MATH 515
SECTION 5: CUSPIDAL AUTOMORPHIC REPRESENTATIONS

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1. AUTOMORPHIC FORMS: DEFINITIONS

Let F be a number field, and $\mathbb{A} = \mathbb{A}_F$ the associated group of adeles. Then $F^\times$ is a discrete subgroup of the group of units $\mathbb{A}^\times$ with the induced topology. A Hecke character is a continuous (unitary) character $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ which is trivial on $F^\times$, and hence can be regarded as a character on the quotients space:

$$\omega : F^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$$

Note that $F^\times$ is a discrete subgroup of $\mathbb{A}^\times$, and hence $F^\times \backslash \mathbb{A}^\times$ has a natural quotient topology.

Definition 1.1. *An automorphic form on $GL(n, F)$ is a smooth function:*

$$\phi : GL(n, \mathbb{A}) \to \mathbb{C}$$

such that the following conditions are satisfied:

(1) $\phi(\gamma g) = \phi(g), \quad \forall g \in GL(n, \mathbb{A}), \gamma \in GL(n, F).$
(2) $\phi(zg) = \omega(z)\phi(g), \quad \forall z \in \mathbb{A}^\times.$
(3) ϕ *is right K-finite.*
(4) ϕ *is Z-finite, where Z is the center of the universal enveloping algebra of*

$$\mathfrak{g}_\infty := \prod_{v|\infty} \mathfrak{g}_v.$$

(5) ϕ *is of moderate growth.*

Note that from now on z will denote an element in $\mathbb{A}^\times$ as well as the diagonal matrix $zI_n \in Z_{\mathbb{A}}$. ω is the *central character* of ϕ. In what follows we make precise the terms used in the definition of an *automorphic form*. From now on, we will use the notation $G_{\mathbb{A}} := GL(n, \mathbb{A})$ and $G_F := GL(n, F)$.

1.1. **Smoothness.**

Let $F_\infty := \prod_{v|\infty} F_v$. Then, for each $g \in G_{\mathbb{A}}$, there exists a neighborhood V of g and a smooth function $f = f_g$ on V such that

$$\phi(h) = f_g(h_\infty); \quad \forall h \in V, \quad h = h_\infty h_f$$

1.2. K**-finiteness.**

The compact subgroups K_v at each place are defined in the following manner:

$$K_v := \begin{cases} O_n(\mathbb{R}), & v|\infty, v = \text{real} \\ U_n(\mathbb{C}), & v|\infty, v = \text{complex} \\ GL(n, O_v), & v < \infty \end{cases}$$

We now define the *global* maximal compact subgroup:

$$K = \prod_v K_v = K_\infty K_f$$

A function ϕ on $G_{\mathbb{A}}$ is right K-finite if and only if the space generated by the right K-translations $\operatorname{Span}\{R(k)\phi(x)|k \in K\}$ is finite-dimensional.

1.3. **Z-finiteness.** Note that Z acts as an algebra of differential operators on the infinite part of the function ϕ, and Z-finiteness means that ϕ lies in a finite dimensional Z-invariant space of functions.

1.4. **Moderate Growth.** Define the norm of a group element:

$$\|g\| = \prod_v \max_{i,j}\{|g_{ij}|_v, |\det g|_v^{-1}\}$$

The function $\phi(g)$ is of moderate growth if, for each Siegel set S, there exists a constant $C = C_S$ and an integer $N \leq 1$ such that:

$$|\phi(g)| \leq C\|g\|^N, \quad \forall g \in S$$

Definition 1.2. *An automorphic form ϕ on $G_{\mathbb{A}}$ is cuspidal if, for any $1 \leq r, s \leq n, r+s = n$, and any $g \in G_{\mathbb{A}}$, the following integral vanishes:*

$$\int_{\operatorname{Mat}_{2\times 2}(F)\backslash \operatorname{Mat}_{2\times 2}(\mathbb{A})} \phi\left(\begin{pmatrix} I_r & X \\ 0 & I_s \end{pmatrix} g\right) dX = 0$$

Here I_r denotes the identity $r \times r$ matrix, while 0 denotes the $n-r \times n-s$ null matrix.

Definition 1.3. *We make the following notations:*

1. $\mathcal{A}(G_F\backslash G_{\mathbb{A}}, \omega)$ =*space of automorphic forms with (unitary) central character* ω.
2. $\mathcal{A}_0(G_F\backslash G_{\mathbb{A}}, \omega)$ =*space of cuspidal automorphic forms with central character* ω.

Remark 1.4. *The space $\mathcal{A}(G_F\backslash G_{\mathbb{A}}, \omega)$ is not preserved by the right action of $G_{\mathbb{A}}$. By taking the right $G_{\mathbb{A}}-$ translates of an element $\phi \in \mathcal{A}(G_F\backslash G_{\mathbb{A}}, \omega)$, the property of right $K-$ finiteness at the infinite places is automatically lost. Hence $\mathcal{A}(G_F\backslash G_{\mathbb{A}}, \omega)$ is a right $G(\mathbb{A}_f)-$ module but not a right $G(\mathbb{A})-$ module.*

To correct this problem, we introduce the following space:

Definition 1.5. *$L^2(G_F\backslash G_{\mathbb{A}}, \omega)$ is the space of functions $\phi : G_{\mathbb{A}} \to \mathbb{C}$ satisfying the following properties:*

1. $\phi(\gamma g) = \phi(g), \quad \gamma \in G_f, g \in G_{\mathbb{A}}$;
2. $\phi(zg) = \omega(z)\phi(g)$;
3. $\int_{Z_{\mathbb{A}}G_F\backslash G_{\mathbb{A}}} |\phi(g)|^2 dg < \infty$.

We also denote by $L^2_0(G_F\backslash G_{\mathbb{A}}, \omega)$ the subset of $L^2(G_F\backslash G_{\mathbb{A}}, \omega)$ which generated by cusp forms.
Note that since the Haar measure on the quotient space $G_F\backslash G_{\mathbb{A}}$ is right $G_{\mathbb{A}}$-invariant, it means that $L^2(G_F\backslash G_{\mathbb{A}})$ is a unitary representation of $G_{\mathbb{A}}$, given byu the action:

$$g, x \in G_{\mathbb{A}}, \phi \in L^2(G_F\backslash G_{\mathbb{A}}, \omega), \quad (R(g)\phi)(x) := \phi(xg)$$

Also, it is easy to see that $L^2_0(G_F\backslash G_{\mathbb{A}}, \omega)$ is an invariant $G_{\mathbb{A}}$-subrepresentation.

2. DECOMPOSITION OF THE CUSPIDAL SPACE

In this section we will prove the following theorem, due to G-G-PS:

Theorem 2.1 (Main Theorem). *The space* $L^2_0(G_F\backslash G_{\mathbb{A}},\omega)$ *is a Hilbert direct sum of* $G_{\mathbb{A}}$*-invariant irreducible subspaces.*

Definition 2.2. *An irreducible* $G_{\mathbb{A}}$*-submodule of* $L^2_0(G_F\backslash G_{\mathbb{A}},\omega)$ *is called a cuspidal automorphic representation.*

Proof. The way we will prove this theorem is by making a detour to the action of the convolution algebra $C_c^\infty(G_{\mathbb{A}})$. It consists of two parts:

Part 1: first, we will prove the following theorem, due G-G-PS:

Theorem 2.3 (A). *Suppose* ϕ *is a smooth function in* $C_c^\infty(G_{\mathbb{A}})$*. Then there exists a constant* $C(\phi) > 0$ *depending on* ϕ *only such that, for any cusp form* $f \in L^2_0(G_F\backslash G_{\mathbb{A}},\omega)$*, the following inequality holds:*

$$\|R(\phi)f(x)\|_\infty \leq C\|f\|_{L^2}$$

Part 2: in the second part we deduce the statement of the Main Theorem from the following theorem of G-G-PS:

Theorem 2.4 (B). *Let* (ρ, V) *be a unitary (continuous) representation of the group* G*. Assume that for all* $\phi \in C_c^\infty(G_{\mathbb{A}})$*, the operator:*

$$\rho(\phi) : V \to V$$

is compact. Then the representation V *is completely reducible, in the sense that it can be written as a Hilbert direct product of irreducible subrepresentations, each occurring with finite multiplicity:*

$$V = \oplus_{\pi\in\hat{G}_{\mathbb{A}}} m_\pi V_\pi$$

where $\hat{G}_{\mathbb{A}}$ *is the unitary dual of* $G_{\mathbb{A}}$*. Here* m_π *represents a finite positive integer which is zero for all but countably many representations* π*.*

3. PROOF OF THEOREM A

3.1. Helping Lemma.

We first have two prove the following helping lemma:

Lemma 3.1. *Assume* ϕ *is a smooth function of compact Support on* $G_{\mathbb{A}}$*. Then there exists a positive constant* $C(\phi)$*, depending on* ϕ *only, such that:*

$$f \in L^2_0(G_F\backslash G_{\mathbb{A}},\omega), \quad \|R(\phi)f\|_\infty \leq C(\phi)\|f\|_{L^2}$$

Proof. We give a proof of this lemma in the case $n = 2$ and $F = \mathbb{Q}$, so $G_F = G_{\mathbb{Q}} = GL(2,\mathbb{Q})$, while $G_{\mathbb{A}} = GL(2,\mathbb{A})$, where $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ is the group pf adeles of $\mathbb{Q}$. We denote by $C_c^\infty(G_{\mathbb{A}})$ the space of finite linear combinations of functions of the type:

$$\phi(g) = \prod_v \phi_v(g_v), \quad \forall g \in G_{\mathbb{A}}, \quad g = (\cdots, g_v, \cdots)$$

where $\phi_v \in C_c^\infty(G_v), \forall v$, such that for almost all places v, $\phi_v = 1_{K_v}$ is the characteristic function of the maximal compact subgroup at the place v. Now, the right action $R(g)$ of the group $G_{\mathbb{A}}$ on $L^2_0(G_F\backslash G_{\mathbb{A}},\omega)$ induces an action of the algebra of smooth functions. Thus the effect of $R(\phi)$ on $f \in L^2_0(G_F\backslash G_{\mathbb{A}},\omega)$ is given by:

$$x \in G_{\mathbb{A}}, \quad (R(\phi)f)(x) := \int_{G_{\mathbb{A}}} \phi(g)f(xg)dg$$

We can further write this as:

$$R(\phi)f(x) = \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \phi_\omega(g)f(gx)$$

where:

$$\phi_\omega(g) := \int_{Z_{\mathbb{A}}} \phi(zg)\omega(z)dz$$

Remark 3.2. *The function $\phi_\omega(g)$ is compactly supported* $(\bmod Z_{\mathbb{A}})$.

We introduce now the notion of a Siegel domain.

Definition 3.3 (Siegel Domain)**.** *For $c, d > 0$, a group element $g = (g_v) \in G_{\mathbb{A}}$ belongs to the Siegel set $\mathfrak{S}_{c,d}$ if and only if the following conditions are satisfied:*

1. *$g_v \in K_v$, if v is non-archimedean;*
2. *$g_\infty = \begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_\infty, \quad z \in \mathbb{R}^\times, \quad y \geq c, \quad 0 \leq x \leq d.$*

We use now the following fact from reduction theory (for reference, see [Bo1]):

Proposition 3.4. *If $c \leq \frac{\sqrt{3}}{2}$ and $d \geq 1$, then:*

$$G_{\mathbb{A}} = G_{\mathbb{Q}}\mathfrak{S}_{c,d}$$

Note that this proposition implies the finiteness of the volume of the quotient space:

$$\mathrm{vol}(Z_{\mathbb{A}}G_{\mathbb{Q}}\backslash G_{\mathbb{A}}) < \infty$$

Because of the Proposition, the statement of the lemma reduces to proving the following:

$$\sup_{g\in\mathfrak{S}_{c,d}} |R(\phi)f(g)| \leq C_\phi \|f\|_{L^2}$$

Remark 3.5. *From now on the element g will be in a fixed Sielgel set $\mathfrak{S}_{c,d}$.*

For such a g we can write:

$$R(\phi)f(g) = \int_{Z_{\mathbb{A}}\backslash G_{\mathbb{A}}} \phi_\omega(g^{-1}h)f(h)dh = \int_{Z_{\mathbb{A}}N_{\mathbb{Q}}\backslash G_{\mathbb{A}}} f(h)\left(\sum_{\gamma\in N_{\mathbb{Q}}} \phi_\omega(g^{-1}\gamma h)\right) dh$$

$$= \int_{Z_{\mathbb{A}}N_{\mathbb{Q}}\backslash G_{\mathbb{A}}} K(g,h)f(h)dh$$

where $K(g,h)$ is the kernel:

$$K(g,h) = \sum_{\gamma\in N_{\mathbb{Q}}} \phi_\omega(g^{-1}\gamma h)$$

3.1.1. *The Kernel K_0.* Define now the kernel:

$$K_0(g,h) := \int_{N_{\mathbb{A}}} \phi_\omega(g^{-1}nh)dn = \int_{\mathbb{A}} \phi_\omega\left(g^{-1}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} h\right) dt$$

Compute the 'action' of K_0 on a cusp form f:

$$K_0 f(g) := \int_{Z_{\mathbb{A}} N_{\mathbb{Q}} \backslash G_{\mathbb{A}}} K_0(g,h) f(h) dh$$

$$= \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \int_{Z_{\mathbb{A}} \backslash A} \int_K \phi_\omega(g^{-1} nak) f(nak) dndadk$$

$$= \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \int_{Z_{\mathbb{A}} \backslash A} \int_K \int_{N_{\mathbb{A}}} \phi_\omega(g^{-1} xnak) f(nak) dxdndadk$$

Remark 3.6. *Here we used the fact that the (left invariant) Haar measure dg decomposes with respect to the Iwasawa decomposition $G = NAK$ as $dg = dndadk$.*

A change of variable $x \mapsto xn^{-1}$ immediately gives us:

$$\int_{Z_{\mathbb{A}} \backslash A} \int_K \int_{N_{\mathbb{A}}} \phi_\omega(g^{-1} xak) \left\{ \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} f(nak) dn \right\} dtdadk$$

$$\equiv 0$$

since f is a cusp form. Combining the previous two equations, we obtain:

$$R(\phi) f(g) = \int_{Z_{\mathbb{A}} N_{\mathbb{Q}} \backslash G_{\mathbb{A}}} K'(g,h) f(h) dh$$

where now the kernel K' is give by the difference:

$$K(g,h) := K(g,h) - K_0(g,h)$$

The idea is that the action of $R(\phi)$ restricted to L_0^2 is given by a modified kernel, whose behavior 'at infinity' will be analyzed in the next section.

3.1.2. *Transformation of the Kernel: Poisson Summation.* We now use the *Poisson Summation Formula* for the kernel:

$$K(g,h) := \sum_{\alpha \in \mathbb{Q}} \phi_\omega \left(g^{-1} \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} h \right)$$

to obtain:

$$K(g,h) = \sum_{\alpha \in \mathbb{Q}} \hat{\Phi}_{g,h}(\alpha)$$

where the Fourier transform is given by:

$$\hat{\Phi}_{g,h}(\alpha) := \int_{\mathbb{A}} \phi_\omega \left(g^{-1} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} h \right) \bar{\psi}(\alpha t) dt$$

Notice the identity:

$$K_0(g,h) \equiv \hat{\Phi}_{g,h}(0)$$

So $R(\phi)$ acts on the space of cusp forms with a kernel given by:

$$K'(g,h) = \sum_{\alpha \in \mathbb{Q}^\times} \hat{\Phi}_{g,h}(\alpha)$$

3.1.3. *Transformation of* $\hat{\Phi}_{g,h}(\alpha)$. In this section we will rely on the fact that g can be taken in the Siegel set $\mathfrak{S}_{c,d}$.

Using the Iwasawa decomposition we can write the elements $g, h \in G_{\mathbb{A}}$ in the following manner:

$$g = \begin{pmatrix} z & \\ & z \end{pmatrix} \begin{pmatrix} y & x \\ & 1 \end{pmatrix} k_g, \quad h = \begin{pmatrix} \zeta & \\ & \zeta \end{pmatrix} \begin{pmatrix} v & u \\ & 1 \end{pmatrix} k_h$$

where:

$$z, \zeta, y, v \in \mathbb{A}^\times, \quad u, x \in \mathbb{A}, \quad k_g, k_h \in K$$

Moreover, $g \in \mathfrak{S}_{c,d}$, so we have:

$$x = x_\infty \in \mathbb{R}, 0 \leq x \leq d, \quad y = y_\infty \in \mathbb{R}^\times, y \geq c$$

The Fourier transform $\hat{\Phi}_{g,h}$ is then given by the integral:

$$\int_{\mathbb{A}} \phi_\omega \left(\begin{pmatrix} z^{-1}\zeta & \\ & z^{-1}\zeta \end{pmatrix} k_g^{-1} \begin{pmatrix} y & \\ & \end{pmatrix} 1 \begin{pmatrix} 1 & t - x + u \\ & 1 \end{pmatrix} \begin{pmatrix} v & \\ & 1 \end{pmatrix} k_h \right) \bar{\psi}(\alpha t) dt$$

$$= \omega(z\zeta^{-1}) \psi(\alpha(u - x)) \int_{\mathbb{A}} \phi_\omega \left(k_g^{-1} \begin{pmatrix} 1 & y^{-1}t \\ & 1 \end{pmatrix} \begin{pmatrix} vy^{-1} & \\ & 1 \end{pmatrix} k_h \right) \bar{\psi}(\alpha t) dt$$

$$= \omega(z\zeta^{-1}) \psi(\alpha(u - x)) |y| \hat{F}_{k_g, k_h, y^{-1}v}(\alpha y)$$

where:

$$F_{k_g, k_h, \eta}(t) = \phi_\omega \left(k_g^{-1} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} \eta & \\ & 1 \end{pmatrix} k_h \right)$$

Note that since ϕ_ω has compact support $(\operatorname{mod} Z_{\mathbb{A}})$, we can write

$$\operatorname{supp} \phi_\omega = \Omega Z_{\mathbb{A}}$$

where Ω is some compact subset of $G_{\mathbb{A}}$. Since k_g, k_h are already confined to the compact subgroup K, it means that tere exists a compact subset $S \subset \mathbb{A}^\times$ such that:

$$F_{k_g, k_h, \eta} \neq 0 \Rightarrow \eta \in S$$

This also implies:

$$y^{-1}v \in \mathbb{A}^\times - S \Rightarrow K'(g, h) = 0$$

Now, since the function :

$$F_{k_g, k_h, y^{-1}v} : t \mapsto \phi_\omega \left(k_g^{-1} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} \begin{pmatrix} y^{-1}v & \\ & 1 \end{pmatrix} k_h \right)$$

has compact support independent of the parameters y, v, k_g, k_h it means that the Fourier transform has rapid decay uniformly in those parameters, so we get:

$$K'(g, h) \ll_{\phi, N} |y|^{-N}$$

for any integer $N \geq 1$, where the implied constant depend on ϕ and N only. We thus obtain:

$$R(\phi) f(g) \ll |y|^{-N} \int_{Z_{\mathbb{A}} N_{\mathbb{Q}} \backslash G_{\mathbb{A}}} |f(h)| dh$$

The last integral is:

$$\mathcal{J} = \int_{N_{\mathbb{Q}} \backslash N_{\mathbb{A}}} \int_{vy^{-1} \in S} \int_K f \left(n \begin{pmatrix} v & \\ & 1 \end{pmatrix} k \right) dn d^\times v dk$$

and it is performed over the compact set:

$$B := \{h = n \begin{pmatrix} v & \\ & 1 \end{pmatrix} k \in Z_{\mathbb{A}} N_{\mathbb{Q}} \backslash G_{\mathbb{A}}; \quad y(g)^{-1} v \in S\}$$

Since $g \in \mathfrak{S}_{c,d}$ so that $|y| \geq c$, it means $v \geq c' > 0$ (c' independent of g), and that will put h in a fixed finite union of right-translates of a Siegel set $\mathfrak{S}$:

$$h \in \cup_{i=1}^{m} \mathfrak{S} \xi_i, \quad \xi_i \in G_{\mathbb{A}}$$

Hence the last integral is:

$$\mathcal{J} \leq \sum_{i=1}^{m} \int_{\mathfrak{S}\xi_i} |f(h)| dh \ll \sum_{i=1}^{m} \|R(\xi_i) f\|_1 = m \|f\|_1 \ll \|f\|_2$$

and the last inequality holds because the set $G_F \backslash G_{\mathbb{A}}$ has finite measure. So that we can conclude with:

$$g \in \mathfrak{S}_{c,d}, f \in L_0^2(G_F \backslash G_{\mathbb{A}}, \omega), \quad |R(\phi) f(g)| \ll_{\phi,N} |y|^{-N} \|f\| 2$$

which certainly implies:

$$\|R(\phi) f(g)\|_{\infty} \ll_{\phi,N} \|f\|_2$$

since the Siegel set $\mathfrak{S}_{c,d}$ was chosen such that $Z_{\mathbb{A}} G_{\mathbb{Q}} \mathfrak{S}_{c,d} = G_{\mathbb{A}}$, and the lemma is proved.

3.2. Compactness of $R(\phi)$.

The compactness of the operator:

$$R(\phi) : L_0^2(G_F \backslash G_{\mathbb{A}}, \omega) \to L_0^2(G_F \backslash G_{\mathbb{A}}, \omega)$$

follows from the lemma by applying Arzela-Ascoli theorem to the one compactification Y of the set $Z_{\mathbb{A}} G_F \backslash G_{\mathbb{A}}$.

Theorem 3.7 (Arzela-Ascoli). *Assume Y is a compact topological space, and consider $\Sigma \subset C(Y)$ is an equicontinuous family which is bounded in the L^{∞} norm. Then the closure of Σ in $C(Y)$ is compact.*

Suppose $\mathcal{B}$ is the unit ball in the Hilbert space $L_0^2(G_F \backslash G_{\mathbb{A}}, \omega)$. Consider the family of functions $\Sigma = R(\phi)\mathcal{B}$. Then Σ has the following properties:

1. $\Sigma \subset C(Y)$. This is because $R(\phi)$ is a convolution operator with smooth kernel (smoothing operator). On the other hand, we saw that each function $R(\phi) f$ decreases rapidly in the cusp, so it can be view as a genuine continuous function on Y.
2. Σ is an equicontinuous family. This immediate, since $R(\phi)$ is a convolution operator.
3. The Lemma says that every function in Σ is bounded in the L^{∞}-norm.

Hence we can apply Arzela-Ascoli theorem to the family Σ and conclude that the σ is pre-compact in $C(Y)$. This makes:

$$R(\phi) : L_0^2(G_F \backslash G_{\mathbb{A}}, \omega) \to C(Y)$$

a compact operator. Since Y is compact, the inclusion:

$$C(Y) \hookrightarrow L_0^2(G_F \backslash G_{\mathbb{A}}, \omega)$$

is a bounded operator. The composition of the previous two operators gives us that:

$$R(\phi) : L_0^2(G_F \backslash G_{\mathbb{A}}, \omega) \to L_0^2(G_F \backslash G_{\mathbb{A}}, \omega)$$

is also compact operator.

Remark 3.8. *The statement of the Main Theorem now follows immediately from Theorem B.*

4. MULTIPLICITY ONE PROPERTY

Question 4.1. *Assume π, V is an irreducible unitary representation of $G_{\mathbb{A}}$ which occurs discretely in $L_0^2(G_F\backslash G_{\mathbb{A}}, \omega)$. What is its multiplicity?*

In other words, Theorem 2.1 says that we have a Hilbert direct sum decomposition:

$$L_0^2(G_F\backslash G_{\mathbb{A}}, \omega) \simeq \hat{\bigoplus}_{\pi\in\widehat{G}_{\mathbb{A}}} m_\pi V_\pi$$

where $\widehat{G}_{\mathbb{A}}$ denotes the unitary dual of $G_{\mathbb{A}}$, and the integer $m_\pi \geq 0$ represents the number of times (multiplicity) a given unitary irreducible representation $\pi \in \widehat{G}_{\mathbb{A}}$ occurs in the decomposition of $L_0^2(G_F\backslash G_{\mathbb{A}}, \omega)$.

The question is to determine the numbers m_π. The following theorem of Shalika answers this question in the case $G = GL(n)$. For reference, see [Sha].

Theorem 4.2 (Shalika)**.** *Let F be a number field, and let $G = GL(n)$ be defined over F. Then the space $L_0^2(G_F\backslash G_{\mathbb{A}}, \omega)$ decomposes as a Hilbert direct product of irreducible unitary representations of $G_{\mathbb{A}}$, each such representation appearing at most once. In other words,*

$$m_\pi \leq 1.$$

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