MATH 515
LECTURE 8: GROUPS OVER FINITE FIELDS

Instructor: Ramin Takloo-Bighash
Notes by Florin Spinu

Assume k is a number field and $\mathbb{A}$ is the group of adeles. Assume that v varies over the places of k, and k_v denotes the completion of k at that place.
In view of the tensor product decomposition theorem, an irreducible, cuspidal, automorphic representation (cuspidal automorphic form) π of $GL_n(\mathbb{A})$ can be written as a restricted tensor product

$$\pi = \otimes'_v \pi_v$$

where π_v is an irreducible (unitary) representation of $GL_n(k_v)$. Moreover, almost all π_v are spherical. Therefore, it is important to understand the unitary dual of $GL_n(k_v)$, or at least the spherical (unitary) dual. Since finite fields provide a good approximation for non-archimedean fields, we will give a description of the representation theory of $GL_2(F)$, when F is a finite field. This theory generalizes to $GL_n(F)$ and $GL_n(k_v)$, modulo technical modifications.

1. Representations of Finite Groups

A representation (π, V) of a finite group G (or a G-module or, equivalently, a $\mathbb{C}[G]$-module) is a group homomorphism $\pi : G \to \mathrm{End}(V)$, where V is a (finite dimensional) complex vector space. For convenience, we will denote the action $\pi(g)v$ simply by gv or by $g * v$, when $g \in G$ and $v \in V$.

1.1. The Induced Representation.

Assume G is a finite group and $H < G$ is a subgroup. Suppose (π, V) is a finite dimensional $\mathbb{C}[H]$-module. Consider the linear space of functions:

$$V^G := \{f : G \to W | f(hg) = \pi(h)(f(g))\}$$

Then G acts on the elements of V^G by right translations, $R(x)f(g) = f(gx)$. We say that the representation (R, V^G) is obtained by inducing the representation π from H to G. We also use the notation $\mathrm{Ind}_H^G \pi$ for it. Note that, we have the following isomorphism of $\mathbb{C}[G]$-modules:

$$V^G \simeq V \otimes_{\mathbb{C}[H]} \mathbb{C}[G]$$

1.2. Explicit Basis.

Define the following elements of V^G, for $g \in G$ and $v \in V$:

$$f_{g,w}(g') = \begin{cases} \pi(g'g^{-1})v, & \text{if} \quad Hg = Hg'; \\ 0, & \text{otherwise.} \end{cases}$$

If $\mathfrak{R} \subset G$ is a collection of coset representatives for $H \backslash G$, and $\{w_i\}_{i=\overline{1\cdots n}}$ is a basis of the complex vector space V, then for any element $F \in V^G$ we have the following identity:

$$F = \sum_{\gamma \in \mathfrak{R}} f_{\gamma, F(\gamma)} = \sum_{\gamma \in \mathfrak{R}} \sum_{1 \le i \le n} a_{\gamma,i} f_{\gamma, w_i}$$

where $a_{\gamma,i}$ come from writing $F(\gamma)$ in the $\{w_i\}$-basis, $F(\gamma) = \sum_i a_{\gamma,i} w_i$. Therefore, the set $\{f_{\gamma,w_i}\}_{\gamma\in\mathfrak{R}, i=\overline{1\cdots n}}$ is a basis for V^G.

Corollary 1.1. *For any H-module U, we have the following Frobenius reciprocity law:*

$$i)\dim \operatorname{Ind}_H^G \pi = [G:H]\dim\pi$$
$$ii)\operatorname{Hom}_G(U, V^G) \simeq \operatorname{Hom}_H(U|_H, V)$$

1.3. **Mackey Theory.**

Theorem 1.2 (Mackey)**.** *Let G a group as above, and let H_1, H_2 be subgroups of G. Assume that V_i is an H_i-module, for $i = 1, 2$. Then the linear space $\operatorname{Hom}_G(V_1^G, V_2^G)$ is naturally isomorphic to the space of functions $\Delta : G \to \operatorname{Hom}_{\mathbb{C}}(V_1, V_2)$ satisfying the equivariance property $\Delta(h_2 g h_1) = h_2\Delta(g)h_1$. Given such a function Δ, the corresponding intertwining operator $L \in \operatorname{Hom}_G(V_1^G, V_2^G)$ is given by*

$$L(f) = \Delta * f(g) := \frac{1}{|G|}\sum_{h\in G}\Delta(gh^{-1})(f(h))$$

Proof. We only note that for L so defined we have, for any $v \in V_1$, the identity:

$$\Delta(g)(v) = [G : H_1]L(f_{g^{-1},v})(e)$$

□

We conclude with the observation that

$$\dim_{\mathbb{C}}\operatorname{Hom}(V_1^G, V_2^G) \le |H_1\backslash G/H_2|(\dim_{\mathbb{C}} V_1)(\dim_{\mathbb{C}} V_2)$$

Corollary 1.3. *If χ_1, χ_2 are 1-dimensional characters of H_1 and H_2, respectively, then:*

$$\dim\operatorname{Hom}_G(\chi_1^G, \chi_2^G) = \#\{\textit{double cosets which support an intertwining operator}\}$$

Lemma 1.4. *A double coset $H_1 g H_2$ supports an intertwining operator if and only if $\chi_1(g^{-1}h_2 g) = \chi_2(h_2), \quad \forall h_2 \in gH_1g^{-1}\cap H_2$.*

Lemma 1.5. *Assume V_1, V_2, V_3 are H_1, H_2, H_3-modules, respectively, where H_1, H_2 and H_3 are subgroups of G. Suppose that $L_1 \in \operatorname{Hom}_G(V_1^G, V_2^G)$ and $L_2 \in \operatorname{Hom}_G(V_2^G, V_3^G)$. Then $L_2 \circ L_1 \in \operatorname{Hom}(V_1^G, V_3^G)$. If, in view of the Theorem 1.2, L_1 corresponds to the function Δ_1 and L_2 corresponds to the function Δ_2, then $L_1 \circ L_2$ corresponds to the convolution of Δ_1 and Δ_2 which is given by*

$$\Delta(g) = \sum_{\gamma\in H_2\backslash G}\Delta_2(g\gamma^{-1})\Delta_1(\gamma) \tag{1}$$

Corollary 1.6. *Suppose (π, V) is an H-module. Then the ring of G-endomorphisms $\operatorname{End}_G(V^G)$ is isomorphic as a G-module to the convolution algebra of functions $\Delta : G \to \operatorname{End}_{\mathbb{C}}(V)$ which satisfy the following equivariance property:*

$$\forall g \in G, h_1, h_2 \in H, \quad \Delta(h_1 g h_2) = \pi(h_1)\Delta(g)\pi(h_2)$$

2. Irreducible Representations of $GL(2, F)$

In this section we assume that $F = \mathbb{F}_q$ is the finite field with q elements. Therefore, the group $G = GL(2, GF)$ is finite with $(q^2 - 1)(q^2 - q)$ elements.

2.1. **Principal Series of $GL(2,F)$.** Assume χ_1, χ_2 are two characters of $F^\times$, $\chi_{1,2} : F^\times \to \mathbb{C}^\times$. Consider the character $\chi = \chi_1 \oplus \chi_2$ defined on the torus $T = \begin{pmatrix} * & \\ & * \end{pmatrix}$. Since the parabolic $B = \begin{pmatrix} * & * \\ & * \end{pmatrix}$ factors as $B = TN$, where $N = \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}$, the character χ extends trivially to B, and is given by $\chi \begin{pmatrix} d_1 & * \\ & d_2 \end{pmatrix} = \chi_1(d_1)\chi_2(d_2)$. Let $B(\chi_1, \chi_2)$ denote the representation obtain by inducing the character χ from B to G:

$$B(\chi_1, \chi_2) = \operatorname{Ind}_B^G \chi$$

We call $B(\chi_1, \chi_2)$ a **principal series** representation of $G = GL(2,F)$.
Mackey theory allows us to determine which of these representations are irreducible and which not. For example, let's assume for the moment that $\chi_1 \neq \chi_2$. Then, according to Theorem1.2, $\operatorname{End}_G(B(\chi_1, \chi_2))$ is isomorphic to the linear space of functions $\Delta : G \to \mathbb{C}$ which satisfy the following equivariance property:

$$\forall g \in G, b_1, b_2 \in B : \quad \Delta(b_2 g b_1) = \chi(b_2)\chi(b_1)\Delta(g) \tag{2}$$

where, as before, $\chi = \chi_1 \oplus \chi_2$. Such a function is clearly determined by the values at representatives of the double coset space $B\backslash G/B$. The Bruhat decomposition gives a set of such representatives, namely G is a disjoint union:

$$G = B \amalg Bw_0B$$

where w_0 is the Weyl element $w_0 = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$. A function Δ satisfying equation (refx1) is thus determined by its values at e and w_0. However, unless $\Delta(w_0) = 0$, condition (2) implies:

$$b_1 w_0 = w_0 b_2 \Rightarrow \chi(b_1) = \chi(b_2)$$

In particular, it requires that

$$\chi \begin{pmatrix} t & \\ & 1 \end{pmatrix} = \chi \begin{pmatrix} 1 & \\ & t \end{pmatrix}, \quad \forall t \in F^\times$$

But this simply means that $\chi_1(t) = \chi_2(t)$, or $\chi_1 = \chi_2$, which contradicts our assumption. Therefore, $\Delta(w_0) = 0$ and hence the space of function satisfying (2) is one-dimensional. By Schur's lemma, the fact that $\dim \operatorname{End}_G(B(\chi_1, \chi_2)) = 1$ implies that $B(\chi_1, \chi_2)$ is an irreducible representation.

It is also easy to see that two principal series $B(\chi_1, \chi_2)$ and $B(\chi_1', \chi_2')$ are not equivalent unless $(\chi_1, \chi_2) = (\chi_1', \chi_2')$ or $(\chi_1, \chi_2) = (\chi_2', \chi_1')$.

Note. Since $[G : B] = |P^1(\mathbb{F}_q)| = q + 1$, it implies that $\dim B(\chi_1, \chi_2) = q + 1$.

We can also find, in the same way, that for the trivial representation χ_0, the principal series $B(\chi_0, \chi_0)$ associated to it is reducible, with a one-dimensional subspace where G acts trivially, and a q-dimensional irreducible subrepresentation, called the *Steinberg representation*. We will denote it by St_q and we have:

$$B(\chi_0, \chi_0) \simeq \chi_0 \oplus St_q$$

Also, for any other character χ_1 of $F^\times$, it is easy to see that $B(\chi_1, \chi_1) \simeq \chi_1 \otimes B(\chi_0 \cdot \chi_0)$, therefore this representation is also a sum of two irreducible representations:

$$B(\chi_1, \chi_1) \simeq \chi_1 \oplus (St_q \otimes \chi_1)$$

Notation. If χ is a character of $F^\times$, we can regard χ as a character on $GL(2,F)$ given by $g \mapsto \chi(\det(g))$. If (π, V) is a representation of $GL(2,F)$, the "twisted" representation

$\pi \otimes \chi$ is the action of $GL(2,F)$ on the space V via $g \mapsto \pi(g)\chi(\det(g))$. To summarize this section, we have:

i) $\chi_1 \neq \chi_2 \Rightarrow B(\chi_1, \chi_2) =$ irreducible$, \dim = q$;

ii) $\chi_0 =$ trivial $\Rightarrow B(\chi_0, \chi_0) \simeq \chi_o \oplus St_q, \dim St_q = q$ (irreducible);

iii) $\forall \chi_1, B(\chi_1, \chi_1) \simeq \chi_1 \oplus (St_q \otimes \chi_1)$.

2.2. **Cuspidal Representations.** In this section we mention without proof the existence of the **cuspidal representations**, i.e. irreducible representations of $GL(2,F)$ which have non-zero, N-invariant vectors. Here N is, as before, the subgroup of upper triangular matrices of $G = GL(2,F)$.

Reference: [Bu], page 405.

The cuspidal representations cannot be found in the list of principal series representations. They can be realized as sub-representations of the *Weil representation*, which we will not describe. However, they can be parametrized as follows: assume E/F is a degree 2 extension, and $\omega : E^\times \to \mathbb{C}^\times$ is a multiplicative character on E which does not factor through the norm map $Nm : E^\times \to F^\times$. For such a character, there exists an irreducible representation $W(\omega)$ of $GL(2,F)$, which occurs as a subrepresentation of W (the Weil representation). $W(\omega)$ is cuspidal and has dimension $q-1$. Two $Gal(E/F)$-conjugate characters ω, ω' give rise to isomorphic representations $W(\omega) \simeq W(\omega')$.

2.3. **Dimension Counting.** To make sure that our list of irreducible representations of $GL(2,F)$ (principal series, one-dimensional, Steinberg, cuspidal) is exhaustive, we have to compute $\sum_\pi (\dim \pi)^2$ and check that it equals $|G|$. Since the number of multiplicative characters on $F^\times$ is exactly $q-1$, it follows that:

- there are $\frac{(q-1)^2-(q-1)}{2} = \frac{(q-1)(q-2)}{2}$ ordered pairs of characters (χ_1, χ_2) such that $\chi_1 \neq \chi_2$. They give rise to irreducible representations in dimension $q+1$;

- there are $q-1$ degenerate (non-isomorphic) principal series $B(\chi_1, \chi_1)$. They contain a 1-dimensional and a q-dimensional irred. subrepresentation;

- there are $q^2 - q$ choices for a character $\omega : E^\times \to \mathbb{C}^\times$ which does not factor through the norm map $Nm : E^\times \to F^\times$. Of these characters, $\frac{q^2-q}{2}$ are Galois non-conjugate. They give rise to the irreducible cuspidal representations in dimension $q-1$.

$$\begin{aligned}\sum_\pi (\dim \pi)^2 &= \frac{(q-1)(q-2)}{2}(q+1)^2 + (q-1)(1+q^2) + \frac{q^2-q}{2}(q_1)^2 \\ &= (q^2-1)(q^2-q) = |GL(2,F)|\end{aligned}$$

Therefore, these are all the irreducible representations of $GL(2,F)$.

2.4. **The Whittaker Model.** Fix ψ a non-trivial additive character of F and regard it as a character on N defined by

$$\psi \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \psi(t)$$

Let $\mathcal{W}$ be the representation obtained by inducing the character ψ from N to G:

$$\mathcal{W} = \operatorname{Ind}_N^G \psi$$

$\mathcal{W}$ is the space of Whittaker functions.

Existence and Uniqueness of the Whittaker Model. Over a finite field, ay irreducible representation π of $GL(2,F)$ occurs as a subrepresentation of $\mathcal{W}$ with multiplicity one. This echoes the principle of existence and uniqueness of the *Whittaker model* in the context of automorphic forms.

To prove that every irreducible representation occurs with multiplicity at most 1 inside $\mathcal{W}$, it is sufficient to show that the ring $\mathrm{End}_G(\mathcal{W})$ is commutative. For, if we write $\mathcal{W}$ as $\mathcal{W} = \oplus_\pi m_\pi \pi$, then

$$\mathrm{End}_G(\mathcal{W}) \simeq \bigoplus_\pi Mat_{m_\pi \times m_\pi}(\mathbb{C})$$

hence it is commutative if and only if $0 \le m_\pi \le 1, \forall \pi$.

We first make use of the Theorem 1.2 to see that $\mathrm{End}_G(\mathcal{W})$ is isomorphic to the convolution algebra of functions $\Delta : G \to \mathbb{C}$ which satisfy:

(3) $$\forall g \in G, n_1, n_2 \in N, \quad \Delta(n_1 g n_2) = \psi(n_1 n_2)\Delta(g)$$

Such a function Δ is clearly determined by its value at representatives of $N\backslash G/N$ A set of such representatives is the set of matrices of the type $\begin{pmatrix} * & \\ & * \end{pmatrix}$ or $\begin{pmatrix} & * \\ * & \end{pmatrix}$.

Since

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} a & \\ & b \end{pmatrix}\begin{pmatrix} 1 & -ba^{-1}x \\ & 1 \end{pmatrix} = \begin{pmatrix} a & \\ & b \end{pmatrix}$$

equation (3) says that, unless $\Delta\begin{pmatrix} a & \\ & b \end{pmatrix} = 0$, we need to have $\psi(x - ba^{-1}x) = 1, \forall x \in F$. This is only possible if $a = b$. Therefore, a function with property (3) is supported on elements of the form:

(4) $$\begin{pmatrix} a & \\ & a \end{pmatrix}, \quad \begin{pmatrix} & b \\ c & \end{pmatrix}, \quad a, b, c \in F^\times$$

Consider now the involution $\tau : GL(2, F) \to GL(2, F)$ given by

$$\tau(g) = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} g^t \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$$

hence $\tau(g_1 g_2) = \tau(g_2)\tau(g_1), \tau^2 = 1$. This involution has the special property that it leaves invariant precisely the group elements listed at (4), which form the support of Δ. Therefore, for two functions Δ_1, Δ_2 which satisfy (3), we can repeatedly apply the involution τ to obtain:

$$\begin{aligned}\Delta_1 * \Delta_2(g) = \Delta_1 * \Delta_2(\tau g) &= \sum_{h \in G} \Delta_1(\tau(g)h^{-1})\Delta_2(h) \\ = \sum_{h \in G} \Delta_1(\tau(h)^{-1}g)\Delta_2(\tau(h)) &= \sum_{h \in G} \Delta(h^{-1}g)\Delta_2(h) \\ = \sum_{h \in G} \Delta_2(gh^{-1})\Delta_1(h) &= \Delta_2 * \Delta_1(g)\end{aligned}$$

Hence the convolution algebra of functions with property (3) is commutative. In view of Lemma 1.5, it follows that $\mathrm{End}_G(\mathcal{W})$ is a commutative algebra. Therefore, every irreducible representation of $GL(2, F)$ occurs as a subrepresentation of $\mathcal{W}$ with multiplicity at most one (*uniqueness* of the Whittaker model).

We prove the *existence* in the case of non-degenerate principal series. For $\chi_1 \ne \chi_2$, by Mackey's theorem, $\mathrm{Hom}_G(B(\chi_1, \chi_2), \mathcal{W})$ is isomorphic to the linear space of functions $\Delta : G \to \mathbb{C}$ which satisfy:

(5) $$\forall g \in G, b \in B, n \in N, \quad \Delta(bgn) = \chi(b)\psi(n)\Delta(g)$$

therefore Δ is determined by its values on a set of double coset representatives of $B\backslash G/N$. A set of such representatives is $\{e, w_0\}$.

Take $g = e$ in equation (5) and get $\Delta(bn) = \chi(b)\psi(n)\Delta(e)$. Unless $\Delta(e) = 0$, this condition implies:

$$bn = b_1 n_1 \Rightarrow \chi(b)\psi(n) = \chi(b_1)\psi(n_1)$$

Since $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$, this means that $\psi(t) = 1$, for any $t \in F$, which is impossible, since we assumed that the additive character ψ is non-trivial. Therefore, a function Δ satisfying (5) is supported on the double coset Bw_0N. However, it can be easily checked that an element $g \in Bw_0N$ can be uniquely written as $g = bw_0n$, with $b \in B$ and $n \in N$. Therefore, the function $\Delta(g)$ given by:

$$\Delta(bw_0n) = \chi(b)\psi(n);$$
$$\Delta(g) = 0, \quad \text{if} \quad g \notin Bw_0N$$

is well defined, and satisfies equation (5).

Therefore, the linear space $\mathrm{Hom}_G(B(\chi_1, \chi_2), \mathcal{W})$ is one-dimensional, which shows that $B(\chi_1, \chi_2)$ occurs as a subrepresentation of $\mathcal{W}$ with multiplicity one.

This proves the existence of the Whittaker model for an irreducible principal series representation. The proof relied on Mackey theory since both representations are induced from one-dimensional characters of subgroups. The proof of the existence of a Whittaker model for a cuspidal representation requires a separate discussion, but can also be determined by dimension counting, since the uniqueness property holds for any representation. We will not pursue this issue further.

References

[Bu] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, Cambridge, 1997.