ON THE ZERO-TEMPERATURE LIMIT OF GIBBS STATES

JEAN-RENÉ CHAZOTTES AND MICHAEL HOCHMAN

ABSTRACT. We exhibit Lipschitz (and hence Hölder) potentials on the full shift $\{0,1\}^{\mathbb{N}}$ such that the associated Gibbs measures fail to converge as the temperature goes to zero. Thus there are "exponentially decaying" interactions on the configuration space $\{0,1\}^{\mathbb{Z}}$ for which the zero-temperature limit of the associated Gibbs measures does not exist. In higher dimension, namely on the configuration space $\{0,1\}^{\mathbb{Z}^d}$, $d \geq 3$, we show that this non-convergence behavior can occur for finite-range interactions, that is, for locally constant potentials.

1. INTRODUCTION

1.1. **Background.** The central problem in equilibrium statistical mechanics or thermodynamic formalism is the description of families of Gibbs states for a given interaction. Their members are parametrized by inverse temperature, magnetic field, chemical potential, etc. The ultimate goal is then to describe the set of Gibbs states as a function of these parameters. The zero temperature limit is especially interesting since it is connected to "ground states" , that is, probability measures supported on configurations with minimal energy [13, appendix B.2].

The purpose of this article is to shed some light on the zero-temperature limit in the case of classical lattice systems, that is, systems with a configuration space of the form $F^{\mathbb{Z}^d}$, where F is a finite set. We consider translation or shift invariant, summable interactions $\Phi = (\Phi_B)_{B\subseteq\mathbb{Z}^d,|B|<\infty}$. Denoting $\mu_{\beta\Phi}$ a Gibbs state of Φ at inverse temperature β, we are interested in the following question:

$$\text{Does the limit } \lim_{\beta\to+\infty} \mu_{\beta\Phi} \text{ exist?}$$

(Limits of measures should be understood in the weak-* sense). In the present setting, it can be pr oven that weak-* accumulation points of $\mu_{\beta\Phi}$ are indeed ground states, in that they are probability measures which minimize $\int \varphi d\nu$ among all shift-invariant measures ν, where $\varphi : F^{\mathbb{Z}^d} \to \mathbb{R}$ is defined as

$$\varphi(x) := \sum_{B \ni 0} \frac{1}{|B|} \Phi_B(x).$$

This function can be physically interpreted as the contribution of the lattice site 0 to the energy in the configuration x. ([1]) In our setting Gibbs states are also equilibrium states: they maximize the quantity

$$P_\beta(\nu) = \int \beta\varphi d\nu + h(\nu)$$

over all invariant probability measures ν on $F^{\mathbb{Z}^d}$. Here $h(\nu)$ is the Kolmogorov-Sinai entropy of ν, and the supremum is called the (topological) pressure.

1.2. **The one-dimensional case.** Let us make a few remarks about the ergodic perspective. Fix the usual metric

$$d(x,y) = 2^{-\min\{k\,:\,x_i=x_j \text{ for } |i|<k\}}$$

on $F^{\mathbb{Z}}$. For a number of reasons, the usual class of "potentials" $\varphi : F^{\mathbb{Z}} \to \mathbb{R}$ which are studied are Hölder continuous ones. First, for these potentials the Gibbs measure is unique. Second, this class of potentials arises naturally in the theory of differentiable dynamical systems (e.g. Axiom A diffeomorphisms): By choosing a suitable Markov partition of the phase space one can code such a diffeomorphism to a subshift of finite type in $F^{\mathbb{Z}}$ [1], and under this coding smooth potentials lift to Hölder ones. And third, Hölder potentials correspond to the natural objects in statistical mechanics, namely "exponentially decaying" interactions (Φ_B) [12, chapter 5]. We also note that the case when φ is locally constant corresponds to interactions of finite range; see below.

There is a trick, due to Sinai, which allows one to reduce the study to a "one-sided" subshift of finite type of $F^{\mathbb{N}}$ and a potential φ which depends only on "future" coordinates [1]. Thus it suffices to study the one-sided full shift, and our question can be formulated as follows:

When φ is Hölder continuous on $F^{\mathbb{N}}$, when does $\lim_{\beta\to+\infty} \mu_{\beta\varphi}$ exist?

The existence of the zero-temperature limit has been verified in a number of situations, but, surprisingly, a systematic study of this question began only recently. When $d = 1$ and φ is locally constant (i.e. the interaction

[1]Since the interaction is translation-invariant, we can take any lattice site. Other definitions are possible [12, section 3.2], but all lead to the same expected value under a given translation-invariant measure.

is finite range), the zero-temperature limit was proved to exist in [3] and was described explicitly in [9, 4]. In this case, the zero-temperature limit is supported on the union of finitely many transitive subshifts of finite type and is a convex combination of the entropy-maximizing measure on them. The case $d = 1$ with F a countable set was studied in [8, 10].

Another class of examples where convergence may be verified arises as follows. Let $X \subseteq F^{\mathbb{N}}$ be a subshift (a closed non-empty shift-invariant set) and define $\varphi = \varphi_X$ by

$$\varphi(y) = -d(y, X) = -\inf\{d(y, x) \,:\, x \in X\}.$$

This is a Lipschitz function on $F^{\mathbb{N}}$ with $\varphi|_X = 0$ and $\varphi \leq 0$. The ground states of φ_X are then precisely the measures supported on X, and it follows that all accumulation points of $(\mu_{\beta\varphi})_{\beta\geq 0}$ are invariant measures supported on X. In particular, when X has only one invariant measure μ (i.e. is uniquely ergodic), all accumulation points coincide, and we have $\mu_{\beta\varphi} \to \mu$ as $\beta \to +\infty$.

The only example of non-convergence of which we are aware is by van Enter and Ruszel [14]. Their example is of a nearest-neighbor potential model, but is defined over a continuous state space F (the circle).

This state of affairs has led to the belief that over finite state spaces convergence should generally hold. Our first result is a counterexample, showing that this is not the case:

Theorem 1.1. *There exist subshifts $X \subseteq \{0,1\}^{\mathbb{N}}$ so that, for the Lipschitz potential $\varphi_X(y) = -d(y, X)$, the sequence $\mu_{\beta\varphi}$ does not converge (weak-*) as $\beta \to +\infty$.*

This theorem holds more generally for one-sided or two-sided mixing shifts of finite type.

Our construction gives reasonable control over the dynamics of X and of the dynamics, number and geometry of the limit measures. An interesting consequence of the construction is that the set of limit measures need not be convex. We discuss these issues more in section 4.

1.3. **The multi-dimensional case.** Our second result concerns higher dimension: non-convergence can also arise when $d \geq 3$, *even for finite-range interactions*. As we noted, in dimension $d = 1$ the zero temperature limit is known to exist in this case [3, 9, 4].

While the methods used in the one-dimensional case are fairly classical and quite well-known in the dynamics community, the study of zero-temperature limits and ground states in higher dimensions turns out to be closely connected to symbolic dynamics, and our results rely heavily on recent progress in understanding of multidimensional subshifts of finite type, where computation theory plays a prominent role. Recall that a shift of finite type $X \subseteq \{0,1\}^{\mathbb{Z}^3}$ is a subshift defined by a finite set L of patterns and the condition that $x \in X$ if and only if no pattern from L appears in x. Given $L \subseteq \{0,1\}^E$ one can define the finite-range interaction $(\Phi_B)_{B\subseteq\mathbb{Z}^d,|B|<\infty}$ by

$$\Phi_E(x) = \begin{cases} 0 & x|_E \in L \\ -1/|E| & \text{otherwise} \end{cases}$$

and $\Phi_B = 0$ for $B \neq E$; the associated potential on $\{0,1\}^{\mathbb{Z}^d}$ is

$$\varphi_L(x) := \sum_{B \ni 0} \frac{1}{|B|}\, \Phi_B(x) = \begin{cases} 0 & x|_E \in L \\ -1 & \text{otherwise.} \end{cases}$$

Clearly an invariant measure μ on $\{0,1\}^{\mathbb{Z}^d}$ satisfies $\int \varphi_L d\mu = 0$ if and only if μ is supported on X; thus the shift-invariant ground states are precisely the translation invariant measures on X. In this sense φ_L is similar to φ_X (although there are significant differences, as we shall see in section 5).

The main result of [6] provides a general method for transferring one-dimensional constructions to higher-dimensional SFTs with corresponding directional dynamics. Using this we are able to adapt the construction from theorem 1.1 to prove:

Theorem 1.2. *For $d \geq 3$ there exist locally constant, that is, finite-range, potentials on $\{0,1\}^{\mathbb{Z}^d}$ such that the associated Gibbs measures does not converge as $\beta \to +\infty$.*

The example we construct is of the form φ_L above. As the results from [6] do not apply to $d = 2$ the statement in that case remains open, but we believe it to be true.

In the next section we construct the subshift X of theorem 1.1. Section 3 contains the analysis and proof of theorem 1.1. Section 4 contains some remarks and problems. Section 5 discusses the multidimensional case.

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2. Construction of X

For each $k \geq 0$ we define by induction integers ℓ_k, and finite sets of blocks $A_k, B_k \subseteq \{0,1\}^{\ell_k}$. The construction uses an auxiliary sequence of integers $N_1, N_2, \ldots$, with $N_1 \ldots N_k$ determining A_i, B_i, ℓ_i for $i \leq k$. Here we treat the N_k as given, but in fact at each stage we are free to choose N_{k+1} based on the construction so far, and during the analysis in the next section we impose conditions on the relation between ℓ_k and N_{k+1}.

Begin with $\ell_0 = 5$, and let

$$\begin{aligned} A_0 &= \{00000, 01000\} \\ B_0 &= \{11111, 10111\}. \end{aligned}$$

Next, given A_{k-1}, B_{k-1} and ℓ_{k-1} and the parameter N_k, let c_k be a block containing every block in $(A_{k-1} \cup B_{k-1})^{2^{\ell_{k-1}+1}}$, e.g. enumerate all these blocks and concatenate them.

We proceed in one of two ways, depending on whether k is odd or even. We denote by ab the concatenation of blocks a, b of symbols, and by a^k the k-fold concatenation of a block a.

- If k is odd, let

$$\begin{aligned} A_k &= \{c_k a^{N_k} \,:\, a \in A_{k-1}\} \\ B_k &= \{c_k b_1 b_2 \ldots b_{N_k} \,:\, b_i \in B_{k-1}\}. \end{aligned}$$

- If k is even, set

$$\begin{aligned} A_k &= \{c_k a_1 a_2 \ldots a_{N_k} \,:\, a_i \in A_{k-1}\} \\ B_k &= \{c_k b^{N_k} \,:\, b \in B_{k-1}\}. \end{aligned}$$

Thus A_k, B_k consist of blocks of the same length, which we denote ℓ_k. Note that ℓ_k can be made arbitrarily large by increasing N_k.

If we assume that N_k is large enough then one can identify the occurrences of c_k in any long enough subword of length $2\ell_k$ of a concatenation of blocks from $A_k \cup B_k$. This is shown by induction: first one shows that one can identify the $A_{k-1} \cup B_{k-1}$-blocks, and then c_k is identifiable because it contains blocks from both A_{k-1} and B_{k-1}.

For a set Σ let Σ^* denote the set of all concatenations of elements from a set Σ. Given a finite set $L \subseteq \{0,1\}^*$ let

$$\langle L \rangle = \bigcup_n T^n(L^{\mathbb{N}})$$

denote the subshift consisting of all shifts of concatenations of blocks from L. Note that if $L' \subseteq L^*$, then $\langle L' \rangle \subseteq \langle L \rangle$. Let

$$L_k = A_k \cup B_k,$$

so that $L_{k+1} \subseteq L_k^*$, and define

$$X = \bigcap_{k=1}^{\infty} \langle L_k \rangle .$$

Alternatively, X is the set of points $x \in \{0,1\}^{\mathbb{N}}$ such that every finite block in x appears as a sub-block in a block from some L_k.

3. Analysis of the zero-temperature limit

We make some preliminary observations. For $u \in L_k$ let

$$f_i(u) = \text{frequency of } i \text{ in } u.$$

Then the following is clear from the construction:

Lemma 3.1. *If N_k/ℓ_{k-1} increases rapidly enough, then $f_0(u) > \frac{2}{3}$ for $u \in A_k$ and $f_0(u) < \frac{1}{3}$ for $u \in B_k$.*

In fact it can be shown that X supports two ergodic measures, respectively giving mass $> \frac{2}{3}$ and $< \frac{1}{3}$ to the cylinder $[0]$.

The construction is designed so that the ratio $|A_k|/|B_k|$ fluctuates between very large and very small. More precisely, one may verify the following:

Lemma 3.2. *If N_k/ℓ_{k-1} is sufficiently large, then*

- *If k is odd then $|B_k| > |A_k|^{100}$.*
- *If k is even then $|A_k| > |B_k|^{100}$.*

The next two lemmas show that for certain values of β the measure $\mu_{\beta\varphi}$ concentrates mostly on blocks from L_k. Let

$$Y_k = \{x \in \{0,1\}^{\mathbb{N}} \,:\, x|_{[i,1+\ell_k-1]} \in L_k \text{ for some } i \in [0,\ell_k - 1]\}.$$

$Y_k \subseteq \{0,1\}^{\mathbb{N}}$ is an open and closed set.

Lemma 3.3. *For $\beta = 2^{3\ell_k}$,*

$$\mu_{\beta\varphi}(Y_k) > 1 - 2^{-\ell_k}.$$

Proof. If $x \notin Y_k$, then we certainly have $d(x, X) > 2^{-2\ell_k}$. Therefore,

$$\begin{aligned}\int \beta\varphi d\mu_{\beta\varphi} &= \int -\beta d(y, X) d\mu_{\beta\varphi}(y) \\ &< -2^{3\ell_k} \cdot 2^{-2\ell_k} \mu_{\beta\varphi}(\{0,1\}^{\mathbb{N}} \setminus Y_k) \\ &= 2^{\ell_k}(\mu_{\beta\varphi}(Y_k) - 1).\end{aligned}$$

Since $h(\mu_{\beta\varphi}) \leq 1$ we have

$$P_\beta(\mu_{\beta\varphi}) \leq \int \beta\varphi d\mu_{\beta\varphi} + 1 \leq 2^{\ell_k}\mu_{\beta\varphi}(Y_k) - (2^{\ell_k} - 1).$$

Finally, choosing ν to be an invariant measure supported on X we have $P_\beta(\nu) = h(\nu) \geq 0$, hence $P_\beta(\mu_{\beta\varphi}) \geq P_\beta(\nu) \geq 0$. Combining these we have the desired inequality. □

Lemma 3.4. *For $\beta = 2^{3\ell_k}$, for all large enough n at least half of the mass of $\mu_{\beta\varphi}$ is concentrated on sequences $u \in \{0,1\}^n$ which can be decomposed as*

$$u = \diamond v_1 \diamond \ldots v_2 \diamond \ldots v_m \diamond \tag{3.1}$$

where $v_i \in L_k$, the symbol $\diamond$ represents blocks of $0, 1$'s (which may vary from place to place), and at least a $(1 - 2^{-\ell_k})$-fraction of indices $j \in [0, n)$ lie in one of the v_i.

Proof. Let

$$Y'_k = \{x \in \{0,1\}^{\mathbb{N}} \,:\, x|_{[0,\ell_k-1]} \in L_k\}.$$

Since $Y_k = \bigcup_{i=0}^{\ell_k-1} T^{-i}Y'_k$, from the previous lemma and shift-invariance of $\mu_{\beta\varphi}$, we see that

$$\mu_{\beta\varphi}(Y'_k) > \frac{1}{\ell_k}(1 - 2^{-\ell_k}).$$

Since $\mu_{\beta\varphi}$ is ergodic (being a Gibbs measure), by the ergodic theorem, for n large enough at least half the mass of $\mu_{\beta\varphi}$ is concentrated on points $x \in X$ such that

$$\frac{1}{n}\#\{i \in [0, n-1] \,:\, T^i x \in Y'_k\} > \frac{1}{\ell_k}(1 - 2^{-\ell_k})$$

Since the beginning of an L_k-block is uniquely determined (because the c_k blocks can be identified uniquely) we also have that if $y \in Y'_k$, then $T^i y \notin Y'_k$ for all $1 \leq i < \ell_k$. Thus if u is the initial n-segment of a point x as above, then there is a representation of u of the desired form. □

Next, we obtain a lower bound on $P_\beta(\mu_{\beta\varphi})$:

Lemma 3.5. *If k is odd and $\beta = 2^{-3\ell_k}$ then*

$$P_\beta(\mu_{\beta\varphi}) > \frac{\log|B_k|}{\ell_k} - 2^{3\ell_k}2^{-\ell_k 2^{\ell_k}}.$$

A similar statement holds for $\langle A_k \rangle$.

Proof. Let ν be the entropy-maximizing measure on $\langle B_k \rangle$. Since

$$P_\beta(\mu_{\beta\varphi}) \geq P_\beta(\nu) = h(\nu) - \int \beta\varphi d\nu$$

and $h(\nu) = \frac{\log|B_k|}{\ell_k}$ it suffices to show that

$$\int \beta\varphi(y)d\nu(y) > -2^{3\ell_k}2^{-\ell_k 2^{\ell_k}}. \tag{3.2}$$

Indeed, if $y \in \langle B_k \rangle$ then $y = ab_1b_2\ldots$ where $b_i \in B_k$ and a is the tail segment of a block in B_k. Since, by construction, every concatenation of $2^{\ell_k}+1$ blocks from B_k appears in X, it follows that the initial segment of y of length $\ell_k 2^{\ell_k}$ appears in X, and therefore $d(y, X) < 2^{-\ell_k 2^{\ell_k}}$, and (3.2) follows. □

The last component of the proof is to show that, for $\beta = 2^{3\ell_k}$, the measures $\mu_{\beta\varphi}$ concentrate alternately B_k and A_k. This is essentially due to the fact that by the lemmas above, $\mu_{\beta\varphi}$ is mostly supported on the blocks of L_k, and because of the appearance of entropy in the variational formula, it tends to give approximately equal mass to these blocks. Since $|B_k|/|L_k| \to 1$ along the odd integers and $|A_k|/|L_k| \to 1$ along the even ones, this implies that $\mu_{\beta\varphi}$ will alternately be supported mostly on B_k and A_k.

Here are the details. Denote by $[u]$ the cylinder set defined by a block $u \in \{0,1\}^*$.

Proposition 3.6. *If N_k increases sufficiently rapidly then for and $\delta > 0$ and all sufficiently large k, if we set $\beta_k = 2^{-3\ell_k}$ then: if k is odd then*

$$\mu_{\beta_k\varphi}\Big(\bigcup_{u\in B_k}[u]\Big) \geq 1-\delta \tag{3.3}$$

and if for k even then

$$\mu_{\beta_k\varphi}\Big(\bigcup_{u\in A_k}[u]\Big) \geq 1-\delta.$$

Proof. We assume that N_k increases rapidly enough for the previous lemmas to hold and furthermore that, writing $H(t) = -t\log t - (1-t)\log(1-t)$,

$$\frac{H(2^{-\ell_k})}{\log|B_k|/\ell_k} \to 0$$

and

$$\frac{2^{3\ell_k}2^{-\ell_k 2^{\ell_k}}}{\log|B_k|/\ell_k} \to 0$$

as $k \to \infty$ along the odd integers, and similarly, with A_k in place of B_k, as $k \to \infty$ along the even integers. This condition is easily satisfied by choosing N_k large enough at each stage, since for fixed k, as we increase N_k the numerator decays to 0 but the denominator does not.

Under these hypotheses we establish the proposition for odd k, the case of even k being similar. Thus, we assume that $|B_k| > |A_k|^{100}$. Suppose that (3.3) fails for some k. For all large enough n lemma 3.4 implies that at least half the mass of $\mu_{\beta_k\varphi}$ is concentrated on points whose initial n-segment is of the form (3.1), and, by the ergodic theorem and the assumed failure of (3.3), if n is large then with $\mu_{\beta_k\varphi}$-probability approaching 1 the fraction of v_i's that belong to A_k in the decomposition (3.1) is at least δ.

For such an n we now perform a standard estimate to bound the entropy of $\mu_{\beta_k\varphi}$. Applying e.g. Stirling's formula, the number of different ways the $\diamond$'s can appear in u is

$$\leq \sum_{r<2^{-\ell_k}\cdot n} \binom{n}{r} \leq 2^{H(2^{-\ell_k})n}.$$

The positions of $\diamond$'s determines the positions of the v_i, and given this, the number of ways to fill in the v_i so that at least a δ-fraction of them come from A_k is bounded from above by

$$\sum_{r=\delta n/\ell_k}^{n/\ell_k} |A_k|^r |B_k|^{n/\ell_k - r} \leq \frac{n}{\ell_k}\cdot |A_k|^{\delta n/\ell_k}|B_k|^{(1-\delta)n/\ell_k}.$$

Using the bound $|A_k| \leq |B_k|^{1/100}$ and setting

$$\delta' = \delta\cdot\frac{99}{100}$$

we get

$$\leq \frac{n}{\ell_k}\cdot |B_k|^{(1-\delta')n/\ell_k}.$$

Thus, for arbitrarily large n, half the mass of $\mu_{\beta_k\varphi}$ is concentrated on a set $E_k \subseteq \{0,1\}^n$ of cardinality

$$|E_k| \leq 2^{nH(2^{-\ell_k})+\log n - \log \ell_k}\cdot 2^{(1-\delta')n\log|B_k|/\ell_k}.$$

It follows from this and the Shannon-McMillan theorem that

$$h(\mu_{\beta_k\varphi}) \leq (1-\delta')\frac{\log|B_k|}{\ell_k} + H(2^{-\ell_k}),$$

hence, since $\varphi \leq 0$, we have

$$P_{\beta_k}(\mu_{\beta_k\varphi}) \leq h(\mu_{\beta_k\varphi}) \leq (1-\delta')\frac{\log|B_k|}{\ell_k} + H(2^{-\ell_k}).$$

Substituting the lower bound from lemma (3.5), we have

$$\frac{\log|B_k|}{\ell_k} - 2^{3\ell_k}2^{-\ell_{k-1}2^{\ell_k}} < (1-\delta')\frac{\log|B_k|}{\ell_k} + H(2^{-\ell_k}).$$

By our assumptions about the growth of N_k the inequality above is possible only for finitely many k. This completes the proof. □

We can now prove theorem 1.1. For $\delta = \frac{1}{100}$ choose the sequence N_k so that the conclusion of the last proposition holds. Since the density of 0's in the blocks $a \in A_k$ is $> \frac{2}{3}$ and the density in the blocks $b \in B_k$ is $< \frac{1}{3}$, it follows that for k large enough and $\beta_k = 2^{-3\ell_k}$,

$$\begin{aligned} \mu_{\beta_k}([0]) &< \frac{1}{3} - \delta \qquad \text{if } k \text{ is odd} \\ \mu_{\beta_k}([0]) &> \frac{2}{3} + \delta \qquad \text{if } k \text{ is even} \end{aligned}$$

Hence $(\mu_{\beta\varphi})_{\beta\geq 0}$ does not weak-* converge.

4. Remarks

4.1. **Topological dynamics of X.** In our example X is minimal. Indeed, any block $a \in L_k$ appears in c_{k+1} and hence in every block in L_{k+1}, so a appears in X with bounded gaps. Note that there are also minimal (non uniquely ergodic) systems X for which the zero-temperature limit exists.

One can easily modify the construction to endow X with other dynamical properties, e.g. one can make X topologically mixing (our example it is not, in fact it has a periodic factor of order 4). There is also a cheap way one can get positive entropy of X (and the limiting measures): form the product of the given example with a full shift.

4.2. **Measurable dynamics of the zero-temperature limits.** In our example, $(\mu_{\beta\varphi})_{\beta\geq 0}$ has two ergodic accumulation points, and one can show that the convex combinations of these two are also accumulation points.

In general, the set of accumulation points need not contain ergodic measures, even when the zero-temperature limit exists. This is true even of locally constant potentials [9, 4], and one can also construct examples which are simpler to analyze. For example, if $X \subseteq \{0,1\}^{\mathbb{N}}$ is a subshift invariant under involution $0 \leftrightarrow 1$ of $\{0,1\}^{\mathbb{N}}$, and if X has precisely two invariant measures μ', μ'' which are exchanged by this involution, then for the potential $\varphi_X(y) = -d(y, X)$ we will have $\lim_{\beta \to +\infty} \mu_{\beta\varphi} = \frac{1}{2}\mu' + \frac{1}{2}\mu''$.

The set of accumulation points also need not be convex. Using the same scheme as above one can construct a subshift $X \subseteq \{1,2,3\}^{\mathbb{Z}}$ with three invariant measures $\mu^{(i)}, i = 1,2,3$, by maintaining three sets of blocks A_k, B_k, C_k at each stage (rather than two). At each step of the construction we choose the smallest of the sets and concatenate its blocks freely, but concatenate the blocks of the others in a constrained way, so that at the next stage the sizes of the selected set is much larger than the other two, which have not changed much in relative size. For each n there are always two sets (the two which are not growing very much at that stage) for which the number of n-blocks in one is much greater than in the other. Thus the Gibbs measures at the appropriate scale will have very small contributions from the smaller of these sets, and the accumulation points of $\mu_{\beta\varphi}$ will lie near the boundary of the simplex spanned by the $\mu^{(i)}$ (in our example there were only two sets and at each step one grew at the expense of the other; thus the relative number of n- blocks achieved all intermediate ratios).

Regarding the ergodic nature of the accumulation points, the same periodicity of order five that obstructs topological mixing causes the ergodic invariant measures on X (i.e. the ergodic zero-temperature limits) to have $e^{-2\pi i/5}$ in their spectrum, but this can be avoided by introducing spacers into the construction. In this way one can make the limiting ergodic measures weak or strong mixing, and possibly K.

Finally, we have the following variant of Radin's argument from [11]. Let μ be an ergodic probability measure for some measurable transformation of a Borel space, and $h(\mu) < \infty$. By the Jewett-Krieger theorem [5] there is a subshift X on at most $h(\mu) + 1$ symbols whose unique shift-invariant measure ν is isomorphic to μ in the ergodic theory sense. For the potential φ_X, all accumulation points of $\mu_{\beta\varphi}$ are invariant measures on X, so they all equal ν; thus $\mu_{\beta\varphi} \to \nu$ as $\beta \to +\infty$. This shows that the zero-temperature limit of Gibbs measures can have arbitrary isomorphism type, subject to the finite entropy constraint, and raises the analogous question for divergent potentials:

Problem. Given arbitrary ergodic measures μ', μ'' of the same finite entropy, can one construct a Hölder potential φ whose Gibbs measures $\mu_{\beta\varphi}$ have two ergodic accumulation points as $\beta \to +\infty$, isomorphic respectively to μ', μ''?

4.3. **Maximization of marginal entropy.** Let φ be a Hölder potential and $\mathcal{M}$ the set of invariant probability measures μ for which $\int \varphi d\mu$ is maximal. It is known that if μ is an accumulation point of $(\mu_{\beta\varphi})_{\beta\geq 0}$ then $\mu \in \mathcal{M}$ and furthermore μ maximizes $h(\mu)$ subject to this condition.

In the example constructed above the potential φ had two φ-maximizing ergodic measures μ', μ'', and the key property that we utilized was that their marginals at certain scales had sufficiently different entropies. In fact, the measure maximizing the marginal entropy on $\{0,1\}^n$ for certain n was alternately very close to μ' and to μ''.

It is an interesting question if such a connection between zero-temperature convergence and marginal entropy exists in general. Let φ be a Hölder potential, and for each n let $\mathcal{M}_n^*$ denote the set of marginal distributions produced by restricting $\mu \in \mathcal{M}$ to $\{0,1\}^n$. The entropy function $H(\cdot)$ is strictly concave on $\mathcal{M}_n^*$, and therefore there is a unique $\mu_n^* \in \mathcal{M}_n^*$ maximizing the entropy function. Let

$$\mathcal{M}_n = \{\mu \in \mathcal{M} \,:\, \mu|_{\{0,1\}^n} = \mu_n^*\}.$$

This is the set of φ-maximizing measures which maximize entropy on n-blocks. Note that the diameter of $\mathcal{M}_n$ tends to 0 as $n \to \infty$ in any weak-* compatible metric. Hence we can interpret $\mathcal{M}_n \to \mu$ in the obvious way.

Problem. Is the existence of a zero-temperature limit for φ equivalent to existence of $\lim \mathcal{M}_n$? More generally, do $(\mu_{\beta\varphi})_{\beta\geq 0}$ and $(\mathcal{M}_n)_{n\geq 0}$ have the same accumulation points?

5. The multidimensional case

In this section we apply the main theorem of [6] to obtain a locally constant potential (i.e. a finite-range interaction) in dimension $d \geq 3$ such that the associated Gibbs measures do not converge as $\beta \to +\infty$; contrast this with the positive result for locally constant potentials in dimension one [3, 9, 4]. Note that in dimension $d \geq 2$, the possibility of failure of the 0-temperature limit to exist for finite-range potentials is known over continuous state spaces [14]; it is the fact that the alphabet is finite and the potential is locally constant that is of interest here. Our methods do not work in $d = 2$, because the results of [6] are not known in that case, but probably a more direct construction is possible.

5.1. **SFTs and their subdynamics.** The metric on $\{0,1\}^{\mathbb{Z}^d}$ is defined by[2]

$$d(x,y) = 2^{-\min\{\|u\|\,:\,x(u)\neq y(u)\}}$$

where $\|\cdot\|$ is the sup-norm. We denote by T the shift action on $\{0,1\}^{\mathbb{Z}^d}$ and write $T_1,\ldots,T_d$ for its generators.

Let

$$E_n = \{-n,\ldots,0,,\ldots,n\}^d$$

denote the discrete d-dimensional cube of side $2n+1$. A subshift X is a shift of finite type (SFT) if there is an n and finite set of patterns $L \subseteq \{0,1\}^{E_n}$ such that

$$X = \{x \in \{0,1\}^{\mathbb{Z}^d} \;:\; \text{no pattern from } L \text{ appears in } x\}.$$

(Note: here L determines the forbidden patterns, which is the opposite of its role in $\langle L\rangle$.) A pattern a is said to be *locally admissible* if it does not contain any patterns from L; it is *globally admissible* if it appears in X, i.e. it can be extended to a configuration on all of $\mathbb{Z}^d$ which does not contain patterns from L. These two notions are distinct, and it is formally impossible to decide in general, given L, whether a locally admissible word is globally admissible.

If we write

$$\varphi_L(y) = \begin{cases} -1 & y|_{E_n} \in L \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

then every invariant measure μ on $\{0,1\}^{\mathbb{Z}^k}$ satisfies $\int \varphi_L d\mu \leq 0$ with equality if and only if μ is supported on X. Thus for any SFT X there is a locally constant potential whose maximizing measures are precisely the invariant measures on X.

Given a subshift $X \subseteq \{0,1\}^{\mathbb{Z}^d}$, we may consider the restricted one-parameter action of T_1 on X. We shall say that the topological dynamical system (X,T_1) is a (one-dimensional) subaction of (X,T). To each partition $C = \{C_1,\ldots,C_m\}$ of $\{0,1\}^{\mathbb{Z}^d}$ into closed and open sets we associate to each $x \in X$ its itinerary x^C given by the action of T_1 and the partition C, i.e.

[2]The dimension of the ambient space is also denoted d but no confusion should arise.

$x \mapsto x^C \in \{1,\ldots,m\}^{\mathbb{Z}}$ is defined by

$$x^C(i) = j \text{ if and only if } T_1^i x \in C_j.$$

The subshift

$$X^C = \{x^C \,:\, x \in X\} \subseteq \{1,\ldots,m\}^{\mathbb{Z}}$$

is a factor, in the sense of topological dynamics, of the subaction (X, T_1).

For a subshift $Y \subseteq \{0,1\}^{\mathbb{Z}}$ write $L_k(Y) \subseteq \{0,1\}^k$ for the set of k-blocks appearing in Y; note that for any sequence $k(i) \to \infty$ the sets $L_{k(i)}$, $i = 1,2,\ldots$, determine Y.

The main result of [6] says that the subaction of SFTs can be made to look like an arbitrary subshift, as long as the subshift is constructive in a certain formal sense. The version we need is the following:

Theorem. *Let A be an algorithm that for each i computes[3] an integer $n(i)$ and a set $L_i \subseteq \{0,1,\ldots,r\}^{n(i)}$ such that $\langle L_i \rangle \supseteq \langle L_{i+1} \rangle$. Then there is an alphabet Σ, an SFT $X \subseteq \Sigma^{\mathbb{Z}^3}$ of entropy 0 and a closed and open partition $C = \{C_0, C_1, \ldots, C_r\}$ of $\Sigma^{\mathbb{Z}^3}$ such that $L_{n(i)}(X^C) = L_i$, and consequently $X^C = \cap \langle L_i \rangle$. Furthermore, the partition elements C_i can be made invariant under the shifts T_2 and T_3.*

To apply this one usually begins with a subshift Y which is has been constructed in some explicit manner, and a computable sequence $n(i)$ (e.g. $n(i) = i$), and derives an algorithm which from i computes $L_{n(i)}(Y)$; one then gets an SFT X and partition C so that $X^C = Y$. This means that for all practical nearly purposes (e.g. the construction of counterexamples) one can realize arbitrary dynamics as the subdynamics of an SFT.[4]

From the result for dimension $d = 3$ the same is easily seen to hold for $d \geq 3$, but it is not known whether this holds in dimension $d = 2$.

5.2. **A modified one-dimensional example.** For notational convenience, for the rest of the paper we concentrate on the case $d = 3$, the general case being similar.

Realizing a specific subshift (such as the one from section 2) as the subaction of an SFT X does not in itself give good control over the Gibbs measures of φ_X or φ_L. Indeed, the size of $L_n(X^C)$ is exponential in n, which implies

[3]A stronger statement can be made in which the computability is replaced with semi-computability of an appropriate family of blocks, and then one obtains (nearly) a characterization; but we do not need this here.

[4]Nevertheless, one should bear in mind that the family of SFTs (and the set of algorithms) is countable.

similar growth of the corresponding set $L_n(X)$, but does not guarantee exponential growth in n^3, which is the appropriate scale for 3-dimensional subshifts. Thus for example we can have $h(X^C) > 0$ but $h(X) = 0$.

In order to use subactions to control entropy of the full $\mathbb{Z}^3$ action we rely on a trick by which the frequency of symbols in X^C can be used to control pattern counts in a certain extension of X. This approach was used in [7, 2].

We begin by modifying the main example of this paper so as to control frequencies rather than block counts. We define a sequence of integers ℓ_k and sets of blocks $A_k, B_k \subseteq \{0,1,2\}^{\ell_k}$ by induction, using an auxiliary sequence $N_1, N_2 \ldots$ of integers.

Start with $\ell_0 = 2$ and $A_0 = \{00, 01\}$, $B_0 = \{00, 02\}$. Next, given k define

$$\begin{aligned} A_k &= \{a^{1+2^{\ell_{k-1}}+N_k} : a \in A_{k-1}\} \\ B_k &= \{b^{1+2^{\ell_{k-1}}} 2^{N_k \ell_{k-1}} : b \in B_{k-1}\} \end{aligned}$$

and for k even define

$$\begin{aligned} A_k &= \{a^{1+2^{\ell_{k-1}}} 1^{N_k \ell_{k-1}} : a \in A_{k-1}\} \\ B_k &= \{b^{1+2^{\ell_{k-1}}+N_k} : b \in B_k\}. \end{aligned}$$

Let ℓ_k be the common length of blocks in the sets above, i.e. $\ell_k = \ell_{k-1}(2^{\ell_{k-1}+1} + N_k)$. Note that $1^{\ell_k} \in A_k$ and $2^{\ell_k} \in B_k$.

As $k \to \infty$ the frequency of 0's in the blocks of A_k, B_k tends to 0, and the frequency of 1's and $2's$ tends, respectively, to 1, and we can control the relative speed at which they do so. More precisely, there is a function $\widetilde{N}_k(\cdot)$ such that given $N_1, \ldots, N_{k-1}$ and $N_k \geq \widetilde{N}_k(N_1, \ldots, N_{k-1})$ we have

$$\begin{aligned} f_0(a) &> 100 f_0(b) \qquad \text{for } k \text{ odd, } a \in A_k, b \in B_k \\ f_0(b) &> 100 f_0(a) \qquad \text{for } k \text{ even, } a \in A_k, b \in B_k. \end{aligned}$$

(Recall that $f_0(x)$ is the frequency of the symbol 0 in x.)

Define

$$Y = \bigcap_{k=1}^{\infty} \langle A_k \cup B_k \rangle .$$

Similarly define

$$Y_1 = \bigcap \langle A_k \rangle$$

and

$$Y_2 = \bigcap \langle B_k \rangle .$$

(Notice that these are decreasing intersections). Note that the only invariant measures on Y are the point masses at the fixed points $1^\infty \in Y_1$ and $2^\infty \in Y_2$. We denote

$$\ell'_k = \ell_{k-1}\left((|A_K| + |B_k|)^{M_k} + \widetilde{N}_k(N_1 \dots N_{k-1})\right) \tag{5.2}$$

(so $\ell_k \geq \ell'_k$) and note that as long as $N_k \geq \widetilde{N}_k(N_1, \dots, N_{k-1})$, the set $L_{\ell'_k}(Y)$ is in fact independent of N_k and depends only on $N_1, \dots, N_{k-1}$. We also note that $\widetilde{N}_k$ can be computed explicitly, and in particular the function $(k, N_1, \dots, N_{k-1}) \mapsto \widetilde{N}_k(N_1, \dots, N_{k-1})$ is a formally computable function.

5.3. **Controlling pattern counts in a 3-dimensional SFT.** We now incorporate the subshift Y constructed above into a 3-dimensional SFT and use the control over the frequency of symbols in Y to gain control of the pattern counts of an associated SFT.

First, some notation: for a subshift $X \subseteq \Sigma^{\mathbb{Z}^3}$ write

$$L_n(X) = \{x|_{E_n} \,:\, x \in X\} \subseteq \Sigma^{E_n}$$

where $E_n = \{-n, \dots, n\}^3$. This is the same notation we used for one-dimensional subshifts, but the meaning will be clear from the context. We remark that if the (topological) entropy of X is 0 then $|L_n(X)| = o(|E_n|)$.

Apply theorem 5.1 to Y (or, rather, to an algorithm that computes a sequence $L_{n(k)}(Y)$; we shall be more precise later about the algorithm used). We obtain a zero-entropy SFT $X \subseteq \Sigma^{\mathbb{Z}^3}$ and $C = \{C_0, C_1, C_2\}$ a T_2, T_3-invariant partition so that $X^C = Y$.

Next, for $x \in X$ and $u = (u_1, u_2, u_3) \in \mathbb{Z}^3$, if $x^C(u_1) = 0$ (i.e. if $T_1^{u_1} x \in C_0$) we "color" the site with one of the two colors $0', 0''$. Otherwise we leave it "blank". Collect all such colorings into a new subshift $\widehat{X}$. Formally, $\widehat{X} \subseteq X \times \{0', 0'', \text{blank}\}^{\mathbb{Z}^3}$ is defined by

$$\widehat{X} = \{(x, y) \in X \times \{0', 0'', \text{blank}\} \,:\, y(u) = \text{blank if } x^C(u_1) \neq 0\}.$$

For $x = (x_1, x_2) \in \widehat{X}$ we also write x^C instead of x_1^C. One may verify that $\widehat{X}$ is an SFT. We write $\widehat{\Sigma} = \Sigma \times \{0', 0'', \text{blank}\}$ for the alphabet of $\widehat{X}$ and write $\widehat{L}$ for the finite set of patterns whose exclusion defines $\widehat{X}$. We may assume that if a pattern over $\widehat{\Sigma}$ is locally admissible for $\widehat{L}$ then the pattern induced from its first component is locally admissible for L.

Notice that, since C_0, C_1, C_2 are invariant under T_2, T_3, the pattern of symbols $0', 0''$ in a point $x \in \widehat{X}$ is the union of affine planes whose direction is spanned by $(0,1,0), (0,0,1)$. The sequence of coordinates at which these planes intersect the x-axis corresponds to the location of 0-s in x^C, and on each plane the symbols $0', 0''$ are distributed as randomly as possible, i.e. given the arrangement of affine planes there is no restriction on the combinations of $0', 0''$ that may appear in them. It follows that if $a \in \{0,1,2\}^{\{-n,\ldots,n\}}$ is a block in Y then

$$\#\{(x,y)|_{E_n} \,:\, (x,y) \in \widehat{X} \text{ and } x^C|_{\{-n,\ldots,n\}} = a\} = 2^{f_0(a)|E_n| + o(|E_n|)}.$$

(The term $o(|E_n|)$ comes from the pattern growth of X, which has entropy 0).

Write

$$\begin{aligned} \widehat{X}_1 &= \{x \in \widehat{X} \,:\, x^C \in Y_1\} \\ \widehat{X}_2 &= \{x \in \widehat{X} \,:\, x^C \in Y_2\}. \end{aligned}$$

Then for k large enough the frequency gap between blocks in A_k and B_k translates into

$$\begin{aligned} |L_{\ell_k}(\widehat{X}_1)| &> |L_{\ell_k}(\widehat{X}_2)|^{1/10} \qquad k \text{ odd} \\ |L_{\ell_k}(\widehat{X}_2)| &> |L_{\ell_k}(\widehat{X}_1)|^{1/10} \qquad k \text{ even.} \end{aligned}$$

Compare this with lemma 3.2.

5.4. **Local versus global admissibility.** For $\varphi = \varphi_{\widehat{X}}$, i.e. $\varphi(y) = -d(y, \widehat{X})$, one can adapt the analysis in section 3 and show that $\mu_{\beta\varphi}$ does not have a limit as $\beta \to +\infty$. Let us review this argument. Fix $\beta = 2^{-3\ell_k}$, and set $p = 1,2$ according to whether k is odd or even, and write $q = 2 - p$ for the other index. First, an in Lemma 3.5, we prove a lower bound on $P_\beta(\mu_{\beta\varphi})$ by constructing a measure ν_k whose blocks (i.e. square patterns) are overwhelmingly drawn from $L_{\ell_k}(X_p)$, making it nearly $\varphi_{\widehat{X}}$-maximizing, and with entropy close to $\frac{1}{|E_{\ell_k}|}|L_{\ell_k}(\widehat{X}_p)|$. This forces the entropy of $\mu_{\beta\varphi}$ to be similar. Second, we use the fact that most of $\mu_{\beta\varphi}$ concentrates on blocks from $L_{\ell_k}(\widehat{X})$ and the fact that $L_{\ell_k}(\widehat{X}_p) \gg L_{\ell_k}(\widehat{X}_q)$ to deduce that in order for $\mu_{\beta\varphi}$ to have entropy near $\frac{1}{|E_{\ell_k}|}|L_{\ell_k}(\widehat{X}_p)|$, it must be mostly concentrated on $\widehat{X}_p$. This argument is similar to that in proposition 3.6.

We are now interested in proving the same thing for the potential $\varphi_{\widehat{L}}$ (given in (5.1)) instead of $\varphi_{\widehat{X}}$. The first part of the analysis above carries over with only minor modifications.

However, the second part runs into difficulties. Notice that $\int \varphi_{\widehat{X}} d\mu \approx 0$ implies that nearly all the $\mu_{\beta\varphi}$-mass is concentrated on patterns in $L_{\ell_k}(\widehat{X})$, but $\int \varphi_{\widehat{L}} d\mu \approx 0$ tells us only that $\mu_{\beta\varphi}$-most blocks on E_{ℓ_k} are *locally* admissible for $\widehat{L}$; they do not have to be *globally* admissible, giving us little control of their structure.

To pull things through, we will make use of the following observation: it is not necessary for us to know that most of the mass of $\mu_{\beta\varphi}$ concentrates on $L_{\ell_k}(\widehat{X})$. Instead, it suffices that it concentrates on $L_{\ell'_k}(\widehat{X})$, where ℓ' is as in equation (5.2). This is because $L_{\ell'_k}(\widehat{X}_p)$ is already much larger than $L_{\ell'_k}(\widehat{X}_q)$, so we can argue as in the first part of the proof of proposition 3.6.

Thus, to complete the construction we want to ensure that if a block $a \in \Sigma^{E_{\ell_k}}$ is locally admissible then $a|_{E_{\ell'_k}}$ is globally admissible, i.e. belongs to $L_{\ell'_k}(\widehat{X})$.

A simple compactness argument establishes the following general fact: For any SFT and $m \in \mathbb{N}$ there is an R so that if $b \in \Sigma^{E_R}$ is locally admissible then $b|_{E_m}$ is globally admissible. In general, however, R depends in a very complicated way on both the SFT and m, and in fact is not formally computable given these parameters. For our purposes we require finer control than this. Luckily, an inspection of the proof in [6] gives the following:

Theorem 5.1. *Let A be an algorithm that from i computes $n(i) \in \mathbb{N}$ and $L_i \subseteq \{0, 1, \ldots, r\}^{n(i)}$ such that $\langle L_i \rangle \supseteq \langle L_{i+1} \rangle$. Denote by τ_i the number of time-steps required for the computation on input i. Then the SFT X from theorem 5.1 can be chosen so that, for $R_i = R_i(|A|, \tau_1, \ldots, \tau_i)$, if $a \in \Sigma^{E_{R_i}}$ is locally admissible then $a|_{E_{n(i)}}$ is globally admissible, and furthermore the function $R_i(\ldots)$ is computable. Here τ_i and A are taken with respect to some fixed universal Turing machine.*

5.5. **Completing the construction: The fine print.** We now specify an algorithm A which, given i, computes sequences $n(i) \in \mathbb{N}$ and $L_i \subseteq \{0, 1, 2\}^{n(i)}$ so that $\langle L_i \rangle \supseteq \langle L_{i+1} \rangle$. The even elements $n(2k)$ are the lengths ℓ_k associated to a sequence N_k in the construction in section 5.2, i.e.

$$N_k = \frac{n(2k) - n(2k-2) \cdot 2^{1+n(2k-2)}}{n(2k-2)}.$$

The odd elements of the sequence are

$$n(2k-1) = \ell'_k = \ell_{k-1} \widetilde{N}_k(N_1, \ldots, N_{k-1}).$$

Note that, having determined $n(i)$, the blocks in Y of length $n(i)$ depend only on $N_1, \ldots, N_{[n(i)/2]}$ and not on any future choices of parameters of the construction. Hence $L_i = L_{n(i)}(Y)$ is well defined given $n(1), \ldots, n(i)$ and may be computed from this data. Thus at the i-th stage of the computation we will write $L_{n(i)}(Y)$ even though strictly speaking Y is not yet defined.

On input i the algorithm is as follows.

Case 0: $i = 1$. Output

$$\begin{aligned} n(1) &= 1 \\ L_1 &= \{0,1,2\}. \end{aligned}$$

Case 1: $i = 2k-1$. Recursively compute $N_1, \ldots, N_{k-1}$, and output

$$\begin{aligned} n(i) &= \ell'_k = \ell_{k-1}\widetilde{N}_k(N_1, \ldots, N_{k-1}) \\ L_i &= L_{n(i)}(Y). \end{aligned}$$

Case 2: $i = 2k$. Recursively compute $N_m, m < k$ and the time $\tau_1, \ldots, \tau_{i-1}$ spent by the algorithm when run on each of the inputs $j = 1, \ldots, i-1$. Let

$$N_k = (\max\{n(i-1), R(|A|, \tau_1, \ldots, \tau_{i-1})\})^2$$

and output

$$\begin{aligned} n(i) &= \ell_k = N_k \ell_{k-1} \\ L_i &= L_{n(i)}(Y). \end{aligned}$$

Realizing such an algorithm (which can simulate itself) is a non-trivial but standard exercise in computation theory.

We can now sketch the remainder of the proof of theorem 1.2. Using A as input to theorem 5.1 we obtain an SFT $X \subseteq \Sigma^{\mathbb{Z}^d}$ and associated partition $C = \{C_0, C_1, C_2\}$ of $\{0,1,2\}^{\mathbb{Z}^3}$, invariant under T_2, T_3, such that $X^C = Y$. Next, form the SFT $\widehat{X}$ as explained above, defined by a set $\widehat{L}$ of excluded patterns.

For $\beta = 2^{3\ell_k}$ let $\mu_{\beta\varphi}$ be the Gibbs measure associated to the potential $\varphi_{\widehat{L}}$. By the definition of Gibbs measures we have $\int \varphi_{\widehat{L}} d\mu_{\beta\varphi} > -c2^{-3\ell_k}$, where $c = \log|\widehat{\Sigma}|$ is the maximal entropy achieved by an invariant measures on the full shift $\widehat{\Sigma}^{\mathbb{Z}^3}$; in section 3 this constant was 1. Thus in a $\mu_{\beta\varphi}$-typical configuration the density of patterns from $\widehat{L}$ is $< c2^{-3\ell_k}$. Hence for $r = \sqrt{\ell_k}$ and large enough k, with $\mu_{\beta\varphi}$-probability $> 1 - 2^{-2\ell_k}$ a configuration x satisfies that $x|_{E_r}$ is globally admissible. By our choice of $\ell_k = n(2k)$ we have $r \geq R(|A|, N_1, \ldots, N_k)$, so $x|_{E_{n(2k-1)}}$ is *globally* admissible. But since $n(2k-1) \geq \ell'_k$, we are in the situation described at the end of the previous subsection, and this is enough to conclude that $\mu_{\beta\varphi}$ is mostly concentrated on $\widehat{X}_1$ or $\widehat{X}_2$, depending on k mod 2; so $\mu_{\beta\varphi}$ diverges along $\beta = 2^{-3\ell_k}$.

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Centre de Physique Théorique, École Polytechnique, 91128 Palaiseau Cedex, France

E-mail address: jeanrene@cpht.polytechnique.fr

Department of Mathematics, Fine Hall, Washington Rd., Princeton NJ 08540, USA.

E-mail address: hochman@math.princeton.edu