{dx}(x^3 y)\,dx = \int 2x^3 e^{x^4}\,dx$$

$$x^3 y = 2\int x^3 e^{x^4}\,dx$$

$$u = x^4$$

$$du = 4x^3\,dx \Rightarrow \tfrac{1}{2}du = 2x^3\,dx$$

$$x^3 y = \tfrac{1}{2}\int e^u\,du$$

$$x^3 y = \tfrac{1}{2}e^u + C$$

$$x^3 y = \tfrac{1}{2}e^{x^4} + C$$

5) Solve for y:

$$y = \frac{\tfrac{1}{2}e^{x^4} + C}{x^3}$$

5 a) $f(x,y) = x^5 + 3x^3y^2 + 3xy^4$

$\frac{\partial f}{\partial x}$ $\quad f(x,y) = x^5 + (3y^2)x^3 + (3y^4)x$

$$\frac{\partial f}{\partial x} = 5x^4 + (3y^2)(3x^2) + 3y^4$$

$$= 5x^4 + 9x^2y^2 + 3y^4$$

$\frac{\partial f}{\partial y}$: $\quad f(x,y) = x^5 + (3x^3)y^2 + (3x)y^4$

$$\frac{\partial f}{\partial y} = 0 + (3x^3)(2y) + (3x)(4y^3)$$

$$= 6x^3y + 12xy^3$$

b) $f(x,y) = \ln(x^2 + 3xy - y^5)$

$$\frac{\partial f}{\partial x} = \frac{2x + 3y}{x^2 + 3xy - y^5} \leftarrow$$ derivative of inside wrt x

$$\frac{\partial f}{\partial y} = \frac{3x - 5y^4}{x^2 + 3xy - y^5} \leftarrow$$ derivative of inside wrt y

c) $f(x,y,z) = e^{x+2y+3z}$

$$\frac{\partial f}{\partial x} = (e^{x+2y+3z})(1) = e^{x+2y+3z}$$

$$\frac{\partial f}{\partial y} = (e^{x+2y+3z})(2) = 2e^{x+2y+3z}$$

$$\frac{\partial f}{\partial z} = (e^{x+2y+3z})(3) = 3e^{x+2y+3z}$$

6 a) $f(x,y) = 3xy - x^2y - xy^2$

$$\frac{\partial f}{\partial x} = 3y - 2xy - y^2 = 0$$

$$2xy = 3y - y^2$$

$$x = \frac{3y - y^2}{2y} = \frac{1}{2}(3 - y)$$

$$\frac{\partial f}{\partial y} = 3x - x^2 - 2xy = 0$$

$$= \frac{3}{2}(3-y) - \left[\frac{1}{2}(3-y)\right]^2 - 2\left[\frac{1}{2}(3-y)\right]y = 0$$

$$= \frac{9}{2} - \frac{3}{2}y - \left[\frac{1}{4}(9 - 6y + y^2)\right] - (3y - y^2) = 0$$

$$= \frac{9}{2} - \frac{3}{\cancel{2}}y - \frac{9}{4} + \frac{3}{2}y - \frac{1}{4}y^2 - 3y + y^2 = 0$$

$$= \frac{3}{4}y^2 - 3y + \left(\frac{18}{4} - \frac{9}{4}\right) = 0$$

$$4\left(\frac{3}{4}y^2 - 3y + \frac{9}{4} = 0\right)$$

$$3y^2 - 12y + 9 = 0$$

$$3(y^2 - 4y + 3) = 0$$

$$3(y-3)(y-1) = 0$$

$$y = 1, 3$$

When $y=1$,

$$x = \frac{1}{2}(3-1) = 1$$

When $y=3$,

$$x = \frac{1}{2}(3-3) = 0$$

2 possible relative max/min

$(1,1)$ and $(0,3)$

Need to find second derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -2y$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -2 .$$

So,

$$D(x,y) = (-2y)(-2x) - (-2)^2$$

$$= 4xy - 4$$

We need to analyze each point

$(1,1)$

$D(1,1) = 4-4=0$

So, we don't know if $(1,1)$ is a relative max or min

$(0,3)$

$D(0,3) = 0-4 = -4 < 0$

$(0,3)$ is NOT a relative max or min

b) $f(x,y) = (x^2+y)e^{y/2}$

$f(x,y) = x^2 e^{y/2} + ye^{y/2}$

$\frac{\partial f}{\partial x} = 2xe^{y/2} = 0$

Since $e^{y/2} \neq 0$ only answer is $x=0$

$\frac{\partial f}{\partial y} = \frac{1}{2}x^2e^{y/2} + \left(y \cdot \frac{1}{2}e^{y/2} + e^{y/2}\right)$

$= e^{\frac{y}{2}}\left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right) = 0$

Since $e^{y/2} \neq 0$ we know we need

$\frac{1}{2}x^2 + \frac{1}{2}y + 1 = 0$

Since we already found $x=0$

$\frac{1}{2}y = -1 \Rightarrow y = -2$

One possible relative max/min at $(0,-2)$

Find second derivatives

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 2e^{y/2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = e^{\frac{y}{2}}\left(0 + \frac{1}{2} + 0\right) + \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right)\left(\frac{1}{2}e^{\frac{y}{2}}\right)$$

$$\underbrace{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = xe^{y/2}}_{\text{same as } \frac{\partial^2 f}{\partial x \partial y} \text{ but this was easier to compute}}$$

Evaluate at $(0, -2)$

$$\frac{\partial^2 f}{\partial x^2} = 2e^{-2/2} = 2e^{-1}$$

$$\frac{\partial^2 f}{\partial y^2} = e^{-1}\left(0 + \frac{1}{2} + 0\right) + (0 - 1 + 1)\left(\frac{1}{2}e^{-1}\right)$$

$$= \frac{1}{2}e^{-1} + 0$$

$$\frac{\partial^2 f}{\partial y \partial x} = 0$$

So,

$$D(0, -2) = (2e^{-1})\left(\frac{1}{2}e^{-1}\right) - 0$$

$$= e^{-2} > 0$$

Also,

$$\frac{\partial^2 f}{\partial x^2} = 2e^{-1} > 0$$

so, $(0, -2)$ is a relative minimum

7. $f(x,y) = x^2 y$

$g(x,y) = x^2 + y^2 - 1 = 0$

Let

$$F(x,y) = f(x,y) + \lambda g(x,y)$$
$$= x^2 y + \lambda(x^2 + y^2 - 1)$$

Find

$$\frac{\partial F}{\partial x} = 2xy + 2x\lambda = 0$$

$$\frac{\partial F}{\partial y} = x^2 + 2y\lambda = 0$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 1 = 0$$

Solve for λ from $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$:

$$\frac{\partial F}{\partial \lambda}: \quad 2xy + 2x\lambda = 0$$
$$2xy = -2x\lambda$$
$$\lambda = -y \quad (\text{or } x = 0)$$

$$\frac{\partial F}{\partial y}: \quad x^2 + 2y\lambda = 0$$
$$2y\lambda = -x^2$$
$$\lambda = \frac{-x^2}{2y}$$

So,

$$-\frac{x^2}{2y} = -y \iff x^2 = 2y^2$$

Plug $x^2 = 2y^2$ into $\frac{\partial F}{\partial \lambda}$:

$$2y^2 + y^2 - 1 = 0$$
$$3y^2 = 1$$
$$y^2 = \frac{1}{3}$$
$$y = \pm\sqrt{\frac{1}{3}}$$

So,

$$x^2 = 2y^2 = 2\left(\pm\sqrt{\frac{1}{3}}\right)^2$$
$$= \frac{2}{3}$$

$$x = \pm\sqrt{\frac{2}{3}}$$

Possible relative max/min when

$$x = \sqrt{\frac{2}{3}} \text{ or } -\sqrt{\frac{2}{3}}$$
$$y = \sqrt{\frac{1}{3}} \text{ or } -\sqrt{\frac{1}{3}}$$

Since

$$f(x,y) = x^2 y$$

$f(x,y)$ is biggest when $y = \frac{1}{\sqrt{3}}$ and smallest when $y = -\frac{1}{\sqrt{3}}$.