MIDTERM 2 REVIEW

1. Definitions

   (a) Sequence
   (b) Convergent sequence
   (c) Divergent sequence
   (d) Decreasing sequence
   (e) Increasing sequence
   (f) Series
   (g) n^{th} partial sum
   (h) Convergent series
   (i) Divergent series
   (j) $k!$
   (k) Alternating series
   (l) Absolutely convergent series
   (m) Conditionally convergent series
   (n) Power series
   (o) Interval of convergence
   (p) Radius of convergence

2. Things that you will need to know that are NOT on your formula sheet:

   (a) Geometric series
   (b) Divergence test
   (c) p-series
   (d) Direct comparison test

3. Determine whether the following sequences convergence of diverge.

   (a) $\left\{ \dfrac{n^2}{e^{2n}} \right\}$
   (b) $\left\{ \left(\dfrac{n+3}{n} \right)^n \right\}$
   (c) $\left\{ (-1)^{n+1} \right\}_{n=4}^{\infty}$

4. Express $6.2\overline{54}$ as a fraction (a ratio of integers to be exact).

5. Determine whether the following series converge or diverge.

   (a) $\displaystyle\sum_{k=0}^{\infty} \frac{7^k}{5^k+1}$
   (b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-5)^{2k}}{k^2 9^k}$
   (c) $\displaystyle\sum_{k=2}^{\infty} \ln\left(\frac{k^2+1}{3k^2+2k-1} \right)$
   (d) $\displaystyle\sum_{k=3}^{\infty} k e^{-k}$
   (e) $\displaystyle\sum_{k=4}^{\infty} \frac{\sqrt{k^5+1}}{k+9}$
   (f) $\displaystyle\sum_{k=5}^{\infty} \left(\frac{2k}{6k-3} \right)^k$

6. Find the sum of the series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^{2k+3}}{5^{k+1}}$

(b) $\displaystyle\sum_{k=3}^{\infty} \frac{1}{k(k+2)}$

7. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{2^k}{k!}$

(b) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{k+3}$

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{\sqrt{k+3}}$

8. Find the radius of convergence and the interval of convergence of the following power series.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{(x-2)^k}{k!}$

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(x+1)^k}{k3^k}$

(c) $\displaystyle\sum_{k=1}^{\infty} k!x^k$

(d) $\displaystyle\sum_{k=1}^{\infty} \frac{(x+4)^k}{6^k}$

9. I also strongly suggest you look at problems from the previous **quizzes**, **homework assignments** and **lecture notes**. Practicing is by far your best way to succeed on the exam!!

MIDTERM 2 FORMULA SHEET

- **Integral Test**: Consider the series $\sum_{k=b}^{\infty} a_k$. Let $a_k = f(k)$ for all k on the interval $[b, \infty)$, where f is a positive, continuous and eventually decreasing function of x on the interval $[b, \infty)$. Then

$$\sum_{k=b}^{\infty} a_k \text{ and } \int_b^{\infty} f(x)dx$$

  either both converge or both diverge.

- **Limit Comparison Test**: Suppose that $\sum a_k$ and $\sum b_k$ are series with positive terms. If

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L, \ (0 < L < \infty)$$

  then either both series converge or both series diverge.

- **Generalized Ratio Test**: Given a series $\sum a_k$ with $a_k \neq 0$, suppose that

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = L$$

  Then:

  1. If $L < 1$, then $\sum a_k$ converges.
  2. If $L > 1$ or L is infinite, then $\sum a_k$ diverges.
  3. If $L = 1$ the test is inconclusive.

- **Generalized Root Test**: Given a series $\sum a_k$, suppose that

$$\lim_{k\to\infty} |a_k|^{1/k} = L$$

  Then:

  1. If $L < 1$, then $\sum a_k$ converges.
  2. If $L > 1$ or L is infinite, then $\sum a_k$ diverges.
  3. If $L = 1$ the test is inconclusive.

- **Alternating Series Test**: An alternating series

$$\sum_{k=1}^{\infty}(-1)^k a_k \text{ or } \sum_{k=1}^{\infty}(-1)^{k+1} a_k$$

where $a_k > 0$ for all k converges if both of the following conditions are satisfied:

1. $\lim_{k\to\infty} a_k = 0$
2. $\{a_k\}$ is a decreasing sequence.

- $\lim_{x\to\infty}\left(a + \frac{b}{x}\right)^x = e^{b/a}$

Midterm 2 Review

3 a) $\left\{\frac{n^2}{e^{2n}}\right\}$

$$\lim_{n\to\infty}\frac{n^2}{e^{2n}} \overset{L'H}{=} \lim_{n\to\infty}\frac{\cancel{2}n}{\cancel{2}e^{2n}} \overset{L'H}{=} \lim_{n\to\infty}\frac{1}{2e^{2n}} = \frac{1}{2\cdot\infty} = 0$$

Converges

b) $\left\{\left(\frac{n+3}{n}\right)^n\right\}$

$$\lim_{n\to\infty}\left(1+\frac{3}{n}\right)^n = e^{3/1} \qquad \text{converges}$$

c) $\left\{(-1)^{n+1}\right\}_{n=4}^{\infty}$

Look a first few terms:

$a_n = (-1)^{n+1}$

$a_4 = (-1)^5 = -1 \qquad a_6 = (-1)^7 = -1$

$a_5 = (-1)^6 = 1 \qquad a_7 = (-1)^8 = 1$

Sequence oscillates b/w +1 and -1 so it can't approach one number $\Rightarrow$ diverges

4. $6.2\overline{54} = 6.2 + 0.054 + 0.00054 + 0.0000054 + \dots$

$$= 6.2 + \underbrace{\frac{54}{10^3} + \frac{54}{10^5} + \frac{54}{10^7}}_{\text{Geometric series}} + \dots$$

$$r = \frac{\cancel{54}}{10^5}\cdot\frac{10^3}{\cancel{54}} = \frac{1}{10^2}$$

$$a = \frac{54}{10^3}$$

Series converges to

$$\frac{a}{1-r} = \frac{\frac{54}{10^3}}{1-\frac{1}{10^2}} = \frac{\frac{54}{10^3}}{\frac{99}{10^2}} \cdot \frac{10^2}{99} = \frac{54}{990}$$

So,

$$6.2\overline{54} = 6.2 + \frac{54}{990} = \frac{62}{10} + \frac{54}{990}$$

$$= \frac{6138+54}{990}$$

$$= \frac{6192}{990}$$

5. a) $\sum_{k=0}^{\infty} \frac{7^k}{5^k+1}$

Compare to $\sum_{k=0}^{\infty} \frac{7^k}{5^k} = \sum_{k=0}^{\infty} \left(\frac{7}{5}\right)^k$

Geometric series with $|r| = \frac{7}{5} > 1 \Rightarrow$ diverges

Need to use Limit Comparison b/c:

$$\frac{7^k}{5^k+1} < \frac{7^k}{5^k}$$

$$\lim_{k\to\infty} \frac{7^k}{5^k+1} \cdot \frac{5^k}{7^k} = \lim_{k\to\infty} \frac{5^k \cdot \frac{1}{5^k}}{5^k+1 \cdot \frac{1}{5^k}} = \lim_{k\to\infty} \frac{1}{1+\frac{1}{5^k}} = 1$$

Since $L = 1$ $(0 < 1 < \infty)$, both series must diverge.

b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-5)^{2k}}{k^2 9^k}$

Use Generalized Ratio Test:

$$\lim_{k\to\infty} \left| \frac{(-5)^{2k+2}}{(k+1)^2 9^{k+1}} \cdot \frac{k^2 9^k}{(-5)^{2k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{(-5)^{2k}}(-5)^2 \cdot k^2 \cdot \cancel{9^k}}{\cancel{(-5)^{2k}}(k+1)^2 \cdot \cancel{9^k} \cdot 9^1} \right|$$

$$= \frac{25}{9} \lim_{k\to\infty} \frac{k^2}{(k+1)^2}$$

$$= \frac{25}{9} > 1$$

Diverges

c) $\displaystyle\sum_{k=2}^{\infty} \ln\left(\frac{k^2+1}{3k^2+2k-1}\right)$

Divergence Test

$$\lim_{k\to\infty} \ln\left(\frac{k^2+1}{3k^2+2k-1}\right) = \ln\left[\lim_{k\to\infty} \frac{k^2+1}{3k^2+2k-1}\right]$$

$$= \ln\left(\tfrac{1}{3}\right) \neq 0$$

Diverges

d) $\displaystyle\sum_{k=3}^{\infty} ke^{-k}$

$ke^{-k} > 0$ for $k \geq 3$

Easier to use Ratio Test!

$e^k \neq 0 \Rightarrow$ continuous

Check if eventually decreasing

$f(x) = xe^{-x}$

$f'(x) = x(-e^{-x}) + e^{-x}$

$= e^{-x}(1-x) = 0$

$x = 1$

$+ \;\big|\; -$

$\;\;1 \quad \uparrow x = 2$

Decreasing on $(1, \infty)$

Can use Integral Test

$$\int_3^\infty xe^{-x}\,dx = \lim_{N\to\infty} \int_3^N xe^{-x}\,dx$$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$= \lim_{N\to\infty} \left[-xe^{-x} + \int e^{-x}dx\right]$$

$$= \lim_{N\to\infty} \left[-xe^{-x} - e^{-x}\Big|_3^N\right]$$

$$= -\lim_{N\to\infty} e^{-x}(x+1)\Big|_3^N$$

$$= -\lim_{N\to\infty} \left(e^{-N}(N+1) - e^{-3}(4)\right)$$

$$= 4e^{-3} - \lim_{N\to\infty} \frac{N+1}{e^N}$$

$$\overset{LH}{=} 4e^{-3} - \lim_{N\to\infty} \frac{1}{e^N}$$

$$= 4e^{-3}$$

Series converges by Integral Test.

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e) $\sum_{k=4}^{\infty} \frac{\sqrt{k^5+1}}{k+9}$

Compare to $\sum_{k=4}^{\infty} \frac{\sqrt{k^5}}{k} = \sum_{k=4}^{\infty} \frac{k^{5/2}}{k^{2/2}} = \sum_{k=4}^{\infty} \frac{1}{k^{3/2}} \to k^{3/2}$

p-series with $p = -\frac{3}{2} \leq 1 \Rightarrow$ diverges

Limit Comparison

$$\lim_{k\to\infty} \frac{\sqrt{k^5+1}}{k+9} \cdot \frac{k}{\sqrt{k^5}} = \lim_{k\to\infty} \frac{(k^5+1)^{1/2}}{k+9} \cdot \frac{1}{k^{3/2}}$$

$$= \lim_{k\to\infty} \frac{(k^5+1)^{1/2}}{k^{5/2}+9k^{3/2}} \cdot \frac{\left(\frac{1}{k^5}\right)^{1/2}}{\frac{1}{k^{5/2}}}$$

$$= \lim_{k\to\infty} \frac{\left(\frac{k^5+1}{k^5}\right)^{1/2}}{1+9/k} = \lim_{k\to\infty} \frac{\left(1+\frac{1}{k^5}\right)^{1/2}}{1+\frac{9}{k}} = \frac{\sqrt{1}}{1} = 1$$

Since $0<1<\infty$, both series diverge.

f) $\sum_{k=5}^{\infty} \left(\frac{2k}{6k-3}\right)^k$

Root test

$$\lim_{k\to\infty} \left[\left(\frac{2k}{6k-3}\right)^k\right]^{1/k} = \lim_{k\to\infty} \frac{2k}{6k-3} = \frac{1}{3} < 1$$

Converges

6. a) $\sum_{k=1}^{\infty} \frac{2^{2k+3}}{5^{k+1}} = \sum_{k=1}^{\infty} \frac{(2^2)^k\, 2^3}{5^k \cdot 5^1} = \sum_{k=1}^{\infty} \frac{8}{5}\left(\frac{4}{5}\right)^k$

$= \frac{8}{5}\left(\frac{4}{5}\right) + \frac{8}{5}\left(\frac{4}{5}\right)^2 + \dots$

$\downarrow$

$a = \frac{32}{25}$ $\quad r = \frac{4}{5}$ $\quad |r| = \frac{4}{5} < 1$ converges

$$= \frac{\frac{32}{25}}{1 - \frac{4}{5}} = \frac{\frac{32}{25}}{\frac{1}{5}} \cdot \frac{5}{1} = \frac{32}{5}$$

b) $\sum_{k=3}^{\infty} \frac{1}{k(k+2)}$ Probably telescoping. Use Partial Fractions

$$\frac{1}{k(k+2)} = \frac{A}{k} + \frac{B}{k+2}$$

$$= \frac{Ak + 2A + Bk}{k(k+2)} = \frac{(A+B)k + 2A}{k(k+2)}$$

$2A = 1 \Rightarrow A = \frac{1}{2}$

$A + B = 0 \Rightarrow \frac{1}{2} + B = 0 \Rightarrow B = -\frac{1}{2}$

$= \frac{1}{2}\sum_{k=3}^{\infty}\left(\frac{1}{k} - \frac{1}{k+2}\right)$

Look at first few partial sums:

$S_3 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right)$

$S_4 = \frac{1}{2}\left[\left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right)\right]$

$$S_5 = \frac{1}{2}\left[\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)\right]$$

$$S_6 = \frac{1}{2}\left[\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\frac{1}{4}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{5}}-\frac{1}{7}\right)+\left(\cancel{\frac{1}{6}}-\frac{1}{8}\right)\right]$$

$$S_7 = \frac{1}{2}\left[\left(\frac{1}{3}-\cancel{\frac{1}{5}}\right)+\left(\frac{1}{4}-\cancel{\frac{1}{6}}\right)+\left(\cancel{\frac{1}{5}}-\cancel{\frac{1}{7}}\right)+\left(\cancel{\frac{1}{6}}-\frac{1}{8}\right)+\left(\cancel{\frac{1}{7}}-\frac{1}{9}\right)\right]$$

* First 2 terms and last 2 terms only remain

$\vdots$

$$S_n = \frac{1}{2}\left[\frac{1}{3}+\frac{1}{4}-\frac{1}{n+1}-\frac{1}{n+2}\right]$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{n+1}-\frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left(\frac{4+3}{12}\right)$$

$$= \frac{7}{24}$$ ✓

7 a) $\sum_{k=1}^{\infty}\frac{2^k}{k!}$ Use Generalized Ratio Test

$$\lim_{k\to\infty}\left|\frac{2^{k+1}}{(k+1)!}\cdot\frac{k!}{2^k}\right| = \lim_{k\to\infty}\left|\frac{\cancel{2^k}\cdot 2^1\cdot \cancel{k!}}{(k+1)\cancel{k!}\,\cancel{2^k}}\right|$$

$$= 2\lim_{k\to\infty}\frac{1}{k+1}$$

$$= 0 < 1$$

Converges absolutely

b) $\sum_{k=1}^{\infty} (-1)^k \frac{\sqrt{k}}{k+3}$

Does this converge? Use Alternating Series Test

1) $\lim_{k \to \infty} \frac{k^{1/2}}{k+3} \cdot \frac{\frac{1}{k}}{\frac{1}{k}} = \lim_{k \to \infty} \frac{\frac{1}{k^{1/2}}}{1 + \frac{3}{k}} = 0$ ✓

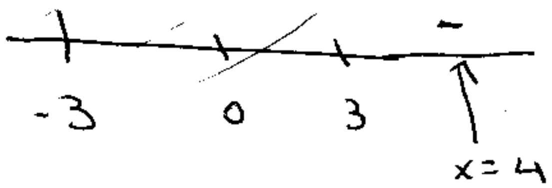
2) Decreasing? $f(x) = \frac{x^{1/2}}{x+3}$

$$f'(x) = \frac{\frac{1}{2}(x+3)\,x^{-1/2} - x^{1/2}}{(x+3)^2}$$

$$= \frac{\frac{x+3}{2\sqrt{x}} - \frac{2x}{2\sqrt{x}}}{(x+3)^2} = \frac{-x+3}{2\sqrt{x}\,(x+3)^2}$$

$-x+3 = 0 \Rightarrow x = 3$

Den $= 0 \Rightarrow x = 0, -3$



Yes

Converges by Alt. Series Test

Does $\sum_{k=1}^{\infty} \left| (-1)^k \frac{\sqrt{k}}{k+3} \right| = \sum_{k=1}^{\infty} \frac{\sqrt{k}}{k+3}$

Compare to $\frac{\sqrt{k}}{k} = \frac{1}{k^{1/2}}$ p-series with

$p = \frac{1}{2} \leq 1 \Rightarrow$ diverges

Limit Comparison

$$\lim_{k\to\infty} \frac{(k)^{1/2}}{k+3} \cdot \frac{k^{1/2}}{1} = \lim_{k\to\infty} \frac{k}{k+3} = 1, \quad 0<1<\infty$$

Diverges

So, $\sum(-1)^k \frac{\sqrt{k}}{k+3}$ is conditionally convergent

c) $\sum_{k=1}^{\infty}(-1)^k \frac{\sqrt{k}}{\sqrt{k+3}}$

1) $\lim_{k\to\infty} \frac{\sqrt{k}}{\sqrt{k+3}} = 1$

Can't use Alt. Series Test. But, diverge by Divergence Test

Also can try Generalized Ratio Test

$$\lim_{k\to\infty} \frac{\sqrt{k+1}}{\sqrt{k+4}} \cdot \frac{\sqrt{k+3}}{\sqrt{k}}$$

$$= \lim_{k\to\infty} \sqrt{\frac{k^2+4k+3}{k^2+4k}} = 1$$, inconclusive

8 a) $\sum_{k=1}^{\infty} \frac{(x-2)^k}{k!}$ Generalized Ratio Test

$$\lim_{k\to\infty} \left| \frac{(x-2)^{k+1}}{(k+1)!} \cdot \frac{k!}{(x-2)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{(x-2)^k}(x-2)\cancel{k!}}{(k+1)\cancel{k!}\cancel{(x-2)^k}} \right|$$

$$= |x-2| \lim_{k\to\infty} \left| \frac{1}{k+1} \right| = 0 < 1$$

Converges $\forall x$:

Interval: $(-\infty, \infty)$

Radius $= \infty$

b) $\sum_{k=1}^{\infty} \frac{(x+1)^k}{k3^k}$ Generalized Ratio Test

$$\lim_{k\to\infty} \left| \frac{(x+1)^{k+1}}{(k+1)3^{k+1}} \cdot \frac{k3^k}{(x+1)^k} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\cancel{(x+1)^k}(x+1)^1 \cdot k \cdot \cancel{3^k}}{(k+1)\cancel{3^k} \cdot 3^1 \cancel{(x+1)^k}} \right|$$

$$= \left| \frac{x+1}{3} \right| \lim_{k\to\infty} \left| \frac{k}{k+1} \right|$$

$$: |x+1| < 3 \Leftrightarrow -3 < x+1 < 3 \Leftrightarrow$$

$$R = 3$$

Test endpoints

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$x = -4 \quad \sum_{k=1}^{\infty} \frac{(-3)^k}{k3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$

1) $\lim_{k \to \infty} \frac{1}{k} = \frac{1}{\infty} = 0$ ✓

2) Decreasing $\frac{1}{k+1} < \frac{1}{k}$ ✓

$f'(x) = \frac{x3^x(0) - 1[x \ln 3 \cdot 3^x + 3^x]}{(x3^x)^2}$

$= \frac{-3^x(1 + x\ln 3)}{(x3^x)^2} = 0$

$1 + x\ln 3 = 0 \Rightarrow x = -\frac{1}{\ln 3} \approx -0.910$

Undefined at $x = 0$

$-\frac{1}{\ln 3}$ 0 $x = 1$ $+$ $+$ $-$

Converges by Alternating Series Test

$x = 2 \quad \sum_{k=1}^{\infty} \frac{3^k}{k3^k} = \sum \frac{1}{k}$ diverges $(p = 1)$

Compare to $\sum_{k=1}^{\infty} \frac{1}{3^k}$ which is a geometric series with $|r| = \frac{1}{3} < 1 \Rightarrow$ converges

Notice that

$k3^k > 3^k$ for $k \geq 1 \Rightarrow$

$\frac{1}{k3^k} < \frac{1}{3^k}$, so they both converge

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$$\boxed{\begin{array}{l}\text{Interval: } -4 \le x < 2 \\ R = \frac{2-(-4)}{2} = 3\end{array}}$$

c) $\sum_{k=1}^{\infty} k!\, x^k$ Generalized Ratio Test

$$\lim_{k\to\infty} \left| \frac{(k+1)!\, x^{k+1}}{k!\, x^k} \right| = \lim_{k\to\infty} \left| \frac{(k+1)\cancel{k!}\, x^k x^1}{\cancel{k!}\, x^k} \right|$$

$$= |x| \lim_{k\to\infty} (k+1) = \infty$$

Diverge for all x except $x = 0$

Interval: $x = 0$

Radius: $R = 0$

d) $\sum_{k=1}^{\infty} \frac{(x+4)^k}{6^k}$ $\sum_{k=1}^{\infty} \left(\frac{x+4}{6}\right)^k$

Geometric series only converges when $|r| < 1$

$$\left|\frac{x+4}{6}\right| < 1 \Leftrightarrow -6 < x+4 < 6$$
$$\Leftrightarrow -10 < x < 2$$

Interval of convergence: $(-10, 2)$

Radius: $R = 6$