

ROHLIN PROPERTIES FOR \mathbb{Z}^d ACTIONS ON THE CANTOR SET

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ABSTRACT. We study the space $\mathcal{H}(d)$ of continuous \mathbb{Z}^d -actions on the Cantor set, particularly questions on the existence and nature of actions whose isomorphism class is dense (Rohlin's property). Kechris and Rosendal showed that for $d = 1$ there is an action on the Cantor set whose isomorphism class is residual; we prove in contrast that for $d \geq 2$ every isomorphism class in $\mathcal{H}(d)$ is meager. On the other hand, while generically an action has dense isomorphism class and the effective actions are dense, no effective action has dense isomorphism class; thus conjugation on the space of actions is topologically transitive but one cannot construct a transitive point. Finally, we show that in the space of transitive and minimal actions the effective actions are nowhere dense, and in particular there are minimal actions that are not approximable by minimal SFTs.

1. INTRODUCTION

Dynamical systems theory studies the asymptotic behavior of automorphisms of some suitable structure, e.g. a measure space, topological space or manifold, and more generally group actions by automorphisms. The space of all such actions often carries a natural topology, and one is led to questions about the distribution of isomorphism types in the space of all actions, and the manner in which certain actions can, or cannot, approximate others. In this way one hopes to achieve some understanding of the relation between different types of dynamics. Such a set-up is also suitable for studying rigidity phenomena, i.e. the transmission of dynamical behaviors from certain actions to actions in some neighborhood of it. For example see [9, 1, 3, 18].

In ergodic theory this point of view is classical and goes back to the work of Rohlin and Halmos [9], who studied the group of automorphisms of a Lebesgue space equipped with the so-called coarse topology. Of interest to us will be Rohlin's theorem that any aperiodic automorphism has a dense isomorphism class. One should interpret this as the assertion that, when viewed at any finite resolution, one cannot distinguish any aperiodic isomorphism type from any other; all dynamical types are "mixed together" rather well. On the other hand, by a theorem of del Junco there is a residual set of automorphisms disjoint from any fixed ergodic

automorphism, and hence no isomorphism class can be residual; it follows from the general theory of Polish group actions that every aperiodic isomorphism type, while dense, is meager. These results hold for actions of more general groups, including \mathbb{Z}^d actions (Rohlin's theorem at least holds for all discrete amenable groups).

A natural analogue in the topological category is the space of continuous actions on the Cantor set, which has received renewed attention recently. We denote the Cantor space by K , and let

$$\mathcal{H} = \text{homeo}(K)$$

be the Polish group of homeomorphisms of K with the topology of uniform convergence. Each $\varphi \in \mathcal{H}$ gives rise to a \mathbb{Z} -action on K , which is the dynamical system associated to it. Glasner and Weiss [7] showed that, as in the ergodic-theory setting, there exist actions $\varphi \in \mathcal{H}$ whose isomorphism class is dense in \mathcal{H} (although it is not true that this is so for every aperiodic φ , as in Rohlin's theorem). More recently this result has been subsumed by a remarkable theorem of Kechris and Rosendal [14], who proved that there is actually a single isomorphism class that is residual. Thus, generically, there is only one \mathbb{Z} -actions on K . This action was later described explicitly by Akin, Glasner and Weiss, and it turns out to be rather degenerate, for example, it is not transitive; but in the Polish space of transitive actions there is also a generic action [12].

The aim of the present paper is to begin the study of the space of \mathbb{Z}^d -actions on K , which are of interest both in themselves and as the topological systems underlying a large number of lattice models in statistical mechanics and probability. We denote this space by $\mathcal{H}(d)$; formally, it is defined by

$$\mathcal{H}(d) = \text{hom}(\mathbb{Z}^d, \mathcal{H})$$

with the Polish topology it inherits as a closed subset of the countable product $\mathcal{H}^{\mathbb{Z}^d}$. The group \mathcal{H} acts on $\mathcal{H}(d)$ by conjugation: if $\varphi \in \mathcal{H}(d)$ is an action $\{\varphi^u\}_{u \in \mathbb{Z}^d}$, and $\pi : K \rightarrow K$ is a homeomorphism, then the conjugation of φ by π is the isomorphic action $\{\pi\varphi^u\pi^{-1}\}_{u \in \mathbb{Z}^d}$. We denote the conjugacy class of φ by $[\varphi]$; this is by definition the set of actions in $\mathcal{H}(d)$ that are isomorphic to φ .

The group \mathbb{Z}^d is said to have the *weak topological Rohlin property* (WTRP) if there is an action in $\mathcal{H}(d)$ with dense conjugacy class; it has the *strong topological Rohlin property* (STRP) if there is a conjugacy class that is a dense G_δ (this is equivalent to there existing a conjugacy class containing a dense G_δ). This terminology has evolved recently in connection with questions about the largeness of conjugacy classes in topological groups and, more generally, largeness of conjugacy classes of the actions of a fixed group; the archetypal example being Rohlin's result on the group of measure-preserving automorphisms. See [8] for a recent survey and extensive bibliography.

As we have seen, \mathbb{Z} has the strong (and therefore also the weak) topological Rohlin property. The mechanism behind this is a rather simple stability phenomenon whereby certain shifts of finite type propagate their structure to nearby actions. To be precise,

Theorem 1. *Let X be a \mathbb{Z}^d -shift of finite type and $\varphi \in \mathcal{H}(d)$ an action that factors into X . Then there is a neighborhood U of φ so that every action $\psi \in U$ factors into X .*

In particular if X is minimal then all actions sufficiently close to X factor *onto* X , and in dimension 1 the same is true if X is a 0-entropy SFT. This partly explains the fact that the generic \mathbb{Z} -system of Kechris and Rosendal as the countable product of all zero-entropy shifts of finite type (the Akin-Glasner-Weiss description is somewhat different). The proof of this also relies on some very special properties of zero-entropy shifts of finite type in dimension 1, particularly the fact that their joinings decompose into countable many disjoint subsystems of the same type. In contrast, in higher dimensions the same stability phenomenon exists but the behavior of shifts of finite type is far more complicated.

Utilizing recent advances in our understanding of multidimensional shifts of finite type, we are able to show that the case $d > 1$ differs from that of $d = 1$:

Theorem 2. *For $d \geq 2$, any action $\varphi \in \mathcal{H}(d)$ has a meager conjugacy class, i.e. \mathbb{Z}^d does not have the strong topological Rohlin property.*

It is much easier to establish that $\mathcal{H}(d)$ has the weak Rohlin property. In fact this is a simple consequence of separability of $\mathcal{H}(d)$, and holds for the space of actions of any discrete groups:

Proposition 3. *For $d \geq 2$, $\mathcal{H}(d)$ has the weak topological Rohlin property, i.e. there are actions with dense conjugacy class.*

However, there is an interesting twist. In $\mathcal{H}(1)$, there are explicit constructions of systems with dense conjugacy class. There are several ways to make precise the notion of an explicit constructions, one of which is the following. Say that $\varphi \in \mathcal{H}(d)$ is *effective* if there is an algorithmic procedure for deciding, given a finite set $F \subseteq \mathbb{Z}^d$ and a family $\{C_u\}_{u \in F}$ of closed and open subsets of K , whether $\bigcap_{u \in F} T^u C_u = \emptyset$. In fact, one can (and we shall) weaken this and demand only that emptiness of this intersection can be semi-decided, in the sense that if it is empty the algorithm must detect this and halt, but may otherwise it need not return a decision (see section 2.3 for a discussion and some other notions of effectiveness). The Kechris-Rosendal can be realized as an effective action in both the stronger and weaker sense, and one can also construct explicitly other actions with dense orbit as Glasner and Weiss did.

In contrast,

Theorem 4. *For $d \geq 2$ there are no effective actions with dense conjugacy class.*

Stated another way, the conjugation action of \mathcal{H} on $\mathcal{H}(d)$ is topologically transitive, and therefore there is a dense G_δ set of actions whose conjugacy class is dense; but it is formally impossible to construct a transitive point.

In spite of the above, note that the effective systems are dense in $\mathcal{H}(d)$ (e.g. the SFTs are dense; see proposition 7 below). But one cannot use the implication separable \implies WTRP, as in proposition 3, to get an effective transitive point, because the separability condition is not effective: there is no recursive dense sequence of effective actions.

Nevertheless, density of effective systems means that in a certain sense the entire space $\mathcal{H}(d)$ is accessible to us. It turns out that this is not the case for some other interesting spaces. Consider for example the Polish space $\mathcal{M}(d) \subseteq \mathcal{H}(d)$ of minimal actions, i.e. actions in which every orbit is dense. Classically such actions have been studied extensively as the analogue of ergodic actions, and there is a rich theory of their structure for arbitrary acting groups [4]. In dimension 1, one can explicitly construct families of minimal actions that are dense in $\mathcal{M}(1)$; indeed, in [12] we showed that the universal odometer, i.e. the unique (up to isomorphism) minimal subsystem of the product of all finite cycles, is generic there, and in particular has a dense conjugacy class; and in other reasonable parametrization one can show that the finite cycles are dense (these fail to be dense in $\mathcal{H}(1)$ only for the technical reason that their phase space is not the Cantor set). On the other hand, the following theorem shows that in higher dimensions the space of minimal actions is in a very strong sense inaccessible to us, even in the approximation sense:

Theorem 5. *For $d \geq 2$, the systems conjugate to minimal effective systems are nowhere dense in $\mathcal{M}(d)$.*

A similar statement holds for transitive systems. Note that a system may be conjugate to an effective system without being effective itself.

Note that SFTs are effective, and it follows that there are minimal actions which cannot be approximated by SFTs. This is somewhat unexpected as well: in dimension $d \geq 2$ SFTs display a wealth of dynamics, including minimal dynamics, and this has led to the impression that they can represent quite general dynamics. Theorem 5 shows that this is far from the case.

All the results above have an analogue in the space of closed subsystems of the shift space $Q^{\mathbb{Z}}$, where Q is the Hilbert cube (the topology is that of the Hausdorff metric). This model was studied in [12] and the methods there can be used to translate the present results to that setting.

The rest of this paper is organized as follows. In the next section we prove the theorems about the WTRP and prove theorem 5. Section 3 is devoted to the STRP. In section 4 we conclude with some open questions.

2. THE WEAK TOPOLOGICAL ROHLIN PROPERTY

In this section we prove theorem 2 and 5. We first develop some basic facts about $\mathcal{H}(d)$.

2.1. Actions versus subshifts. When discussing an action φ on K we shall abbreviate and write φ for the associated dynamical system (K, φ) . When dealing with a subshift X of a symbolic space $\{1, 2, \dots, k\}^{\mathbb{Z}^d}$ or $K^{\mathbb{Z}^d}$, we denote by $\sigma = \{\sigma^u\}_{u \in \mathbb{Z}^d}$ the shift action and denote the system (X, σ) simply by X . We refer the reader to [19] for basic definitions from topological dynamics.

There is a close connection between the space of actions of \mathbb{Z}^d on K and the space of subsystems of a shift space, with the Hausdorff metric, and we shall have occasion to work with both settings. This connection was explored in [12]. Our presentation focuses on the space \mathcal{H} but appeals to the subshift model at some points to make use of symbolic constructions and invariants.

2.2. Topology of $\mathcal{H}(d)$ and projection into SFTs. Let e_1, \dots, e_d denote the standard generators of \mathbb{Z}^d . For concreteness let us fix a complete metric d on K , and for $\varphi, \psi : K \rightarrow K$ let

$$d(\varphi, \psi) = \max_{x \in K} d(\varphi(x), \psi(x)) + \max_{y \in K} d(\varphi(y), \psi(y))$$

This is a complete metric on \mathcal{H} , and one verifies that the metric

$$d(\varphi, \psi) = \max_{i=1, \dots, d} d(\varphi^{e_i}, \psi^{e_i})$$

is a complete metric on $\mathcal{H}(d)$ compatible with the topology defined in the Introduction.

Given a partition $\alpha = \{A_1, \dots, A_n\}$ of K into clopen sets and an action $\varphi \in \mathcal{H}(d)$, we write $c_\alpha : K \rightarrow \{1, \dots, n\}^{\mathbb{Z}^d}$ for the coding map that takes $x \in K$ to its α -itinerary, i.e. to the sequence $(x_u)_{u \in \mathbb{Z}^d} \in \{1, \dots, n\}^{\mathbb{Z}^d}$ with $x_u = i$ if and only if $\varphi^u x \in A_i$. We write $c_{\alpha, \varphi}$ when we want to make explicit the dependence on φ . We also write $\widehat{c}_\alpha(\varphi)$ for the image of the map $c_{\alpha, \varphi}$, which is a subshift of $\{1, \dots, n\}^{\mathbb{Z}^d}$. Thus \widehat{c}_α maps actions to subshifts, and $c_\alpha = c_{\alpha, \varphi}$ is the factor map from (K, φ) to $(\widehat{c}_\alpha(\varphi), \sigma)$.

Recall that an SFT is a subshift $X \subseteq \Sigma^{\mathbb{Z}^d}$ (Σ finite) defined by a finite set of finite patterns a_1, \dots, a_n and is the set of configurations $x \in \Sigma^{\mathbb{Z}^d}$ that do not contain occurrences of any of the a_i .

Although the SFT condition appears syntactic, it is an isomorphism invariant. That is, if two subshifts are isomorphic and one is an SFT, so is the other (defined by some other set of patterns). We shall thus also refer to actions $\varphi \in \mathcal{H}(d)$ as SFTs if they are isomorphic to SFTs.

Proposition 6. *Suppose $\alpha = \{A_1, \dots, A_n\}$ is a clopen partition of K and $\varphi \in \mathcal{H}(d)$ is mapped via \widehat{c}_α into a shift of finite type $X \subseteq \{1, \dots, n\}^{\mathbb{Z}^d}$ (i.e. $\widehat{c}_\alpha(\varphi) \subseteq X$). Then there is a neighborhood of φ in $\mathcal{H}(d)$ whose members are mapped via \widehat{c}_α to subsystems of X .*

Proof. Suppose $X \subseteq \{1, \dots, n\}^{\mathbb{Z}^d}$ is specified by disallowed patterns b_1, \dots, b_k of diameter $< r$. Since A_i are clopen, it follows that whenever ψ is an action close enough to φ then $\varphi^u(x), \psi^u(x)$ belong to the same atom A_i for every $u \in [-r, r]^d$ and $x \in K$. Thus $c_{\alpha, \psi}(x)$ does not contain any of the b_i and so $c_\alpha(\psi) \subseteq X$. \square

Proposition 7. *The shifts of finite type are dense in $\mathcal{H}(d)$.*

Proof. Let $\varphi \in \mathcal{H}$, $\varepsilon > 0$, and choose a clopen partition $\alpha = \{A_1, \dots, A_n\}$ of K whose atoms are of diameter $< \varepsilon$. Let $X \subseteq \{1, \dots, n\}^{\mathbb{Z}^d}$ be the SFT specified by the condition that if for some $1 \leq i, j \leq n$ and $1 \leq k \leq d$ there is no $y \in K$ such that $y \in A_i$ and $\varphi^{e_k} y \in A_j$, then whenever $x \in X$ and $x(u) = i$ then $x(u + e_k) \neq j$. Let $X' = X \times Y$ where Y is the full shift (this is only to ensure that X' has no isolated points). Choose a homeomorphism $\pi : X \times Y \rightarrow K$ mapping $[i] \times Y$ onto A_i (here $[i]$ is the cylinder set $[i] \subseteq \{1, \dots, n\}^{\mathbb{Z}^d}$), and let $\psi = \pi \sigma \pi^{-1}$. One verifies that $d(\varphi, \psi) < \varepsilon$ and clearly ψ is conjugate to the SFT X' . \square

2.3. Effective dynamics. There are a number of ways that one can define what it means for a dynamical system is computable. We shall adopt a rather weak one; at the end of this section we briefly discuss its relation to other notions of computability.

A sequence (a_n) of integers is *recursive* (R) if there is an algorithm A (formally a Turing machine) that, upon input $n \in \mathbb{N}$, outputs a_n . A set of integers is *recursively enumerable* (RE) if it is the set of elements of some recursive sequence. By identifying the integers with other sets we can speak of recursive sequences of other elements. For example, since $\mathbb{N} \cong \mathbb{N}^2$ (and the bijection can be made effective), we can speak of recursive sequences of pairs of integers; and in the same way of sequences of finite sequences of integers.

We shall assume from here on that the Cantor set is parametrized in an explicit way. We shall use several such parametrization, representing K as $\{0, 1\}^{\mathbb{N}}$, $\{1, 2, \dots, k\}^{\mathbb{Z}^d}$ and $(\{0, 1\}^{\mathbb{N}})^{\mathbb{Z}^d} = K^{\mathbb{Z}^d}$. All three may be identified by explicit homeomorphisms in such a way that a family of cylinder sets in one is R or RE if and

only if the corresponding family of cylinder sets in the other parametrization are also R or RE, respectively.

A subset $X \subseteq K$ is *effective* if its complement is the union of a recursive sequence of cylinder sets. Effective sets are automatically closed and have been extensively studied in the recursion theory literature, see e.g. [16].

Definition 8. A closed, shift-invariant subset $X \subseteq \{1, 2, \dots, k\}^{\mathbb{Z}^d}$ or $X \subseteq K^{\mathbb{Z}^d}$ is effective if it is an effectively closed set.

Note that this is not an isomorphism invariant. This is clear from cardinality considerations, since there are countably many effective subsets (each is defined by some algorithm), but there are uncountable many ways to embed the full shift $\{0, 1\}^{\mathbb{Z}^d}$, which is effective, in $K^{\mathbb{Z}^d}$. However, effectiveness is preserved under symbolic factors and is thus an invariant for symbolic systems:

Proposition 9. *A symbolic factor of an effective system is effective.*

For a proof see [11, Proposition 3.3]. It is crucial that by a symbolic factor we mean a factor that is a subsystem of $\{1, \dots, k\}^{\mathbb{Z}^d}$ for some k . Although a subsystem $Y \subseteq \{x_1, \dots, x_k\}^{\mathbb{Z}^d}$ for points $x_i \in K$ is isomorphic to a symbolic system, such a Y may or may not be effective as a subsystem of $K^{\mathbb{Z}^d}$, depending on the points x_i (to see this it is enough to consider fixed points of the shift action on $K^{\mathbb{Z}^d}$).

Since we are working in the space of actions, rather than the space of subshifts, we introduce the following:

Definition 10. Let $\varphi \in \mathcal{H}(d)$. A finite sequence $\{(C_i, u_i)\}_{i=1}^n$, where C_i is a cylinder set and $u_i \in \mathbb{Z}^d$, is φ -disjoint if

$$\bigcap_{u \in F} \varphi^u A_u = \emptyset$$

φ is *effective* if the set of φ -disjoint sequences is RE, or in other words, if there is an algorithm that can recognize a disjoint sequences in finite time (but may or may not identify non-disjoint ones).

This definition is related to effective subshifts as follows. Given an action $\varphi \in \mathcal{H}(d)$ and $x \in K$ let $\pi_\varphi(x) \in K^{\mathbb{Z}^d}$ be the point $(\pi_\varphi(x))_u = \varphi^u x$. Then $\pi_\varphi : K \rightarrow \pi(K)$ embeds (K, φ) as the subsystem $(\pi_\varphi(K), \sigma)$ of $(K^{\mathbb{Z}^d}, \sigma)$ (recall that σ denotes the shift action). One may verify that φ is effective if and only if $\pi_\varphi(X)$ is effective.¹

Another way one might define effectiveness of an action φ is to require that the maps φ^u are computable. More precisely, φ is computable if there is an algorithm

¹Note that if $Y \subseteq K^{\mathbb{Z}^d}$ is a closed and shift invariant Cantor set which is effective, it can happen that Y is not of the form $Y = \pi_\psi(K)$ for any action ψ on K . For example, this is the case when $Y \subseteq \{x_1, \dots, x_k\}^{\mathbb{Z}^d}$ is an infinite subshift for some fixed $x_1, \dots, x_k \in K$.

that, given $u \in \mathbb{Z}^d$, an integer n and a point $x \in K$, reads finitely many bits x_i of x and outputs $(\varphi^u x)_n$. This definition is similar to that of Braverman and Cook [6], and implies continuity of φ^u and that the moduli of continuity are computable. From this one can deduce that this notion is strictly stronger than effectiveness in the sense of definition 10. Other definitions for effectiveness for sets, functions and dynamical system have received some attention recently; see [?, ?, ?].

2.4. Weak topological Rohlin Property. Recall that a perfect space is one without isolated points. The non-effectiveness part of theorem 3 relies on the following:

Theorem 11. *No effective \mathbb{Z}^d -action factors into every perfect SFT.*

The proof of this result is essentially identical to the proof given in [10], where it was shown that if there were an effective system that factors onto every SFT, then it could be used as part of an algorithm that decides whether a given SFT is empty, and this is undecidable by Berger's theorem [5, 15]. Two modifications to the proof are needed to deduce the version above. First, an inspection of the proof in [10] shows that it does not use the fact that the factor map is onto; thus the same proof works with the present hypothesis that the map is into. Second, to prove the version above we must show that, given the rules of an SFT which is either empty or perfect, it is undecidable whether it is empty. But if we could decide this, we could decide whether an arbitrary SFT were empty, since an SFT X is empty if and only if $X \times \{0, 1\}^{\mathbb{Z}^d}$ is empty, and the latter is either empty or perfect.

Corollary 12. *If $\varphi \in \mathcal{H}(d)$ has dense conjugacy class then it is not effective.*

Proof. By proposition 7 φ would factor into every $\psi \in \mathcal{H}(d)$ that is conjugate to an SFT; hence it would factor into every perfect SFT, and the conclusion follows from theorem 11. \square

To complete the proof of theorem 3 it remains to establish that there is a dense conjugacy class in $\mathcal{H}(d)$. The argument is similar to that given in [2] for the measure-preserving category, and works for any countable group. Since $\mathcal{H}(d)$ is separable, we may choose a dense sequence $\varphi_1, \varphi_2, \dots$ in it, and let $\psi = \times_i \varphi_i$ be the product action on K^{\aleph_0} , i.e.,

$$\psi(x_1, x_2 \dots) = (\varphi_1(x_1), \varphi_2(x_2), \dots)$$

It suffices to show that the closure of $[\psi]$ contains all the φ_i . We work with the parametrization $K = \{0, 1\}^{\mathbb{N}}$. Fix an integer $n \in \mathbb{N}$; by uniform continuity of $\varphi_1^{e_1}, \dots, \varphi_i^{e_d}$ there is a $k(n)$ so that each $j = 1, \dots, d$ the first n coordinates of $\varphi_1^{e_j}(x)$ depend only on the first $k(n)$ coordinates of x for. Choose a partition

I_1, I_2, \dots of \mathbb{N} into infinite sets with $\{1, \dots, k(n)\} \subseteq I_1$. Let $\pi_m : \mathbb{N} \rightarrow I_m$ be order-preserving isomorphisms and let $\tau = \tau_n$ be the action on $K^{\mathbb{N}^0}$ which acts on K^{I_m} like $\pi_m \varphi_m \pi_m^{-1}$. The action τ is conjugate to ψ , and the first n coordinates of $\tau^{e_j}(x)$ and $\varphi_1^{e_j}(x)$ agree for all $x \in K$. This proves the claim and completes the proof of theorem 3.

3. THE STRONG ROHLIN PROPERTY

In this section we prove theorem 2. The key fact that we use is that if there were a generic system $\theta \in \mathcal{H}(2)$, then it has countably many symbolic factors X_1, X_2, \dots , since every symbolic factor of θ arises from a clopen partition of K and there are only countable many of these. We shall construct an SFT Y and associated action φ so that, in a neighborhood $U \subseteq \mathcal{H}(d)$ of φ , a generic $\psi \in U$ has a symbolic factor distinct from the X_i 's, and hence is not conjugate to θ . This factor will be the projection of ψ into Y .

The main property we want of Y is that its subsystems can be easily perturbed. This will be accomplished by making the space of subsystems of Y be very rich. More precisely, there will be a sofic factor Z of Y whose subsystems are not isolated, and furthermore if $X \subseteq Z$ is effective then X also has no isolated subsystems. This is what will allow us to perturb the projections of actions into Y . The control over subsystems will be achieved using recursive methods.

3.1. Medvedev degrees and dynamics. Given $X \subseteq \{0, 1\}^{\mathbb{N}}$, a function $f : X \rightarrow \{0, 1\}^{\mathbb{N}}$ is *computable* if there is an algorithm A such that, when given as input a point $x \in X$ (technically, x is an oracle for the computation) and an integer k , outputs the first k coordinates of $f(x)$. Note that x is an infinite sequence of 0 and 1's, but the algorithm will perform finitely many operations before halting so it will only read a finite number of these bits. Which bits it chooses to read will depend on the bits it has already read and on k . Thus if x' differs from x on coordinates which were not used then running the algorithm on x' , k will give the same result as x, k . It follows easily that a computable function is continuous in the induced topology.

An effective subset $Y \subseteq \{0, 1\}^{\mathbb{N}}$ is reducible to an effective subset $X \subseteq \{0, 1\}^{\mathbb{N}}$ if there is a computable function $f : X \rightarrow Y$ (not necessarily onto). We denote this relation by $X \succ Y$. One should interpret this as follows: suppose we want to show that Y is not empty by producing in some manner a point $y \in Y$. If $X \succ Y$ and if we can produce a point $x \in X$ then we can, by applying the computable function f , obtain the point $y = f(x)$. Thus X is at least as complicated as Y , in the sense that demonstrating that $X \neq \emptyset$ is at least as hard as demonstrating that $Y \neq \emptyset$. Notice that if $X \subseteq Y$ then $X \succ Y$ (the identity map is computable), and that if

$y \in Y$ is computable as a function $y : \mathbb{N} \rightarrow \{0, 1\}$ (i.e. if there is an algorithm that given k computes the k -th coordinate of y) then $X \succ Y$ for all X , because there is a computable function $X \rightarrow \{y\}$, i.e. the map that doesn't use the input x at all and simply computes the components of y .

We say that X, Y are Medvedev equivalent if $X \succ Y$ and $Y \succ X$. The equivalence class of X is denoted $m(X)$ and called is Medvedev degree of X . By the above, there is a minimal Medvedev degree consisting of all effective sets containing computable points. There is also a maximal element. There are infinitely many Medvedev degrees, and they form a distributive lattice. Overall, the structure of this lattice is still rather mysterious, although the theory of Medvedev degrees is classical in recursion theory; see [16].

Medvedev degrees were introduced into the study of SFTs by S. Simpson [17], who observed that, since a factor map between SFTs is given by a sliding block code, the factoring relation $X \rightarrow Y$ between SFTs implies $m(X) \succ m(Y)$. This is true more generally for effective symbolic systems and leads to the question, which is still far from understood, of the relation between the Medvedev degree of an effective system and its dynamics. One such connection is the following, which will be central to our argument:

Proposition 13. *If an effective subshift is minimal then it has minimal Medvedev degree.*

The proof follows from [11], proposition 9.4, where it was shown for SFTs.

We can now prove theorem 5. Suppose Y is an SFT with non-minimal Medvedev degree, and let $Y_0 \subseteq Y$ be a minimal subsystem. Let φ be an action conjugate to (Y_0, σ) . Thus there is an open neighborhood $U \subseteq \mathcal{H}(d)$ of ψ so that every $\psi \in U$ factors into Y .

We claim that U does not contain any minimal SFTs. Indeed, if $\psi \in U$ were an action conjugate to a minimal SFT then every symbolic factor of (K, ψ) is effective and has minimal Medvedev degree. But this would imply that Y contains an effective subshift with minimal degree and so itself has minimal degree, contrary to assumption.

The same argument shows that the effective systems are nowhere dense in the space of transitive actions.

3.2. Construction of the SFT Y . We construct an SFT Y factoring onto a sofic shift $Y \rightarrow Z$, so that Z is the union of its minimal subsystems and has nontrivial Medvedev degree.

Let $\Omega \subseteq \{0, 1\}^{\mathbb{N}}$ be an effective, closed set of non-trivial Medvedev degree. To each $\omega \in \Omega$ we assign the point $y_\omega \in \{0, 1\}^{\mathbb{Z}}$ defined as follows. First choose an effective enumeration of the integers: $n(1), n(2), \dots$. Select the coordinates of y_ω

that form the arithmetic progression of period 2 passing through $n(1)$, and assign to them the symbol $\omega(1)$. Next, choose the arithmetic progression of period 4 passing through the first of the $n(i)$ that is not yet colored, and assign to these coordinates the symbol $\omega(2)$. At the k -th step, color with the symbol $\omega(k)$ the coordinates belonging to the arithmetic progression that passes through the first uncolored $n(i)$. Let Z_ω denote the orbit closure of z_ω and set $Z' = \cup_{\omega \in \Omega} Z_\omega$. The map $\omega \mapsto z_\omega$ is a computable function $\Omega \rightarrow Z'$, and there is also a recursive function $Z' \rightarrow \Omega$. One may verify Z' is closed and is effective. Thus Z', Ω are Medvedev equivalent.

To obtain an SFT Y from Z' , we rely on the construction in section 6 of [13]:

Theorem 14. *There is a \mathbb{Z}^2 sofic shift Z such that $(Z, \sigma^{e_1}) \cong Z'$ and σ^{e_2} acts as the identity on Z .*

Let Y be an SFT factoring onto the sofic shift Z and let $\rho : Y \rightarrow Z$ be the factor map. It is easily verified that Y has the required properties. We mention that $m(Y)$ is non-trivial because $Y \succ Z$ and $m(Z) = M(Z') \neq 0$.

3.3. Subsystems and extensions of Y . Recall that if X is a metric space with metric d , then the Hausdorff distance between compact subsets $A, B \subseteq X$ is defined by the condition that $d(A, B) < \varepsilon$ if and only if for each $a \in A$ there is a $b \in B$ with $d(a, b) < \varepsilon$ and the same with the roles of A, B reversed. The topology induced by the Hausdorff metric is independent of the metric we began with, is compact when X is, and is totally disconnected if X is.

Note that if (X, φ) is a dynamical system then the space of subsystems is closed in the Hausdorff metric. The following is elementary:

Lemma 15. *Let $W \subseteq \{1, \dots, k\}^{\mathbb{Z}^d}$. If W is an SFT then the subshifts of W which are SFTs are dense among the subsystems of W ; and similarly if W is effective then its effective subsystems are dense.*

Proof. We prove the SFT case, the effective case being similar. Suppose W is defined by disallowed patterns $\bar{b} = b_1, \dots, b_m$. Fix a subsystems $X \subseteq W$, which we must show is an accumulation point of SFTs. X is defined by an infinite sequence of disallowed patterns $b_1, \dots, b_m, b_{m+1}, b_{m+2}, \dots$ extending the sequence \bar{b} . Let X_n be the SFT defined by excluding the patterns b_1, \dots, b_n ; for $n \geq m$ we have $X_n \subseteq W$ and $\cap X_n = X$. It follows that $d(X, X_n) \rightarrow 0$, as desired. \square

Let $\rho : Y \rightarrow Z$ be the factor and systems constructed in the previous section.

Lemma 16. *If $X \subseteq Z$ is effective, then in the space of subsystems of X no minimal subsystem is isolated in the Hausdorff metric.*

Proof. Suppose $X' \subseteq X$ were an isolated minimal system. By the previous lemma the effective subsystems of X are dense, so X' is an effective minimal system and therefore has degree 0 by proposition 13, implying the same for X and therefore for Z , a contradiction. \square

Lemma 17. *If $X \subseteq Z$ is effective then a minimal subsystem of X is isolated if and only if its distance from every other minimal subsystem of X is bounded away from 0.*

Proof. Clearly if $X_0 \subseteq X$ is isolated then its distance from every other subsystem, and in particular the minimal ones, is bounded away from 0.

Conversely, suppose there are systems arbitrarily close to X_0 ; we must show that there are minimal systems arbitrarily close to X_0 . Let $\varepsilon > 0$ and $x_0 \in X_0$, and choose a finite $F \subseteq \mathbb{Z}^d$ so that $\{\sigma^u x_0\}_{u \in F}$ is ε -dense in X_0 . Let X_1 be a system ε -close to X_0 , and close enough that there is a point $x_1 \in X_1$ such that $d(\sigma^u x_0, \sigma^u x_1) < \varepsilon$ for $u \in F$. The orbit closure X'_1 of x_1 is minimal since all subsystems of Z are. The proof will be completed by showing that X'_1 is within distance 2ε of X_0 . To see this, note that if $x \in X_0$ then $d(x, \sigma^u x_0) < \varepsilon$ for some $u \in F$, hence $d(x, T^u x_1) < 2\varepsilon$; and on the other hand if $x' \in X'_1$ then $x' \in X_1$, so, since $d(X_0, X_1) < \varepsilon$, there is a $x \in X$ with $d(x, x') < \varepsilon$. This implies $d(X_0, X'_1) < 2\varepsilon$, as required. \square

Proposition 18. *Let Y be the SFT cover of Z constructed above, W an SFT and $\pi : W \rightarrow Y$ a shift-commuting map into a subsystem of Y . Then for every $\varepsilon > 0$ there is an SFT $W_0 \subseteq W$ such that $d(W_0, W) < \varepsilon$ in the Hausdorff metric and $\pi(W_0) \neq \pi(W)$.*

Proof. Consider the diagram

$$\begin{array}{ccc} & & W \\ & & \pi \downarrow \\ Y & \supseteq & X = \pi(W) \\ \rho \downarrow & & \rho \downarrow \\ Z & \supseteq & X' = \rho(X) \end{array}$$

X' is a sofic shift, so it is effective. Let C_1, \dots, C_n be a partition of W into cylinder sets of diameter $< \varepsilon$. Since Z is the disjoint union of its minimal subsystems so is X' . Hence by lemma 17, none of the minimal subsystems of X' is isolated, and since X is totally disconnected so is the space of minimal subsystems. We can therefore partition X' into clopen, pairwise disjoint invariant subsystems X'_1, \dots, X'_{n+1} . For each C_i there is at least one X'_j such that $C_i \cap (\pi\rho)^{-1}(X'_j) \neq \emptyset$. Thus without loss of generality, $(\pi\rho)^{-1}(X'_i) \cap C_i \neq \emptyset$, and so $W'_0 = \cup_{i=1}^n \pi^{-1}(X_i)$ satisfies the desired properties except it is not an SFT. But the subsystems that are SFTs are dense

among the subsystems of W by lemma ???; we may therefore choose a system W_0 with the requisite properties. \square

It remains to translate this approximation lemma to the space $\mathcal{H}(d)$.

Corollary 19. *Let Y, W and $\pi : W \rightarrow Y$ be as in the previous lemma, and let $\varphi \in \mathcal{H}(d)$ be conjugate to W by coding with respect to a partition $\alpha = \{A_1, \dots, A_n\}$ of K . Then for every $\varepsilon > 0$ there is a SFT ψ with $d(\varphi, \psi) < \varepsilon$, and ψ factors via $c_{\alpha, \psi}$ to an SFT $W_0 \subseteq W$ with $\pi(W_0) \neq \pi(W)$.*

Proof. Since α generates for φ , there is an r so that the atoms of $\beta = \vee_{\|u\| < r} \varphi^u \alpha$ are of diameter $< \eta$ for a parameter η we shall specify later. By the previous lemma, we may choose a subshift $W_0 \subseteq W$ so that $K_0 = c_{\alpha, \varphi}^{-1}(W_0)$ intersects each atom of β . We define $\psi_0 = \varphi|_{K_0} : K_0 \rightarrow K_0$; notice that $(K_0, \psi_0) \cong (W_0, \sigma)$.

Since W_0 is an SFT it is effective, and since $W_0 \subseteq W$ and W has nontrivial degree, W_0 has no isolated points. Therefore $K_0 \cap B$ is topologically a Cantor set for each atom $B \in \beta$ and we may choose a homeomorphism $\rho : K \rightarrow K_0$ satisfying $\rho(B) = K_0 \cap B$ for $B \in \beta$. It follows that $d(x, \rho(x)) < \eta$ for $x \in K$, so if η is small enough, the action $\psi = \rho^{-1} \psi_0 \rho$ will satisfy $d(\varphi, \psi) < \varepsilon$. Finally, the β -itineraries of a point $x \in K$ are the same for the actions ψ and ψ_0 since $\rho(B) = K_0 \cap B$. Thus $c_{\beta, \psi}$ is a factor map $(K, \psi) \rightarrow W_0$, and the lemma follows. \square

3.4. The strong topological Rohlin Property. We now have all the parts we need to prove our main theorem.

Theorem 20. *For $d \geq 2$ every isomorphism class in $\mathcal{H}(d)$ is meager.*

Proof. Fix $\theta \in \mathcal{H}(d)$ and let $\varphi \in sH(d)$ be conjugate to the SFT $Y \subseteq \{1, \dots, k\}^{\mathbb{Z}^d}$ constructed above, via a partition α . Using proposition 6 choose a neighborhood U of φ such that $\widehat{c}_\alpha(\psi) \subseteq Y$ for $\psi \in U$. We shall show that there is a residual subset $V \subseteq U$ of systems which are not isomorphic to θ .

Let Y_1, Y_2, \dots be an enumeration of all the subsystems of Y that are factors of θ . It suffices to show that for every $i = 1, 2, 3, \dots$ there is a dense open set $V_i \subseteq U$ consisting of actions ψ with $\widehat{c}_\alpha(\psi) \neq Y_i$, for then $V = \bigcap V_i$ is a dense G_δ and if $\psi \in V$ then $\widehat{c}_\alpha(\psi) \neq Y_i$ for all i , implying that $\psi \not\cong \varphi$.

Fix i and let $\psi \in U$ be an SFT. If $\widehat{c}_\alpha(\psi) \neq Y_i$ then clearly any action ψ' sufficiently close to ψ will also have $c_\alpha(\psi') \neq Y_i$. Thus we must show that the SFTs ψ with this property are dense in U . We already know that the SFTs are dense, so let $\psi \in U$ be an SFT and suppose $\widehat{c}_\alpha(\psi) = Y_i$. Then for every $\varepsilon > 0$ we can apply corollary 19 to get an SFT action ψ' within ε of ψ , so that $\widehat{c}_\alpha(\psi') \subseteq \widehat{c}_\alpha(\psi)$ and $\widehat{c}_\alpha(\psi') \neq \widehat{c}_\alpha(\psi)$.

To conclude the proof we use the general fact that any orbit of a Polish group acting transitively on a Polish space is either meager or co-meager [8]. So far we

have shown that $[\theta]$ is not residual, because it is not residual in the open set U ; so $[\theta]$ is meager. \square

4. TWO PROBLEMS

The picture emerging from these results is that the space of \mathbb{Z}^d actions on K is mostly inaccessible to us. The closure of the space of effective systems may be better behaved. Here are a couple of questions about this space.

Recall that an action $\varphi \in \mathcal{H}(d)$ is strongly irreducible if there is an $R > 0$ such that, for every pair of open sets $\emptyset \neq A, B \subseteq K$, we have $\varphi^u A \cap B \neq \emptyset$ for every $u \in \mathbb{Z}^d$ with $\|u\| \geq R$. The class of SFTs with this property has been widely studied in thermodynamics as the class with the best hope of developing something of a thermodynamic formalism, and in symbolic dynamics as a fairly manageable class where embedding and factoring relations may be well behaved (note that the factor of a strongly irreducible system is itself strongly irreducible).

Problem. Can every strongly irreducible action be approximated by a strongly irreducible SFT?

In dimension 1 the answer is affirmative. Note that strongly irreducible SFTs, like minimal SFTs, have Medvedev degree 0 [13, Corollary 3.5]. Thus a negative answer would follow if we could construct an SFT of non-trivial degree having some strongly irreducible subsystem (which of course will not be effective).

With regard to the space of minimal systems, we have shown that the (relative) closure of the effective systems, and thus of the minimal SFTs, has empty (relative) interior. It is still open if these closures are the same. In other words,

Problem. Can every minimal effective system be approximated by a minimal SFT?

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