Broadening the appeal of the math major at Princeton

DRAFT - Feb 7, 2001

1 Introduction

The quality of the undergraduate education for Princeton mathematics majors is excellent, and has been so for many years. The number of students majoring in mathematics is very small, however. The table below gives statistics for Princeton and a few other institutions, for the last five years. The normalized number of math majors is defined as $1,000 \cdot \#\{\text{seniors majoring in math}\}/\#\{\text{seniors university-wide}\})$.

School	Year	# of senior math majors	# of seniors	normalized # of math majors
Princeton	1997	11	1113	9.88
	1998	12	1132	10.60
	1999	15	1200	12.50
	2000	14	1121	12.49
	2001 (Est.)	11	1112	9.89
Harvard	1997	19	1887	10.07
	1998	17	1845	9.21
	1999	21	1893	11.09
	2000	24	1919	12.51
	2001 (Est.)	20	1950	10.26
Berkeley	1997	82	5412	15.15
	1998	89	5633	15.80
	1999	95	6169	15.40
	2000	114	6175	18.46
	2001 (Est.)	125	6200	20.16
Yale	1997	20	1298	15.41
	1998	18	1304	13.80
	1999	15	1352	11.09
	2000	24	1356	17.79
	2001 (Est.)	20	1360	14.71
Chicago	1997	27	811	33.29
	1998	24	877	27.36
	1999	43	894	48.09
	2000	44	956	46.03
	2001 (Est.)	39	1000	39.00

The normalized number of math majors at Princeton is lower than at the other institutions, except for Harvard; it should be noted, however, that in absolute numbers Harvard has between 50 and 100% more majors, for a mathematics faculty that is about 2/3 the size of Princeton's. At Princeton, the mathematics major thus seems to have a rather low appeal when compared to other institutions. In the past few months, a committee consisting of Manjul Bhargava (graduate student and instructor), Jordan Ellenberg (assistant professor), Robert Fernholz (graduate alumn; chief investment officer with INTECH, Princeton), Simon Kochen (director of undergraduate studies), Mark Tygert (senior, majoring in mathematics) and Ingrid Daubechies (chair of committee) have met and discussed several ways in which the math major at Princeton could attract and retain a larger number of students. This is the report of that committee. It contains a discussion of where additional mathematics majors might be found, as well as curriculum and other suggestions which we hope the Mathematics Department will consider and possibly implement.

1.1 Where are the "missing" math majors?

Anecdotal evidence suggests that a large fraction of students who enter Princeton with the stated intent of majoring in mathematics decide during their freshman or early in their sophomore year to pursue a different major instead. (We could not obtain the numbers to check quantitatively whether this fraction is larger at Princeton than elsewhere.)

There are several factors that can explain a large attrition among "intended math majors". At the end of high school, many students have had more exposure to mathematics than to many other possible majors; after they become more familiar with the range of possible directions that their education at Princeton can take, they may opt for another major than mathematics that suits them better and of which they hadn't thought earlier. Moreover, even though students may have had a relatively high exposure to mathematics in high school, it is typically very different, especially in rigor, from the mathematics at a serious college level. Learning what "true mathematics" is, some students may find that the subject in which they excelled in high school has "transformed" into one in which they are not very good, and they wisely decide to major in another department. Finally, some of the "intended" math majors may never have intended to major in mathematics, even though they declared this intent to add gravitas to their application.

There are also students, however, who honestly intended to major in mathematics, who love math, who show clear talent and are excellent in the last math course in which we see them, but who get "turned off" somehow. Some of these students then sign up for a Certificate in Applied Mathematics (which has had more enrollments than the math major in recent years: 20 seniors in 2001, 16 in 2000, 20 in 1999, versus resp. 11, 14, 15 senior math majors in the same years). There are also other students who sign up for engineering, but find courses too "recipe-like" to their taste and switch to a BA; some of them would be excellent and talented math majors, but only rarely do they end up in the math department. We should do a better job at attracting and retaining such students than we are doing presently.

Apart from anecdotal evidence that there are such students, we can also look at the paths taken by our present math majors. Our brightest math majors almost unfailingly apply to graduate school in mathematics. Although good graduate schools in economics, statistics and computer science welcome applicants who have majored in mathematics (and in fact may prefer such applicants to students who majored in economics, statistics or computer science, respectively), hardly any of our majors follow this path. (With the exception of computer science, which some of our majors in recent years have picked as their area for graduate study.) Of course we don't want to argue that we should convince the majors who want to go to grad school in mathematics that they should choose another option - the point here is that we should **also** have, among our math majors, many more bright and mathematically talented students who do not seek to become professional mathematicians, such as e.g. future economists, statisticians, theoretical computer scientists with a special taste for mathematics. At present, they probably do take some of our 200-level courses (203/204), but don't explore further within the mathematics department offerings. If we are more successful at retaining such strong students, then we will probably also, on occasion, recruit some of them into pursuing mathematics even beyond their undergraduate degree.

1.2 Summary of our suggestions

Our first set of suggestions concerns curriculum changes: we propose the introduction of new courses at several different levels.

- Different entries into the math major than the MAT 215-217-218 sequence or the MAT 203-204 + MAT 215 sequence. At present, MAT 215 holds the key to entering the math major. It serves two purposes: 1) it exposes the students to rigorous mathematics, probably for the first time; 2) it is the basis for their more advanced analysis training, to be carried on further at the 300-level. We propose that other entries into the math major be possible, via rigorous 200-level courses, to be developed, that would not be based on analysis. For students entering the math major without having taken 215, their analysis training needs would be addressed by redesigning 314.

- Special 300- or 400-level courses developed and taught by junior faculty. By giving junior faculty the opportunity to develop and teach special 300/400 level courses in fields or topics that they find especially appealing and that are not normally offered in our standard curriculum, a wider range of courses would be available to majors.

- Units on specialized topics in mathematics that would be developed and taught by graduate students, with some supervision by faculty. Two half-semester units or one semester-long unit could be the equivalent of a junior seminar.

Section 2 gives more details on each of these proposals. Note that the proposed teaching initiatives would also fit very naturally within the larger framework of the VIGRE "philosophy" of the Department.

In addition to the proposed new curriculum development, additional initiatives, aimed at building a better contact between undergraduates (especially, but not only, in the 200-level courses) and the mathematics department, will certainly help in raising the appeal of the mathematics major. In the final section, we list suggestions for contact-enhancing or community-building initiatives.

2 Proposed curriculum changes and additions

2.1 Non analysis based courses giving acces to the math major

For students who have never had previous exposure to proofs, 215 is sometimes a huge turnoff. Some students feel they have to work extremely hard to produce complicated proofs of theorems that they "already know" from high school. The value and elegance of proof simply does not resonate with these students, and they leave the major. Here follow two descriptions of courses that we expect would help retain such students in the major, to be offered in parallel with Mat 215, as alternatives for the ramp into the math department.

2.1.1 Mat 216: Numbers, equations and proofs

This course would introduce the notion of theorem and rigorous proof through the detailed study of elementary number theory. We hope Math 216 would attract algebraically-minded students who might shy away from Math 215. The course might also have the side effect of attracting some upperclassmen not majoring in math, as Math 101 at Harvard does.

There might be some overlap with Math 323 in the definition of groups and fields, but other than that Math 216 would fit in well with our curriculum. Since we don't usually offer an undergraduate number theory course, Math 215 would cover some foundational material (primes and unique factorization, the arithmetic of $\mathbf{Z}/p\mathbf{Z}$, Diophantine equations) which is skipped or glanced over in our current curriculum. About half of a committee of undergraduates formed last year to discuss the math major felt that adding a freshman-level course in elementary number theory would help the department; the other half weren't sure it would help, but agreed that it wouldn't hurt.

Below is a skecth for a sample syllabus for the course. Any comments on whether the proposed course would be too fast, too slow, too concrete, or too abstract would be most welcome. (Please commuinicate them to Jordan Ellenberg, author of this subsection.) Enough room has been left so that the instructor can spend the last three weeks working through a somewhat more advanced topic according to his or her interests. This should help keep up faculty interest in teaching the course.

Students often seem to feel that Math 215 doesn't "teach them how to come up with a proof." It isn't clear how effective direct instruction in proof techniques can be; however, in light of these complaints, I think it would be good to teach 216 with an eye towards exposing those strategies of thinking which all mathematicians know, but few speak of in books. In addition to the basic techniques of induction and proof by contradiction, these might include: reduction to a simpler case, working backwards from the desired result, working an example and generalizing the argument, replacing the problem by a generalization, and conjecturing sensibly.

Assigning a written paper in addition to the final exam would be an excellent place for our students to get a first taste of the challenge of mathematical writing.

Syllabus

Week 1 The hypotenuse problem: How many ways can n be expressed as a^2+b^2? Fermat's characterization of sums of two squares.

Week 2 Arithmetic of $\mathbf{Z}[i]$. Unique factorization in $\mathbf{Z}$ and $\mathbf{Z}[i]$. Infinitude of primes in $\mathbf{Z}$ and $\mathbf{Z}[i]$.

Week 3 Solutions mod p to $x^2 + y^2 = 1$ and $x^4 + y^4 = 1$. Arithmetic of $\mathbf{Z}/n\mathbf{Z}$. Chinese remainder theorem. What is i mod p?

Week 4 Solutions in $mathbfQ$ to $x^2 + y^2 = 1$ and $x^4 + y^4 = 1$. Definition of field. What do fields have in common? What kinds of fields are there? Possible sizes of a finite field. Fundamental theorem of algebra.

Week 5 Solutions in fields to $x^2 = a$. Quadratic residues. Structure of $(\mathbf{Z}/p\mathbf{Z})^*$. Definition of cyclic group. Primitive roots. (Optional: public-key cryptography.) Pascal's triangle mod p and Fermat's little theorem.

Week 6 : Quadratic reciprocity. Chinese remainder theorem and structure of $(\mathbf{Z}/n\mathbf{Z})^*$.

Week 7 Arithmetic functions: $\phi(n)$ and $r(n)$. Principle of Inclusion and Exclusion, and its application to computation of $\phi(n)$. The notion of an asymptotic approximation: asymptotics of $\phi(n), r(n), \pi(n)$. The probability that two integers are relatively prime.

Week 8 Mobius inversion. Generating functions attached to arithmetic functions.

Week 9 Waring's problem. Sums of three squares. Minkowski's theorem and the four-square theorem.

Week 10-12 An advanced topic to be chosen by the instructor. Possibilities include: sphere packing and error-correcting codes, p-adic numbers, arithmetic of quadratic forms, transcendental numbers and Liouville's theorem, Conway's numbers, finite projective plane and combinatorial designs, elliptic curves, transfinite numbers....

Textbook

There are many excellent elementary number theory books. One could teach this course out of Hardy and Wright's *An Introduction to the Theory of Numbers*, which contains all the material described above, in addition to much interesting historical detail, is very well written, and will give the students a strong impression that they are studying "the real thing." Joseph Silverman's *A Friendly Introduction to Number Theory* is quite good, and more elementary than Hardy and Wright. Another possibility would be the book of Niven-Zuckerman-Montgomery.

2.1.2 Mat 214: Introduction to Proofs

The primary goal of the course is teaching students to appreciate the importance and elegance of proofs, as well to write proofs, while at the same time introducing fundamental tools and notions of mathematics. The course does not concentrate on just one branch of mathematics (e.g., only analysis), but introduces various kinds of rigorous thinking from a selection of mathematical disciplines, typically including (but not limited to) set theory, logic, number theory, group theory, and analysis.

For example, a sample course outline might be as follows.

- Begin with some simple puzzles and examples that immediately convince students that proofs are "cool", e.g., problems of parity: why can't one domino-tile a checkerboard with one square removed (easy); why can't one domino-tile a checkerboard with two diagonally-opposite squares removed (harder, but simple and pleasing once the proof is revealed), etc.
- Start introducing the basic tools of mathematical proof, keeping in mind neat examples and problems. Quantifiers, basic set theory and logic.
- Number theory: Pattern recognition, proof by induction. Modular arithmetic and applications to Diophantine equations.
- Group theory: Symmetries, platonic solids, and basic theory of finite groups.
- Analysis: motivation for and basics of rigorous analysis, including epsilon-delta definition of continuity.

A course such as this might allow students not only to appreciate the beauty and value of proof, but also to realize to some extent their own strengths and interests in mathematics.

Traditionally, the course Math 101 at Harvard (which inspired this proposed course) consists partly of students considering mathematics as a possible major, and partly of students from computer science, chemistry, physics, philosophy and other disciplines who are interested in getting an overview of the methods of mathematical thinking.

A sample of a successful course outline in Math 101 taught recently by Curtis McMullen (who chose to concentrate on set theory, group theory, and knot theory) can be found at

http://abel.math.harvard.edu/ ctm/home/text/class/harvard/101/00/html/index.html

2.1.3 Impact of entering the math major via 214 or 216 on the analysis training at the 300 level and beyond

Students who would choose the 203/204 + (216 or 214) path to enter the math major would have learned about rigorous proofs, and would therefore be ready to venture further in serious mathematics courses. If they decide to take the 330/331/332 analysis sequence developed by Eli Stein, they would first have to take 215. If they decide to design their math course portfolio less concentrated on analysis, then they would nevertheless need to study real and complex analysis, even though they didn't take 215. This could be addressed by redesigning 314 so that it covers material that would normally be seen in 215 and 218, as well as Lebesgue integration. (This describes basically the material in "baby Rudin", possibly dropping the inverse and implicit function theorem chapter.) Although this seems like a lot of material, we feel confident that this could be done with these students, who will already have a greater mathematical maturity after having gone through at least one rigorous course. For complex analysis, these math majors would still take 319.

2.2 A variable assortment of 300/400-level courses taught by junior faculty

A few years ago, Dennis McLaughlin, while a junior faculty member in the mathematics department, developed a financial mathematics course, that was very successful when he taught it. It attracted a large number of students: there was great interest, and at that time there were no other financial mathematics courses offered within the university. After McLaughlin left, it was difficult for the mathematics department to continue staffing this course, since no other (junior or senior) faculty member could teach it with the same enthusiasm as McLaughlin. In addition, financial math courses were also developed elsewhere (Economics and the newly created Operations Research and Financial Engineering department), which undercut the demand for the math department course. In the long run, this course was therefore not the success initially hoped for. However, viewed as an example of a short-term course, developed by an enthusiastic junior faculty member who was given the opportunity to do so, and who could then teach the course as long as he stayed at Princeton and there was demand, it is in fact a striking success story.

We propose that junior faculty be offered the possibility of developing courses at the 300- or 400-level, which need not (but may) be kept on the books when they leave - this should be evaluated on a case-by-case basis for every course. In the year the course is developed, the junior faculty would have teaching relief; in subsequent years, it would be guaranteed that they can teach the course as one of their four yearly course assignments. This would be a valuable teaching experience for the junior faculty; at the same time the math department and its majors would benefit by having a wider range of courses, and having additional enthusiastically taught offerings. There would be no pressure on junior faculty to participate in this initiative; junior faculty who are interested would be invited to submit a proposal for their course to the department.

2.3 Special-topics units taught by graduate students to juniors/seniors

This proposal is inspired by the "tutorials", developed and taught at Harvard by graduate students. These special-topics units could cover 1/2 or a full semester; they would take the form of lectures by the graduate student as well as interactive sessions with the enrolled undergraduates, in which they report on their readings or their own work, or in which assignments get worked out. Undergraduates would get credit for these: two 1/2-semester units or one full-semester unit could count as junior seminar. Whether a unit is full- or half-semester length depends on the topic and the graduate student; it is important that the pace for 1/2-semester units not be too fast as the result of cramming too much into too little time. The initiative should probably be under the (loose?) supervision of a faculty member so as to satisfy the requirements of the Dean of the College.

Again, there would be no pressure on graduate students to participate. A student developing and offering such a unit would not be asked to grade or teach for another course at the same time. Interested students would be asked to submit proposals to the department, which would then decide which ones to implement in each given semester. At Harvard, there are every year more proposals than slots: even though the "tutorials" require more work from the graduate TA, they are also more satisfying. They also provide excellent teaching experience to the TA in charge.

Examples of tutorials taught in recent years at Harvard include:

- the symmetric group, its representations, and applications;
- computational algebraic geometry;
- model theory;
- p-adic numbers;
- selected topics in combinatorial number theory;
- knots and braids.

As with the special courses taught by junior faculty, we expect that the success of our tutorials will lie not only in the variety of topics chosen, but also in the enthusiasm of the instructors.

3 Other, non-curriculum initiatives

In discussions among members of the committee, or of committee members with students and faculty, many other suggestions came up:

- Active Math Club. This year is seeing an exceptionally active Math Club. This has happened in the past as well, and it is great when it happens. It provides a strong support group for the students, and helps keeping their mathematical interest alive. There have been many years also in which the math Club was virtually inexistent. Because of the unavoidable rapid turnover of departmental majors, every year in which the Math Club is very active has to reinvent the Math Club wheel from scratch. It would be good to have a faculty member who is very interested in interacting with undergraduates outside classes be in charge of keeping the Math Club archives, and of helping to start it up again every year. This faculty member could also help the Math Club in organizing outreach activities if they wish.

- "Marketing": Make sure undergraduates know that many career paths appreciate math majors especially. Redesign math brochure, make web page "Looking for an exciting and very rewarding major?" (or such) to be clicked from Math portal. Should have information about the major but also comments from alumni about appreciation of math majors.

- Have speakers at start of Spring semester, to talk to sophomores, about majoring in math and going on to interesting careers. This could include alumni who now work at Goldman-Sachs and the like, but also alumni who went on to graduate school in other disciplines requiring a solid mathematics background. (This requires keeping track of/staying in contact with alumni ..!)

- Early contact: Can we try to contact undergraduates with interest in mathematics, sending them material about the math major even before they show up at Princeton? (For instance: can we send them a letter during the summer?)

- Show that there is mathematical life beyond 203/204: Have an "ad" session near the end of the semester for each 203 and 204 section, in which students get a glimpse of the further adventures in mathematics that they can experience in further courses. (presentation by the instructor of such a further course)

- A "club" for 203/204 students: Such a club would foster community spirit, and be a framework in which students could talk about mathematics. We could even go so far as to have a T-short contest: ask students to submit designs for a "mathematical" T-shirt, and give all the club members a T-shirt with winning design.

- Portfolio of summer opportunities for undergrads: There are many interesting REU opportunities for students interested in mathematics. We should advertize them, and help students apply.

- Drop the comprehensive exams. They don't serve any purpose now, and other departments (e.g. Physics) dropped them for this reason years ago.