

A NOTE ON UNIVERSALITY IN MULTIDIMENSIONAL SYMBOLIC DYNAMICS

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ABSTRACT. We show that in the category of effective \mathbb{Z} -dynamical systems there is a universal system, i.e. one that factors onto every other effective system. In particular, for $d \geq 3$ there exist d -dimensional shifts of finite type which are universal for 1-dimensional subactions of SFTs. On the other hand, we show that there is no universal effective \mathbb{Z}^d -system for $d \geq 2$, and in particular SFTs cannot be universal for subactions of rank ≥ 2 . As a consequence, a decrease in entropy and Medvedev degree and periodic data are not sufficient for a factor map to exist between SFTs.

We also discuss dynamics of cellular automata on their limit sets and show that (except for the unavoidable presence of a periodic point) they can model a large class of physical systems.

1. INTRODUCTION

1.1. Universality for shifts of finite type. A basic problem about any class of dynamical systems is to understand the factoring relation between its members. Much of ergodic theory and topological dynamics, and particularly the theory of one-dimensional shifts of finite type (SFTs), has been motivated by the hope, which for some classes is satisfied, that the factoring relation reduce to some simple parameter, such as entropy, periodic point data or spectrum. For higher dimensional SFTs, which are the main subject of this note, partial results are known under certain mixing conditions [7], but it has become progressively clearer that the invariants which dictate the factoring relation in dimension 1 are only a part of the picture in dimensions $d > 1$.

We begin by reviewing some definitions; see also section 2. A \mathbb{Z}^d *shift of finite type* (SFT) is a subshift X of the full d -dimensional shift $\Sigma^{\mathbb{Z}^d}$ over Σ , defined by excluding all configurations containing patterns from some fixed finite set. More precisely, by a (d -dimensional) pattern we mean a coloring of a finite subset of \mathbb{Z}^d . For $F \subseteq \mathbb{Z}^d$ and $a \in \Sigma^F$, we say that the pattern a appears in a configuration $x \in \Sigma^{\mathbb{Z}^d}$ if $(T_u x)|_F = a$ for some $u \in \mathbb{Z}^d$; here, T^u is the shift by u . For L a set of

d -dimensional patterns over Σ , set

$$X_L = \{x \in \Sigma^{\mathbb{Z}^d} : \text{no element of } L \text{ appears in } x\}$$

An SFT is a subset of the form X_L for a *finite* set L .

A *subaction* (or subdynamic) of a \mathbb{Z}^d -SFT is the restriction of the \mathbb{Z}^d -action to a subgroup $\mathbb{Z}^k < \mathbb{Z}^d$. Thus the full action and its subactions share the same phase space, which is a disconnected compact metric space, but it is important to note that the subactions are not necessarily symbolic. For example, if $\mathbb{Z} < \mathbb{Z}^2$ as the first component, then the \mathbb{Z} -subaction of the full \mathbb{Z}^2 -shift $\{0,1\}^{\mathbb{Z}^2}$ is isomorphic to the \mathbb{Z} -shift over the Cantor set $\{0,1\}^{\mathbb{Z}}$, which is not expansive; this may be seen by thinking of columns of symbols as points in the Cantor set, so each 2-dimensional configuration becomes a linear sequence of points in the Cantor set and the subaction shifts these points. In particular a subshift of an SFT is may no longer be an SFT; a finite-infinite subgroup always gives an SFT but even the conditions under which a general subaction of an SFT is expansive are not known. We use the unqualified term SFT to refer to an SFT with the full action.

Returning to our subject, in this paper we consider a basic question about the factoring relation for SFTs, namely, whether there is a universal SFT that factors onto all others. We are actually interested in the broader question of universality for the class of subactions of SFTs. More precisely, for each $k \leq d$, we ask whether there is a \mathbb{Z}^d -SFT X so that the \mathbb{Z}^k -subaction on X factors onto the \mathbb{Z}^k -subaction of every other \mathbb{Z}^d SFT. Such an X , if it exists, we call a (d, k) -universal SFT.

One can immediately rule out the existence of (d, d) -universal SFTs on the basis of topological entropy, which does not increase upon passage to a factor, and so, since every SFT has finite entropy but there are SFTs of arbitrarily large entropy, no universal one can exist.

However, for $k < d$ it is not clear what one should expect. As we saw in the case of the full shift, a subaction may have infinite entropy, so this poses no restriction. There is a restriction of a recursive nature, and that is that the subaction of an SFT is effective. Recall that an *effective \mathbb{Z}^k -dynamical system* (EDS) is a subshift of the full \mathbb{Z}^k -shift over the Cantor set whose complement is the union of a formally computable sequence of basic open sets. The fact that a subaction of an SFT is effective was proved in [4], along with a partial converse: any effective \mathbb{Z}^k system can be realized as the factor of a \mathbb{Z}^k -subaction of a \mathbb{Z}^{k+2} -SFT (in fact the extension can be made quite small, but we will not use this). We refer to [4] for further details.

Thus, the questions of whether (d, k) -universal SFTs exist is closely related to the existence of universal systems in the class of effective systems;¹ and the class of effective systems, though countable, includes essentially every type of dynamics we can “describe”. This would seem to indicate that we should not expect universal dynamics to exist, since they do not for general systems (at least if we stay in the context of separable spaces).

Another relevant piece of information was recently provided by S. Simpson [13], who introduced Medvedev degrees as an invariant of SFTs. Simpson associates to each SFT X the Medvedev degree $m(X)$ of its phase space, which is a measure of the recursive-theoretic complexity of the phase space as a subset of the full shift without reference to the dynamics (see section 3). Since factor maps between SFTs are sliding block codes they are computable, and therefore do not increase Medvedev degree. Simpson also showed that every Medvedev degree is realized as a 2-dimensional SFT. It follows that the factoring relation between SFTs is at least as complicated as the order relation between Medvedev degrees, and the latter is still not well understood.

What is relevant to our question, however, is that there exists a maximal Medvedev degree. Thus, from the point of view of recursion theory, there is no obstruction to the existence of SFTs whose (sub)actions factor onto a very broad class of systems; indeed, any set with maximal Medvedev degree at least maps (via a computable function) into, every effective set, and so into every SFT (this map has nothing to do with dynamics, but even so its existence is non-trivial).

It turns out that the existence of universal effective systems depends on the rank of the action. For \mathbb{Z} -actions, such a system exists:

Theorem 1.1. *There exists a universal effective \mathbb{Z} -system, that is, an effective \mathbb{Z} -system that factors onto every other effective \mathbb{Z} -system.*

In particular, for every $d \geq 3$ there exist \mathbb{Z}^d -SFTs whose \mathbb{Z} -subaction factors onto the \mathbb{Z} -subaction of every other \mathbb{Z}^d -SFT; i.e. there are $(d, 1)$ -universal SFTs for every $d \geq 3$.

It remains an open problem whether there exist $(2, 1)$ -universal SFTs, i.e. \mathbb{Z}^2 -systems whose 1-dimensional subaction is universal. This would follow if every \mathbb{Z} -EDS could be realized as the subaction of a \mathbb{Z}^2 -SFT; this is problem open [4].

The universal effective \mathbb{Z} -system in the theorem is constructed by taking the product of all non-empty \mathbb{Z} -EDS (this is a countable product so no topological difficulties arise). The fact that the non-empty \mathbb{Z} -EDS can be effectively enumerated rests on the fact that one can decide whether a \mathbb{Z} -SFT is empty based on the list

¹One must be careful what notion of morphism one chooses for effective systems, since not every factor map is an effective factor map. However in this paper both definitions lead to the same results, since a factor map from an EDS to a symbolic system is automatically effective.

of forbidden patterns which defines it. In contrast, it is a classical result of Berger that this problem is formally undecidable for \mathbb{Z}^2 -SFTs [12, 1], so this construction cannot be imitated in higher dimensions. This is not a shortcoming of the method because

Theorem 1.2. *If $d \geq 2$ then there is no universal effective \mathbb{Z}^d system, and there are no (d, k) -universal SFTs for $k \geq 2$.*

To prove this, we show that the existence of a universal system could be used as a component in an algorithm that would decide whether an arbitrary finite set of patterns L defines a non-empty SFT, which would contradict Berger's theorem.

1.2. Cellular automata. Another class of dynamical systems defined by local rules are cellular automata. Recall that a d -dimensional cellular automaton (CA) is a function $f : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$ which commutes with the shift action. This is well known to be equivalent to being defined by a local, finite transition rule. See [8] for definitions and a recent survey of the subject.

We next present two applications of the results from [4] to the dynamics of CA. The first is analogous to the universality question for SFTs, i.e.: are there universal CA? This question seems to be more difficult than for SFTs, except for the case $d = 1$ where again entropy considerations show that no universal object can exist. However, using the relation between CA and SFTs (see e.g. [4], section 5.1), we can show that for $d \geq 3$ there exist CA whose dynamics is very close to the universal effective \mathbb{Z} -system of theorem 1.1.

In discussing dynamics of CA one must first overcome the fact that their action is in general neither invertible nor surjective (we are interested in invertible dynamics, though the question makes sense also in other categories). The limit set Λ_f of a CA f is the largest set on which f acts surjectively:

$$\Lambda_f = \bigcap_{n=1}^{\infty} f^n(\Sigma^{\mathbb{Z}^3})$$

If f act injectively on Λ_f then the action is effective; in any case, the natural extension of this system is effective.

Note that the limit action always contains a periodic point; this imposes certain restrictions on the dynamics which can occur on limit actions. However this is the only limitation. Combining theorem 1.1 with the results of [4] we have:

Corollary 1.3. *There exists a 3-dimensional cellular automaton such that, after removing from Λ_f a fixed point and its basin of attraction, is a universal \mathbb{Z} -EDS, and in particular factors onto the natural extension of every CA.*

It is known that there are CA f such that for any other CA g , one can encode the configurations of g into configurations of f in a spatially homogeneous way and

so that the action of f simulates the action of g (for a precise definition see [11]). This notion is not directly related to universality in our sense.

Our second application concerns one of the motivations for studying CA in the first place, namely that they provide a simple model for evolution of physical systems. We would like to show that they live up to this expectation in the sense that, for a reasonably large class of such systems, we can find a CA which models their dynamics very closely. We note that although much has been made of the fact that CA can perform universal computation, this in itself does not say much about their dynamics. The dynamics of a computer simulating a dynamical system is quite distinct from the dynamics of the system it is simulating.

Since effective systems can be modeled as limit actions of CA (minus the basin of attraction of a fixed point), we proceed by showing that a large class of systems can be extended to EDS. This may be viewed as an effective dynamical Hausdorff-Alexandroff theorem. For simplicity we restrict our attention to attractors of maps of \mathbb{R}^n , with the aim establishing it under reasonably simple.

The following definition is adapted from [2] where it is proposed as a natural model of effective computation over the reals. A function f defined on some subset of \mathbb{R}^d is *effective* if there is an algorithm which, upon being given input $n \in \mathbb{N}$ and an the infinite array encoding the binary representations of d numbers $x_1, \dots, x_d \in \mathbb{R}$, reads a finite number of bits from the input and outputs d rational numbers y_1, \dots, y_d such that

$$\|f(x_1, \dots, x_d) - (y_1, \dots, y_d)\|_\infty \leq \frac{1}{n}$$

Definition 1.4. Suppose that

- (1) $U \subseteq \mathbb{R}^d$ is an open set,
- (2) $f : U \rightarrow U$ is an effective map,
- (3) $X \subseteq U$ is a closed attractor for f , i.e. there is an open set $V \subseteq U$ so that $\overline{f(V)} \subseteq V$ and $X = \bigcap f^n V$.
- (4) $f|_X$ is invertible.

Then we say that X is an effective attractor of f .

As before, the presence of periodic points in the limit action of a CA prevents CA from modeling arbitrary dynamics, but this is in a sense the only obstruction.

Theorem 1.5. *If $X \subseteq \mathbb{R}^d$ is an effective attractor for f then there is a 3-dimensional cellular automaton g such that, after removing from Λ_g a fixed point and its basin of attraction, the action of g factors onto (X, f) .*

For the proof one shows that effective attractors can be extended to EDSs, and applies the machinery from [4]. As we have indicated, the statement above is in

a sense the best possible for invertible dynamics without fixed points. It is true that in some ways, the dynamics of the CA given by the theorem do not “look like” the original system: tracing back to the construction in [4] one sees that the basin of attraction that we have thrown out is dense, and for product measures on the configuration space typical points will converge under the CA action to the fixed point. However, all the invariant measures on the system, except for the point measure on the fixed point, are pullbacks of measures from the original system. Furthermore, under the technical assumption that the effective attractor to be modeled satisfies the small boundary property, the EDS extending it can be made to be injective on the complement of a universally null set. Thus from the point of view of stationary dynamics the CA looks very much like an extension of the original system.

Organization. In section 3 we recall some of the recursive-theoretic machinery we will need and define effective dynamics. In section 4 we prove theorem 1.1, and in section 5 prove theorem 1.2. Section 6 discusses realization of effective attractors. Section 7 contains some open problems.

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2. DEFINITIONS AND NOTATION

A \mathbb{Z}^d -dynamical system is the action of \mathbb{Z}^d by homeomorphisms on a compact metric space X ; the action of $u \in \mathbb{Z}^d$ is denoted usually by $T^u : X \rightarrow X$. A factor map between systems X, Y acted on by the same group is a continuous, onto map $\pi : X \rightarrow Y$ which commutes with the action in the sense that $\pi T^u = T^u \pi$ for every u .

Let Σ be a finite set of symbols. The space $\Sigma^{\mathbb{Z}^d}$ of colorings of \mathbb{Z}^d by Σ is called the *full d -dimensional shift over Σ* , or just the full shift, and its points are called *configurations*. Topologically the full shift is a Cantor set, and it comes equipped with a natural \mathbb{Z}^d action, called the *shift action*, in which $u \in \mathbb{Z}^d$ acts via the translation $T_u : \Sigma^{\mathbb{Z}^d} \rightarrow \Sigma^{\mathbb{Z}^d}$ defined by

$$(T^u x)(v) = x(u + v)$$

A subset $X \subseteq \Sigma^{\mathbb{Z}^d}$ which is closed and invariant to the shift (i.e. $T^u X = X$ for $u \in \mathbb{Z}^d$) is called a *subshift*, or a *symbolic system*. By the Curtis-Hedlund-Lyndon theorem [3], factor maps between subshifts of the same dimension (but possibly distinct alphabets) are given by a *block code*: if $Y \subseteq \Delta^{\mathbb{Z}^d}$, $X \subseteq \Sigma^{\mathbb{Z}^d}$ and $\pi : Y \rightarrow X$ is a factor map, then there is a finite $F \subseteq \mathbb{Z}^d$ and a function $\pi_0 : \Delta^F \rightarrow \Sigma$, so that π acts on each site of $x \in \Delta^{\mathbb{Z}^d}$ by applying π_0 to the local neighborhood of

the site: $(\pi x)(u) = \pi_0((T^u x)|_F)$. The diameter of F is called the *window size* of π . Conversely, any such map $\pi_0 : \Delta^F \rightarrow \Sigma$ gives rise to a factor map π in this way (the image is automatically a subshift).

3. RECURSIVE SETS AND EFFECTIVE DYNAMICAL SYSTEMS

3.1. Some recursion theory. We require some basic facts from recursion theory; see [6] for a formal introduction. Recursion theory provides a classification of certain subsets of \mathbb{N} according to the extent to which the set may be described algorithmically. By an algorithm we mean a finite set of instructions which can be carried out automatically, i.e. by computer program or, more formally, a Turing machine.

A subset $A \subseteq \mathbb{N}$ is *recursive* (R) if there is an algorithm which, given $n \in \mathbb{N}$, outputs “yes” if $n \in A$ and “no” otherwise. A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is recursive if there is an algorithm which, given n , outputs $f(n)$.

A set A is *recursively enumerable* (RE) if there is an algorithm which, on input n , returns “yes” if $n \in A$ and otherwise runs forever. Alternatively, a non-empty set $A \subseteq \mathbb{N}$ is RE if there is an algorithm which, given $n \in \mathbb{N}$, outputs $a_n \in \mathbb{N}$ so that $A = \{a_n : n \in \mathbb{N}\}$; in other words, it is the image of a recursive function.

By fixing a bijection between \mathbb{N} and another countable set U , we can speak of R and RE subsets of U . Thus we will speak of R and RE subsets of pairs of integers, finite sequences or patterns over a finite set Σ , etc. We will always assume that the objects have been placed in correspondence with \mathbb{N} in some effective way (for the purpose of classifying subsets as R or RE, two identifications which can be algorithmically reduced to each other are equivalent).

Since there are countably many algorithms there are countably many R and RE sets. We note that every recursive set is RE, but not vice-versa. However, the examples of this tend to be rather artificial, e.g. the set of provable theorems in number theory (Gödel’s theorem), or the set of halting Turing machines (Turing’s theorem).

The following standard facts will be useful:

Lemma 3.1. *Let U be recursive and $L \subseteq U$ is an RE set. Let $R \subseteq U \times V$ be a recursive set and let*

$$M = \{b \in V : (a, b) \in R \text{ for some } a \in L\}$$

Then M is RE.

Proof. Let A be an algorithm that on input $a \in U$ halts if $a \in L$ and runs forever otherwise. Let B be the algorithm which, upon input $b \in V$, iterates over all pairs $(n, a) \in \mathbb{N} \times U$, and for each pair runs the algorithm A for n steps (or until it halts)

on the input a . If A halts before n steps are up, it checks whether $(a, b) \in R$, and if so it halts; otherwise it continues to the next pair (n', a') . It is easily seen that this algorithm halts on input b if and only if $b \in M$, so M is RE. \square

Lemma 3.2. *If a set $L \subseteq U$ is RE and $U \setminus L$ is RE then L is R.*

Proof. Let A and B be algorithms which, given $x \in U$, halt if $x \in L$ or $x \in U \setminus L$, respectively, and otherwise run forever. Consider the algorithm which accepts x as input and iterates over $n \in \mathbb{N}$. For each n it simulates n steps of the computation of A on input x , and if that computation halted it announces that $a \in L$ and halts. Otherwise it simulates n steps of the computation of B on x and if that computation halts, it announces $x \notin L$ and halts. If neither simulation terminates, it proceeds to the next n . Clearly, our algorithm always halts and gives the correct answer. \square

3.2. Effective subshifts and EDS. Returning to dynamics, let $K = \{0, 1\}^{\mathbb{N}}$ denote the Cantor set, and for finite $I \subseteq \mathbb{N}$ and $a \in \{0, 1\}^I$ let $[a]$ denote the cylinder set determined by a , i.e.

$$[a] = \{x \in K : x(i) = a(i), i \in I\}$$

it will be convenient to write \mathcal{P} for the set of finite patterns of the form $\{0, 1\}^I$, $I \subseteq \mathbb{N}$; so \mathcal{P} parameterizes the cylinder sets of K . The cylinder sets form a closed and open basis for the topology of K .

Let $\Omega = \Omega_d = K^{\mathbb{Z}^d}$, which topologically is again a Cantor set. A basis for the topology of Ω is given by the generalized cylinder sets $[\bar{a}]$, where $\bar{a} : E \rightarrow \mathcal{P}$ for some finite $E \subseteq \mathbb{Z}^d$, and

$$[\bar{a}] = \prod_{u \in \mathbb{Z}^d} V_u \quad \text{where } V_u = [a(u)] \text{ for } u \in E \text{ and } V_u = K \text{ otherwise}$$

We write \mathcal{P}^{*d} for the set of such maps $\bar{a} : E \rightarrow \mathcal{P}$, $E \subseteq \mathbb{Z}^d$ finite. The sets $[\bar{a}]$ for $\bar{a} \in \mathcal{P}^{*d}$ form a closed and open basis for Ω .

As usual, let $\{T^u\}_{u \in \mathbb{Z}^d}$ denote the shift action of \mathbb{Z}^d on Ω . A subshift $X \subseteq \Omega$ is, as usual, a nonempty, closed subset which is invariant under the shift action.

Every subshift (in fact every closed subset) is the complement of a countable union of cylinder sets. If the complement is the union of a recursive sequence of cylinder sets, we say it is effective. To be precise, fix an effective enumeration of \mathcal{P} and use this to enumerate the elements of \mathcal{P}^{*d} , which parameterizes a basis for Ω .

Definition 3.3. An effective subshift is a subshift $X \subseteq \Omega$ such that $X = \Omega_L$ for some recursively enumerable $L \subseteq \mathcal{P}^*$, where

$$\begin{aligned} \Omega_L &= \{x \in \Omega : x \notin T^u[\bar{a}] \text{ for every } a \in L \text{ and } u \in \mathbb{Z}^d\} \\ &= \Omega \setminus \bigcup_{\bar{a} \in L} \bigcup_{u \in \mathbb{Z}^d} T^u[\bar{a}] \end{aligned}$$

or, equivalently, if the set

$$\{a \in \mathcal{P}^{*d} : X \cap [\bar{a}] = \emptyset\}$$

is recursively enumerable.

Since there are countably many algorithms, there are countably many EDS, representing only countable many isomorphism types of systems. In spite of this we do not know of any “natural” invariant of dynamical systems which cannot be realized as an EDS.

In [4] we showed:

Theorem 3.4. *The subaction of an SFT is an EDS.*

More important for our present discussion is the partial converse obtained there:

Theorem 3.5. *If X is a \mathbb{Z}^d -EDS then there is a $(d+2)$ -SFT Y and a \mathbb{Z}^d -subaction of Y which factors onto (X, \mathbb{Z}^d) .*

4. UNIVERSAL EFFECTIVE \mathbb{Z} -SYSTEMS

In this section we prove the following result:

Theorem 4.1. *There exists a \mathbb{Z} -EDS which factors onto every other \mathbb{Z} -EDS.*

Together with theorem 3.5 this proves theorem 1.1.

A sequence $L_n \subseteq \mathbb{N}$ of sets is *uniformly recursively enumerable* if $\cup_{n=1}^{\infty} L_n \times \{n\} \subseteq \mathbb{N}^2$ is RE, or equivalently, if there is an algorithm A which, given input $i, j \in \mathbb{N}$, halts if $j \in L_i$ and otherwise runs forever.

Lemma 4.2. *Let $L_n \subseteq \mathcal{P}^{*d}$ be a uniformly RE sequence of sets. Write $X_n = \Omega_{L_n}$. Then the product system $\prod_{n=1}^{\infty} X_n$ is an EDS.*

Proof. The proof is based on the fact that $K \cong \prod_{n=1}^{\infty} K$, and this isomorphism can be made effective. Fix some recursive bijection $\varphi : \mathbb{N}^2 \rightarrow \mathbb{N}$, and let

$$I_n = \{k \in \mathbb{N} : k = \varphi(i, n) \text{ for some } i \in \mathbb{N}\}$$

which is a partition of \mathbb{N} into disjoint infinite sets. We may identify I_n with \mathbb{N} via $\varphi(\cdot, n) : \mathbb{N} \rightarrow I_n$. Let $\pi_n : K \rightarrow K$ be the projection $x \mapsto x|_{I_n}$ where identify $x|_{I_n}$ with a point in K using this bijection of I_n and \mathbb{N} . Extend π_n to patterns over K pointwise, so if $a \in K^E$ for a finite set $E \subseteq \mathbb{Z}^d$ then $(\pi_n(a))(u) = \pi_n(a(u))$, $u \in \mathbb{Z}^d$. Then $x \mapsto (\pi_1(x), \pi_2(x), \dots)$ is a homeomorphism from Ω to $\prod_{n=1}^{\infty} \Omega$.

Let $L \subseteq \mathcal{P}^{*d}$ be defined by

$$L = \{\bar{a} \in \mathcal{P}^{*d} , \pi_n([\bar{a}]) \subseteq [\bar{b}] \text{ for some } n \in \mathbb{N} \text{ and } \bar{b} \in L_n\}$$

A moments thought shows that $\Omega_L \cong \prod_{n=1}^{\infty} \Omega_{L_n}$. In order to prove the lemma it suffices to show that L is RE. Below we provide the details.

First, note that if $a, b \in \mathcal{P}$ then we can check whether $[a] \subseteq [b]$ in K : If $a : I \rightarrow \{0, 1\}$ and $b : J \rightarrow \{0, 1\}$ this amounts to verifying that $J \subseteq I$ and $b(i) = a(i)$ for $i \in J$.

Similarly, if $\bar{a}, \bar{b} \in \mathcal{P}^{*d}$ then we can decide whether $[\bar{a}] \subseteq [\bar{b}]$ in Ω : If $\bar{a} : E \rightarrow \mathcal{P}$ and $\bar{b} : F \rightarrow \mathcal{P}$, we only need to check that $F \subseteq E$ and that $[\bar{a}(u)] \subseteq [\bar{b}(u)]$ for $u \in F$; by the above this inclusion is decidable.

Thus, if $\bar{a}, \bar{b} \in \mathcal{P}^{*d}$ and $n \in \mathbb{N}$, then we can decide whether $\pi_n([\bar{a}]) \subseteq [\bar{b}] = \emptyset$ or not.

The fact that L is RE now follows from lemma 3.1, since by assumption $L_* = \cup_{n \in \mathbb{N}} \{n\} \times \{L_n\}$ is RE, and

$$L = \{\bar{a} \in \mathcal{P}^{*d}, \pi_n([\bar{a}]) \subseteq [\bar{b}] \text{ for some } (n, \bar{b}) \in L_*\} \quad \square$$

We also need the following:

Lemma 4.3. *Given a finite $L \subseteq \mathcal{P}^{*1}$, it is decidable whether $\Omega_L = \emptyset$ or not.*

Proof. Suppose L is given. Let $I \subseteq \mathbb{N}$ be large enough that if $\bar{a} \in L$, $\bar{a} : E \rightarrow \mathcal{P}$, then every pattern $\bar{a}(i)$, $i \in E$ is supported in I . It is not hard to check that $\Omega_L \neq \emptyset$ if and only if there is an infinite sequence $x = (x(n))_{n \in \mathbb{Z}}$ over the alphabet $\{0, 1\}^I$ so that, for every $k \in \mathbb{N}$ there is no $\bar{a} \in L$, $\bar{a} : E \rightarrow \mathcal{P}$, which satisfies $x(k+i)(n) = (\bar{a}(i))(n)$ for $i \in E$ and all n at which $\bar{a}(i)$ is defined. Thus, deciding whether Ω_L is empty is equivalent to deciding whether a certain \mathbb{Z} -SFT over the alphabet $\{0, 1\}^I$ is empty (the last condition, though cumbersome, is a finite, local restriction symbols in x and is equivalent to excluding a finite number of patterns). Clearly the reduction from the first problem to the second is computable, and the second problem is decidable, since it is equivalent to deciding whether there are any cycles in a finite graph associated to the given SFT in an effective way (see e.g. [9]). Thus the original problem is decidable as well. \square

Proof. (of theorem 4.1) We show that there is a uniformly recursive sequence $L_n \subseteq \mathcal{P}^{*1}$ so that $X_n = \Omega_{L_n} \neq \emptyset$ for every n and every \mathbb{Z} -EDS appears as one of the X_n 's. Given such a sequence, the product $\prod_{n=1}^{\infty} X_n$ is universal for EDS and is itself an EDS by the previous lemma.

Let $(A_n)_{n=1}^{\infty}$ be a fixed recursive enumeration of all algorithms and let

$$L'_n = \{\bar{b} : A_n \text{ halts on input } \bar{b}\}$$

The sequence L'_n is uniformly RE, because given n and \bar{b} , in order to determine if $\bar{b} \in L'_n$ one first computes A_n (we can because the sequence A_n is recursive) and then simulates the computation of A_n on input \bar{b} , halting only if this computation

halts. This achieves the first of our goals, since clearly the sequence $\Omega_{L'_n}$ will contain all effective subshifts. The problem is that some $\Omega_{L'_n}$'s will be empty. We therefore will define an RE sequence L_n with $L_n = L'_n$ if $\Omega_{L'_n} \neq \emptyset$, and with $\Omega_{L_n} \neq \emptyset$ in any case.

First, given $n, k \in M$ and $\bar{a} \in \mathcal{P}^{*d}$ we say that \bar{a} is (n, k) -recognized if the algorithm A_n halts on input \bar{a} within k steps. Note that this condition is recursive, since A_n can be computed from n and then its computation on input \bar{a} can be simulated for k steps to see if it halts.

Choose an enumeration $\bar{b}_1, \bar{b}_2, \dots$ of \mathcal{P}^{*d} . Given k let

$$L_{n,k} = \{ \bar{b}_i : 1 \leq i \leq k \text{ and } \bar{b}_i \text{ is } (n, k)\text{-recognizable} \}$$

Clearly the $L_{n,k}$ can be computed given n, k , they are increasing in k , and their union is L'_n .

Finally, let

$$L_n = \{ \bar{a} \in \mathcal{P}^{*d} : \Omega_{L_{n,k}} \neq \emptyset \text{ and } \bar{a} \in L_{n,k} \text{ for some } k \in \mathbb{N} \}$$

By lemma 3.1 we see that L_n is RE. Also, if $\Omega_{L'_n} \neq \emptyset$ then $\Omega_{L_{n,k}} \neq \emptyset$ for every k so $\bar{a} \in L_n$ if and only if $\bar{a} \in L'_n$, or in other words, $L_n = L'_n$. On the other hand, if $\Omega_{L'_n} = \emptyset$ then by compactness there is a k for which $\Omega_{L_{n,k}} = \emptyset$; let k_0 be the minimal such k . One sees that $\bar{a} \in L_n$ if and only if $\bar{a} \in L_{n,k_0-1}$, so $L_n = L_{n,k_0-1}$ and by definition $\Omega_{L_{n,k_0}} \neq \emptyset$. This completes the proof. \square

5. NONEXISTENCE OF UNIVERSAL SFTS

The proof from the last section cannot be adapted to \mathbb{Z}^d -EDS because, for $d \geq 2$, one cannot decide in general if a d -dimensional SFT is empty; this is Berger's theorem. Although this in itself is not a proof that no (d, k) -universal SFTs exist for $k > 1$, the proof in fact involves showing that if one did exist then it could be used as a component in an algorithm for deciding the emptiness of SFTs, contradicting Berger's theorem.

Although the proof that universal \mathbb{Z}^d -EDS don't exist for $d \geq 2$ is not conceptually difficult, it will be more transparent to first establish the weaker claim that there are no (d, d) -universal SFTs, i.e. no \mathbb{Z}^d -SFTs which factor onto every other \mathbb{Z}^d -SFT. This follows easily from entropy considerations, but the proof we give is of a recursive nature.

Fix d . It will be convenient to consider SFTs over alphabets which are subsets of \mathbb{N} ; this allows us to examine sets of SFTs without restricting the alphabet size, and is no restriction since any SFT is isomorphic to one over the alphabet \mathbb{N} .

For a language L over Σ , we say that a pattern $a \in \Sigma^E$ is L -admissible if, whenever $b \in \Sigma^F$ is a pattern in L and $F + u \subseteq E$, the pattern b does not appear at u in a ; i.e. $a(u + v) \neq b(v)$ for some $v \in F$.

We return to the question of (d, d) -universal SFTs. It is well-known that the set

$$\mathcal{L} = \{L : L \text{ is a finite set of finite patterns over } \mathbb{N} \text{ and } X_L = \emptyset\}$$

is RE. To see this consider the algorithm that is given as input a finite set L over a finite $\Sigma \subseteq \mathbb{N}$, and iterates over $n \in \mathbb{N}$; for each n it checks if there exist L -admissible $a \in \Sigma^{[-n;n]}$. If none exist it announces that $X_L = \emptyset$ and halts. Otherwise, it goes on to the next n . Clearly if $X_L \neq \emptyset$ then the algorithm will not halt, and a compactness argument shows that if $X_L = \emptyset$ it will.

Thus, in order to prove the claim about non-existence of (d, d) -universal SFTs, we show that, if there is some finite L^* so that $X = X_{L^*} \subseteq \Sigma^{\mathbb{Z}^d}$ factors onto every \mathbb{Z}^d SFT, then the set

$$\mathcal{M} = \{L : L \text{ is a finite set of finite patterns over } \mathbb{N} \text{ and } X_L \neq \emptyset\}$$

is RE. Since \mathcal{L}, \mathcal{M} are complementary in the space of finite sets of patterns over \mathbb{N} , and both are RE, it follows that they are recursive (lemma 3.2); this would contradict Berger's theorem.

The following algorithm establishes that \mathcal{M} is RE. As input it accepts a finite set L of patterns over \mathbb{N} and decides if X_L is empty. Recall that L^* is assumed to be a finite set of patterns so that X_{L^*} factors onto every 0-entropy SFT; we shall use L^* in constructing our algorithm. Let $R \in \mathbb{N}$ be such that each pattern in L^* and L is supported in $[-R, R]^d$.

Algorithm. For each triple r, k, φ with $r, k \in \mathbb{N}$, $r > R + k + 1$, and $\varphi : \Sigma^{[-k;k]^d} \rightarrow \Delta$, do

- (1) Enumerate all L^* -admissible patterns in $\Sigma^{[-r;r]^d}$. Call them a_1, \dots, a_N
- (2) If $\varphi(a_i)|_{[-R;R]^d}$ is L -admissible for every a_i , $1 \leq i \leq N$, announce that $X_L \neq \emptyset$ and halt.

To see that this works, note that if the algorithm halts in (2) for some triple (r, k, φ) then the image under φ of any point in X_{L^*} gives a point in X_L , implying that $X_L \neq \emptyset$. Conversely, if $X_L \neq \emptyset$ then by universality of X_{L^*} there is some k and factor map $X_{L^*} \rightarrow X_L$ which is defined locally by some $\varphi : \Sigma^{[-k;k]^d} \rightarrow \Delta$. A compactness argument shows that for these k, φ , if the condition in (2) fails for every r then there is a point $x \in X_{L^*}$ with $\varphi(x) \notin X_L$, a contradiction.

We now deal with the general case.

Proof. (of theorem 1.2) The proof that there is no universal EDS follows the same argument as above, the only difference being that factor maps are no longer sliding

block codes, which makes the notation more cumbersome. If (X, \mathbb{Z}^d) is a totally disconnected system and $\pi : X \rightarrow Y \subseteq \Sigma^{\mathbb{Z}^d}$ is a factor map to a subshift Y , then the factor map is determined by the partition

$$U_\sigma = \{x \in X_L : f(x)_0 = \sigma\} \quad \sigma \in \Sigma$$

of X into closed and open sets, and conversely if $\{U_\sigma\}_{\sigma \in \Sigma}$ is such a partition then it defines a factor map π into $\Sigma^{\mathbb{Z}^d}$, where $(\pi x)(u) = \sigma$ if and only if $T^u x \in U_\sigma$. Thus in order to adapt the proof above to general EDS we iterate over partitions rather than sliding block codes; note that the finite partitions of Ω into closed and open sets can be effectively enumerated.

Fix d and suppose $L \subseteq \mathcal{P}^{*d}$ is an RE set and that Ω_L factors onto every \mathbb{Z}^d -EDS. As before, we obtain a contradiction by showing that the set \mathcal{L}' above is RE.

Let L_n be a recursive increasing sequence of sets with $\cup L_n = L$, which exists since L is RE. Consider the following algorithm, which as input accepts a finite set M of finite patterns over \mathbb{N} :

- (1) Let R be an upper bound on the diameter of the patterns in M .
- (2) For each triple r, n and $(U_\sigma)_{\sigma \in \Sigma}$, with $r, n \in \mathbb{N}$, $r > R$ and (U_σ) a closed and open partition of Ω , do:
 - If for every $x \in \Omega$ the condition

$$\forall \|u\| \leq n \quad T^u x \notin \Omega \setminus \bigcup_{\bar{a} \in L_n} \bigcup_{\|u\| \leq n} T^u[\bar{a}]$$

implies that $f(x)|_{[R, R]^d}$ is M -admissible, announce that $Y \neq \emptyset$ and halt.

Although the condition in (a) is not phrased as a finite condition and may appear hard to check, it is actually a question about the non-emptiness of the intersection of finitely many cylinder sets which are given in the data, and can therefore be effectively checked. The proof that this algorithm produces the correct output is the same as the one given above for SFTs, and we omit it. \square

6. REALIZATION OF EFFECTIVE ATTRACTORS

Let $U \subseteq \mathbb{R}^d$ be open and let $f : U \rightarrow U$ be an effective map with attractor $X \subseteq U$ as in definition 1.4. We wish to show that there is an EDS which extends (X, f) ; theorem 1.5 then follows.

Let us say that a dyadic rational is one of the form $\frac{k}{2^n}$; a dyadic interval is a closed interval of the form $[\frac{k}{2^n}, \frac{k+1}{2^n}]$; and by a dyadic cell is a product $I_1 \times \dots \times I_d$ of dyadic intervals. An open dyadic cell we mean the interior of a dyadic cell, i.e. the product of open dyadic intervals.

Given a binary representation \bar{x} of $x \in \mathbb{R}$ let $D_N(\bar{x})$ denote the set of all $y \in \mathbb{R}$ whose first N binary digits after the “decimal” point agree with \bar{x} ; this is a closed dyadic interval of length 2^{-N} . For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and binary representations \bar{x}_i of x_i we define $D_N(\bar{x}) = D_N(\bar{x}_1) \times \dots \times D_N(\bar{x}_d)$, which is a dyadic cell with $x \in D_N(\bar{x})$. There are at most 2^d binary representations of (x_1, \dots, x_d) , each giving rise to a dyadic cell containing x .

Lemma 6.1. *Let $f : U \rightarrow U \subseteq \mathbb{R}^d$ be an effective map. Then there is an algorithm which, given a closed dyadic cell $D \subseteq U$ and an integer n , outputs a finite set of rational points in \mathbb{R}^d which are $\frac{1}{n}$ -dense in $f(D)$.*

Proof. Let A be the algorithm given by the definition of an effective function; for a point $x \in \mathbb{R}^d$ with binary representation \bar{x} and for given n let $A(x, n)$ denote the approximation of $f(x)$ produced by A on inputs \bar{x}, n , so $\|A(\bar{x}, n) - f(x)\| < \frac{1}{n}$. Let $N = N(\bar{x}, n)$ denote the number of digits from \bar{x} that is used by A in computing $A(\bar{x}, n)$. If $x' \in D_N(\bar{x})$ and \bar{x}' is a binary representation of $x' \in \mathbb{R}^d$ agreeing with \bar{x} for the first N digits, then $A(\bar{x}', n) = A(\bar{x}, n)$, since the algorithm halts before having a chance to detect that $\bar{x} \neq \bar{x}'$. Hence, for $x' \in D_N(\bar{x})$ we have $\|f(x') - A(\bar{x}, n)\| = \|f(x') - A(\bar{x}', n)\| < \frac{1}{n}$, and so $\|f(x') - f(x)\| < \frac{2}{n}$.

There are at most 2^d binary representations \bar{x} of x , and for each we get an N as above. Let $N^* = N^*(\bar{x}, n)$ be the maximum of these numbers and let $U_n(x)$ be the interior of the union of dyadic cells of side 2^{-N^*} containing x . This is an open set containing x , and from the discussion above we see that the diameter of $f(U_n(x))$ is $< \frac{5}{n}$, since the image of each cell has diameter $< \frac{2}{n}$ and is within distance $\frac{1}{n}$ of $f(x)$. Note that for a dyadic point $r \in \mathbb{Q}^d$, both $N^*(r, n)$ and $U_n(r)$ are computable.

If $D \subseteq U$ is a closed dyadic cell and $n \in \mathbb{N}$, then there is a finite cover of D by sets of the form $U_n(r)$, and such a cover can be computed by iterating over all finite collections of the sets $U_n(r)$ for r a dyadic rational, until such a collection is found that covers D . If $\{U_n(r_i)\}_{i=1}^M$ is such a collection, then as we have seen, the set $\{A(\bar{r}_i, n)\}_{i=1}^M$ is $\frac{5}{n}$ -dense in $f(D)$, where $A(\bar{r}, n)$ is the output of the algorithm on input n and the binary representation \bar{r} of r . \square

Lemma 6.2. *Let X be the attractor of an effective map $f : U \rightarrow U \subseteq \mathbb{R}$. Then there is an algorithm which, when given a dyadic cell $D = I_1 \times \dots \times I_d \subseteq U$ as input, halts if $X \cap D = \emptyset$, and otherwise runs forever.*

Proof. By assumption there is an open set $V \subseteq U$ with $\overline{fV} \subseteq V$ and $X = \bigcap f^n V$. By covering X by small open dyadic cells and then taking their closure, we see that there is a finite set of closed dyadic cells C_1, \dots, C_N which cover X and are contained in V . Denoting by C their union, we have $X = \bigcap f^n C$. Thus for any dyadic cell $D \subseteq U$ we have $D \cap X = \emptyset$ if and only if, for some n , $D \cap f^n C_i = \emptyset$ for $i = 1, \dots, N$.

We do not claim that C can be found effectively but it exists and can be described by finite data, and we may use it in the algorithm that we now present. As input the algorithm takes a dyadic cell D . It then iterates over n and for each n it computes a finite set of points F which are $\frac{1}{n}$ -dense in $f(C)$ (this can be done by the previous lemma). If every point in f has distance $> \frac{1}{n}$ from D the algorithm halts; otherwise it proceeds to the next n . \square

In the same way we have

Lemma 6.3. *Let X be the attractor of an effective map $f : U \rightarrow U \subseteq \mathbb{R}$. Then there is an algorithm which, given as input two (closed) dyadic cells D', D'' , halts $f(D') \cap D'' = \emptyset$ and otherwise runs forever.*

We can now prove theorem 1.5:

Proof. Let $f : U \rightarrow U$ be an effective map with attractor $X \subseteq U$. Since X is bounded and we can assume that U is bounded, and without loss of generality $U \subseteq [0, 1]^d$. Let $K = \{0, 1\}^{\mathbb{N}}$ be the Cantor set and $\pi : K \rightarrow [0, 1]^d$ be given by $x \mapsto (x_1, \dots, x_d)$ where

$$x_i = \sum_{n=0}^{\infty} 2^{-n+1} x(dn + i)$$

Let $Y \subseteq \Omega = K^{\mathbb{Z}}$ be those points ω so that

$$\pi(\omega(n)) \in X \quad , \quad n \in \mathbb{Z}$$

and

$$\pi(\omega(n+1)) = f(\pi(\omega(n)))$$

Clearly Y is a subshift and $\pi : Y \rightarrow X$ is a factor map from Y to X . It remains to show that Y is an effective subshift.

Recall the definition of \mathcal{P}^{*1} from section 3. Note that if $a \in \{0, 1\}^{\{1, 2, \dots, dk\}}$ then $\pi([a])$ is a dyadic cell in $[0, 1]^d$. Consider the set of generalized cylinder sets

$$L = \left\{ \bar{a} \in \mathcal{P}^{*1} \left| \begin{array}{l} \bar{a} : \{n, n+1\} \rightarrow \{0, 1\}^{\{1, 2, \dots, dk\}} \text{ for some } n, k \in \mathbb{N} \\ \text{and either } f(\pi([\bar{a}(n)])) \cap \pi([\bar{a}(n+1)]) = \emptyset \\ \text{or } \pi([\bar{a}(n)]) \cap X = \emptyset \text{ or } \pi([\bar{a}(n+1)]) \cap X = \emptyset \end{array} \right. \right\}$$

One may verify that $Y = \Omega_L$; the proof will be complete if we show that L is RE. In order to do so it suffices to show that there is an algorithm which, given $\bar{a} : \{n, n+1\} \rightarrow \{0, 1\}^{\{1, 2, \dots, dk\}}$, halts if $\pi([\bar{a}(n)]) \cap X = \emptyset$ or $\pi([\bar{a}(n+1)]) \cap X = \emptyset$ or $f(\pi([\bar{a}(n)])) \cap \pi([\bar{a}(n+1)]) = \emptyset$, and otherwise runs forever. But this is a direct consequence of the lemmas preceding this proof. \square

7. OPEN PROBLEMS

We conclude with a few problems which arise in connection with this work.

We have shown that universality cannot occur for effective \mathbb{Z}^d systems. This can be rephrased as follows: If $d > 2$ then for every effective \mathbb{Z}^d -system X , there is a \mathbb{Z}^d -SFT Y so that X does not factor onto Y . If we relax the effectiveness requirement, a stronger result is true: under mild assumptions on the original system X , there is a system disjoint from X in the sense of Furstenberg.

Problem. Given a minimal \mathbb{Z}^d -EDS, does there exist a \mathbb{Z}^d -EDS disjoint from it? (one may of course ask this about other classes).

(minimality is required to avoid trivial counterexamples). The same question may be asked for minimal SFTs.

The non-existence of universal EDS was demonstrated using purely recursion-theoretic considerations. It is not clear how to handle some restricted interesting classes of EDS. Of particular interest are the minimal SFTs, nontrivial examples of which were constructed in [10] and more examples follow from [5]. We note that for SFTs, minimality implies zero entropy, so entropy cannot rule out a universal minimal SFT. We also note that, besides being an interesting class dynamically, minimal SFTs have the additional feature that the set of patterns appearing in a minimal SFT is recursive, and the extension problem is decidable for them, i.e. given a locally admissible pattern one can decide if it can be extended to an infinite configuration [4]. Thus this class does not exhibit the recursive complexity of SFTs in general.

Problem. Are there universal systems in the class of minimal SFTs? (and if so, in what dimensions?)

Finally, a recursive product of effective systems is effective. One can ask a related question about SFTs:

Problem. If a recursive product SFTs has finite entropy, can it be extended to an SFT?

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