MATH 515
LECTURE 11: REPRESENTATIONS OVER P-ADIC FIELDS
LANGLANDS CLASSIFICATION

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1. Introduction

In this section we assume that F is a local, non-archimedean field, e.g. $F = \mathbb{Q}_p$. The group is $G = GL_n(F), n \geq 2$, with center $Z = Z(F)$.
Definitions Assume (π, V) is an irreducible, admissible representation of G, with central character ω, where $\omega : F^\times \to \mathbb{C}^\times$ is a multiplicative character.
1) π is a **square integrable** representation, or L^2, if and only if, for any $v, w \in V$, the matrix coefficient $c_{v,w}(g) :=< \pi(g)v, w >$ is in L^2, in the sense that:

$$\int_{Z\backslash G} |c_{v,w}(g)|^2 |\omega(g)|^{-1} dg < \infty$$

with dg the Haar measure.
2) π is **supercuspidal** if for any $v, w \in V$, the matrix coefficient $c_{v,w}(g)$ has compact support $(\bmod Z)$.
3) π is a **tempered** representation if, for any $v, w \in V$, the matrix coefficient $c_{v,w}(g)$ is in $L^{2+\epsilon}$, for an arbitrarily small $\epsilon > 0$.

2. Parabolic Induction

Assume P is the standard parabolic subgroup of G of type $(n_1, \cdots, n_r)$, where $\sum_{i=1}^r n_i = n$. Let U be the radical unipotent of P and M the Levi subgroup, $P = MU$.

$$M = \begin{pmatrix} \boxed{*} & & \\ & \ddots & \\ & & \boxed{*} \end{pmatrix}, \quad U = \begin{pmatrix} I_{n_1} & * & * \\ & \ddots & * \\ & & I_{n_r} \end{pmatrix}$$

Hence $M \simeq \prod_{i=1}^r GL_{n_i}(F)$. Assume now that (σ_i, V_i) is an irreducible, admissible representation of $GL_{n_i}(F)$. Then $\sigma := \otimes_{i=1}^r \sigma_i$ is an irreducible, admissible representation of M. We extend σ trivially to P, such that the unipotent radical U acts trivially. We denote by V the representation space of ρ, $V := \otimes_{i=1}^r V_i$. Consider now the linear space of locally constant functions $F : G \to V$ satisfying the property

$$F(pg) = \delta(p)^{\frac{1}{2}} \sigma(p)(F(g)) \tag{1}$$

where $\delta(p)$ is the modulus determined by the Haar measure of the group P (see the note below). The group G acts on this set of functions by right translations, $\rho(g)F(x) := F(xg), x, g \in G$. This defines a representation ρ of G, which is obtained by inducing the representation σ from P to G. We use the following notation:

$$\rho := \mathrm{Ind}_P^G \sigma := \mathrm{Ind}(G, P; \sigma_1, \cdots, \sigma_r)$$

Note. Assume dp is the left-invariant Haar measure on the group P. Then for an arbitrary $x \in P$, the right translation of the Haar measure, $R(x)dp := d(px)$ is still a left-invariant measure. Since the Haar measure is unique, $R(x)dp$ must be a constant multiple of dp, and we denote this constant multiple by $\delta^{-1}(x)$:

$$R(x)dp := \delta^{-1}(x)dp$$

This definition determines a group homomorphism $\delta : P \to \mathbb{R}_{>0}$. The group P is not unimodular, which means that $\delta \neq 1$. We also have an explicit formula for the modulus:

$$\delta(p) = \det\left(\mathrm{Ad}(p)|_{\mathfrak{u}_P}\right)$$

where $\mathfrak{u}_P$ is the Lie algebra of the unipotent radical U of P. The insertion of the modulus δ in the definition of the parabolic induction has the role of preserving unitarity, i.e. f σ is unitary, then $\mathrm{Ind}_P^G \sigma$ is unitary as well.

3. Square Integrable Representations

3.1. **Construction.** Bernstein and Zelevinskii [BZ] gave the following description of the **square integrable** representations:
Assume $n = kr$ and consider the standard parabolic subgroup P of the type $(r, r, \cdots, r)$ of G. Let M be the Levi subgroup and U the unipotent radical, $P = MU$ and $M \simeq GL_r(F) \times \cdots \times GL_r(F)$. Let τ be a *supercuspidal* representation of $GL_r(F)$. Define the parabolically induced representation of G:

$$\rho := \mathrm{Ind}(G, P; \tau \otimes |*|^{\frac{r-1}{2}}, \cdots, \tau \otimes |*|^{-\frac{r-1}{2}})$$

where $|*|$ stands for the absolute value, $|*| : F^\times \to \mathbb{R}$. Then ρ contains a unique, irreducible, sub-representation π, which is **square integrable**. Moreover, any square integrable representation can be obtained in this way.
Note. We remark the analogy between the construction of a square-integrable representation and the construction of the Steinberg representation for GL_2 over a finite field. It is essentially the same construction.

3.2. **Formula for the L-factor.** Assume

$$\pi \hookrightarrow \mathrm{Ind}_P^G(\sigma \otimes |*|^{\frac{r-1}{2}}, \cdots, \sigma \otimes |*|^{-\frac{r-1}{2}})$$

is a square-integrable representation. Then:

$$L(s, \pi) = L(s, \tau)$$

Note. This is a degenerate case in the following sense: in principle, the L-factor associated to an irreducible representation of $GL_n(F)$ should be a polynomial of degree n in q^{-s}, where q is the residue characteristic of F; this formula shows that the L-factor of a square-integrable representation is a polynomial of a lower degree.
Example. Assume π is the square-integrable representation of $GL_3(\mathbb{Q}_p)$ which is induced from

$$\begin{pmatrix} |*| & & \\ & 1 & \\ & & |*|^{-1} \end{pmatrix}$$

Then $L(s, \pi) = \zeta_p(s) = (1 - p^{-s})^{-1}$.

4. Tempered Representations

4.1. **Construction.** The following result is due to Jacquet:
Suppose that P is the standard parabolic of the type $(n_1, \cdots, n_r)$. Suppose that σ_i is square-integrable representation of $GL_{n_i}(F)$ with unitary central character. Then

$$\pi := \mathrm{Ind}_P^G(\sigma_1, \cdots, \sigma_r)$$

is a **tempered**, irreducible representation of G.

4.2. **Formula for the L-factor.**

$$L(s, \pi) = \prod_{i=1}^{r} L(s, \sigma_i)$$

5. Langlands Classification

5.1. **Construction.** Assume F is a local fields, archimedean or otherwise. Let $Q = M_Q U_Q$ be a standard parabolic subgroup of type $(n_1, \cdots, n_r)$. For $1 \leq i \leq r$, let τ_i be an irreducible, tempered representation of $GL_{n_i}(F)$. Suppose that $t_1 \geq \cdots \geq t_r$ is a sequence of integer numbers. Then:

$$\eta := \mathrm{Ind}_Q^G(\tau_1 \otimes |*|^{t_1}, \cdots, \tau_r \otimes |*|^{t_r})$$

has a unique irreducible quotient representation, say

$$J(\tau, t) = J(G, P; \tau_1, \cdots, \tau_r; t_1, \cdots, t_r)$$

Moreover, any irreducible admissible representation of G is obtained this way.

5.2. **Formula for the L-factor.**

$$L(s, \pi) = \prod_{i=1}^{r} L(s, \tau_i \otimes |*|^{t_i}) = \prod_{i=1}^{r} L(s + t_i, \tau_i)$$

Note. This theorem is due to Langlands in the case F is archimedean, and to Bernstein and Zelevinskii and Silberger, in the case F is non-archimedean.

References

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