The Geometry of Trigonometric Sums from a Dynamical Viewpoint

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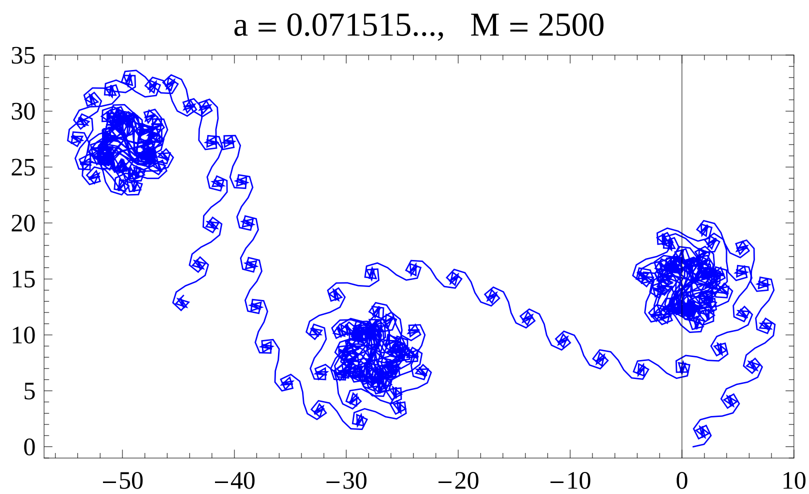
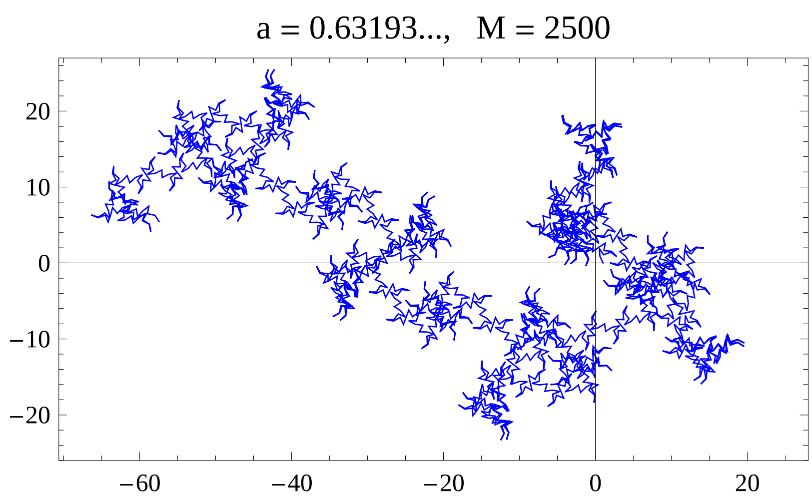
Quadratic Trigonometric Sums

Consider $a \in (-1, 1]\backslash\{0\}$ and $N \in \mathbb{N}$ and define the **quadratic trigonometric sum**

$$\mathcal{S}_a(N) := \sum_{n=0}^{N-1} \exp(\pi i \, a \, n^2) \ \in \mathbb{C}.$$

We are interested in the geometry of the set $\{\mathcal{S}_a(N)\}_{N=1}^{M}$ as $M \to \infty$.

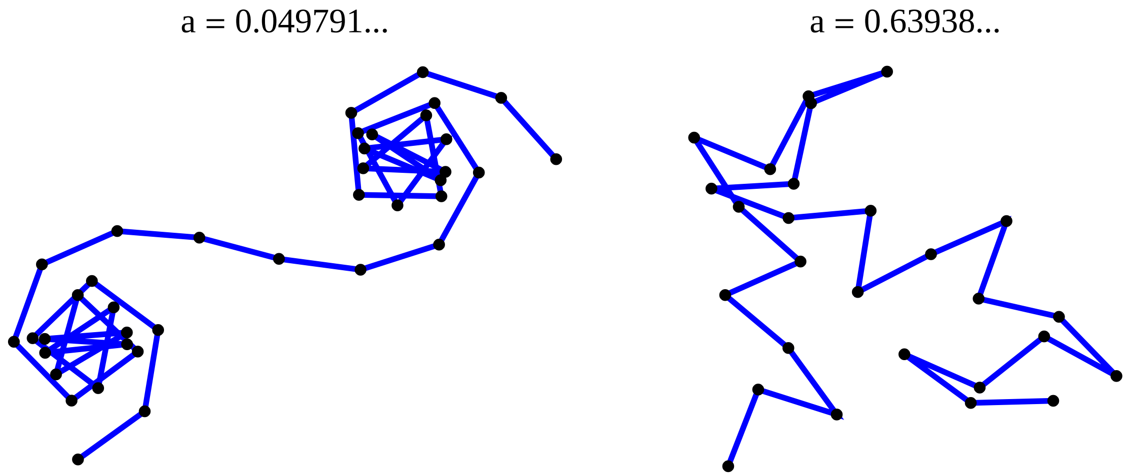




(Incomplete) List of References:

Hardy-Littlewood (1914,‘23), Weyl (1914,‘16), van der Corput (1923), Mordell (1926), Friedler-Jurkat-Körner (1977), Jurkat-van Horne (1981, ‘82,‘83), Dekking-Mendès France (1981), Mendès France (1983,‘84), Deshouillers (1985), Berry-Goldberg (1988), Moore-van der Poorten (1989), Coutsias-Kazarinoff (1987,‘98), Marklof (1999), Forrest (1996,2000), Fedotov-Klopp (2005), Flaminio-Forni (2006), Fayad (2006), Greshonig-Nerurkar-Volný (2007), ...

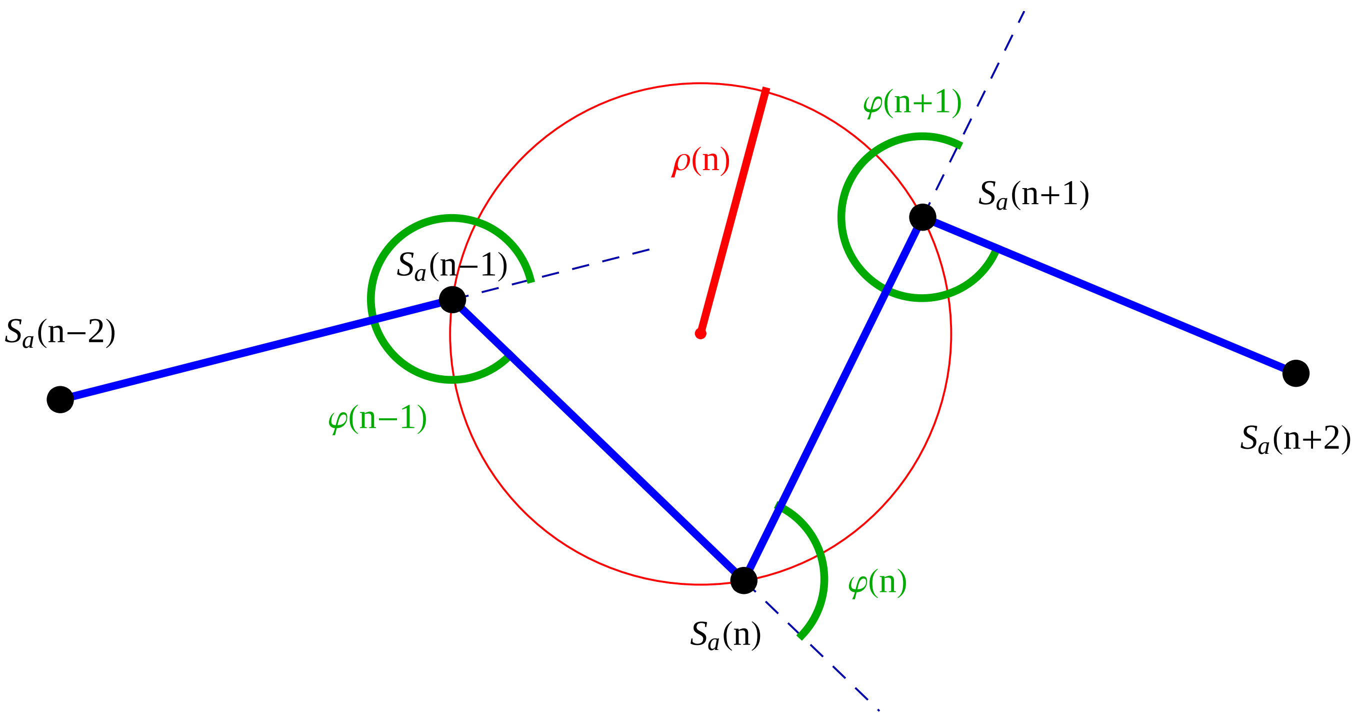
How to explain the presence/absence of spiral-like structures ("curlicues") at several scales? Let's first focus on the smallest scale (level 0).



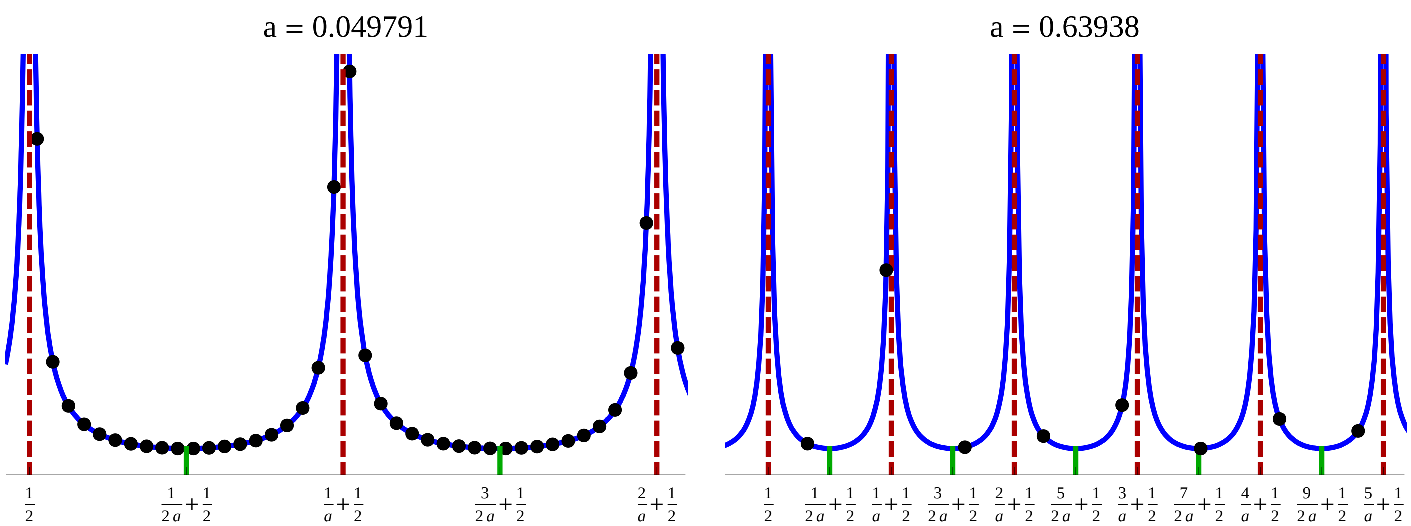
Following Coutsias and Kazarinoff (1987, '98) we introduce the *local discrete radius of curvature* $\rho_a(n)$ = the radius of the circle passing through the three consecutive points $\mathcal{S}_a(n-1)$, $\mathcal{S}_a(n)$, $\mathcal{S}_a(n+1)$. We get

$$\rho_a(n) = \frac{1}{2}\left|\csc\left(\frac{\pi\, a\,(2n-1)}{2}\right)\right|.$$

Since $t \mapsto |\csc(t)|$ is π-periodic, $\nu \mapsto \rho_a(\nu)$ is $\frac{1}{a}$-periodic. Moreover, it has vertical asymptotes at $\frac{k}{a} + \frac{1}{2}$ and local minima at $m_k := \frac{2k+1}{2a} + \frac{1}{2}$, $k \in \mathbb{Z}$.



The geometric structure at **level 0** comes from the integer sampling of the function $\nu \mapsto \rho_a(\nu)$.

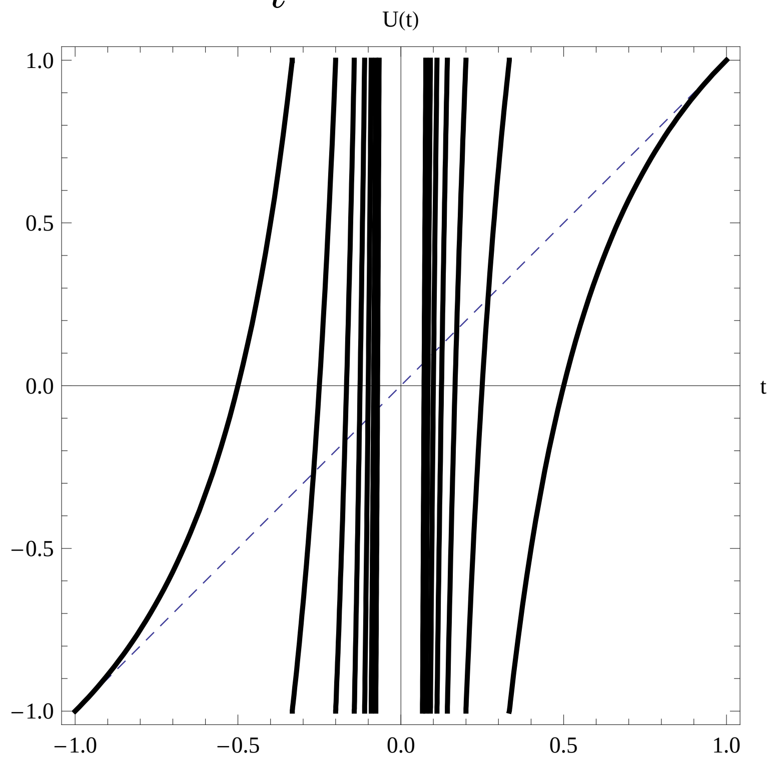


We partition the integers into blocks $\left(m_l, m_{l+1}\right]$ of length $\sim \frac{1}{|a|}$. Therefore, $|a|$ small $\Rightarrow$ spirals (“curlicues”), while $|a| \sim 1 \Rightarrow$ no spirals.

How about the geometric structure at **higher levels**?

Approximate Renormalization Formula

Consider the map $U : (-1,1]\backslash\{0\} \to (-1,1]\backslash\{0\}$ given by $U(t) = -\frac{1}{t} \,(\mathrm{mod}\, 2)$.



Setting $a_1 = U(a)$ and $N_1 = \lfloor |a|\, N \rfloor$ we get the **approximate renormalization formula** (ARF)

$$\left| \mathcal{S}_a(N) - e^{\frac{\pi}{4} i} \, |a|^{-\frac{1}{2}} \, \mathcal{S}_{a_1}(N_1) \right| \le C_1 \, |a|^{-\frac{1}{2}} + C_2,$$

where C_1, C_2 are universal constants.

References: Hardy-Littlewood (1914), Mordell (1926), Wilton (1926), Coutsias-Kazarinoff (1998).

Sketch of the proof of (ARF).

For every $0 < \delta, \gamma < 1$,

$$\mathcal{S}_a(N) = \sum_{n=-\infty}^{+\infty} e^{\pi\, i\, a\, n^2}\, \mathbf{1}_{[-\delta, N-\gamma]}(n) =$$

by Poisson Summation Formula

$$= \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{\infty} e^{-2\pi\, i\, m\, x}\, e^{\pi\, i\, a\, x^2}\, \mathbf{1}_{[-\delta, N-\gamma]}(x)\, \mathrm{d}x =$$

$$= \sum_{m=-\infty}^{+\infty} \int_{-\delta}^{N-\gamma} e^{\pi\, i (a\, x^2 - 2m\, x)}\, \mathrm{d}x \approx$$

by Stationary Phase Method choose δ, γ s.t.

$$\approx \sum_{m=0}^{\lfloor |a| N \rfloor - 1} \int_{-\infty}^{\infty} e^{\pi\, i (a\, x^2 - 2m\, x)}\, \mathrm{d}x =$$

$$= \sum_{m=0}^{\lfloor |a| N \rfloor - 1} \frac{e^{-\pi\, i\, \frac{m^2}{a}}}{|a|^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{\pi\, i\, y^2}\, \mathrm{d}y =$$

$$= \frac{e^{\frac{\pi}{4} i}}{|a|^{\frac{1}{2}}} \sum_{m=0}^{\lfloor |a| N \rfloor - 1} e^{-\pi\, i\, \frac{m^2}{a}} =$$

$$= e^{\frac{\pi}{4} i}\, |a|^{-\frac{1}{2}}\, \mathcal{S}_{-\frac{1}{a}}(\lfloor |a| N \rfloor) = e^{\frac{\pi}{4} i}\, |a|^{-\frac{1}{2}}\, \mathcal{S}_{a_1}(N_1).$$

$\square$

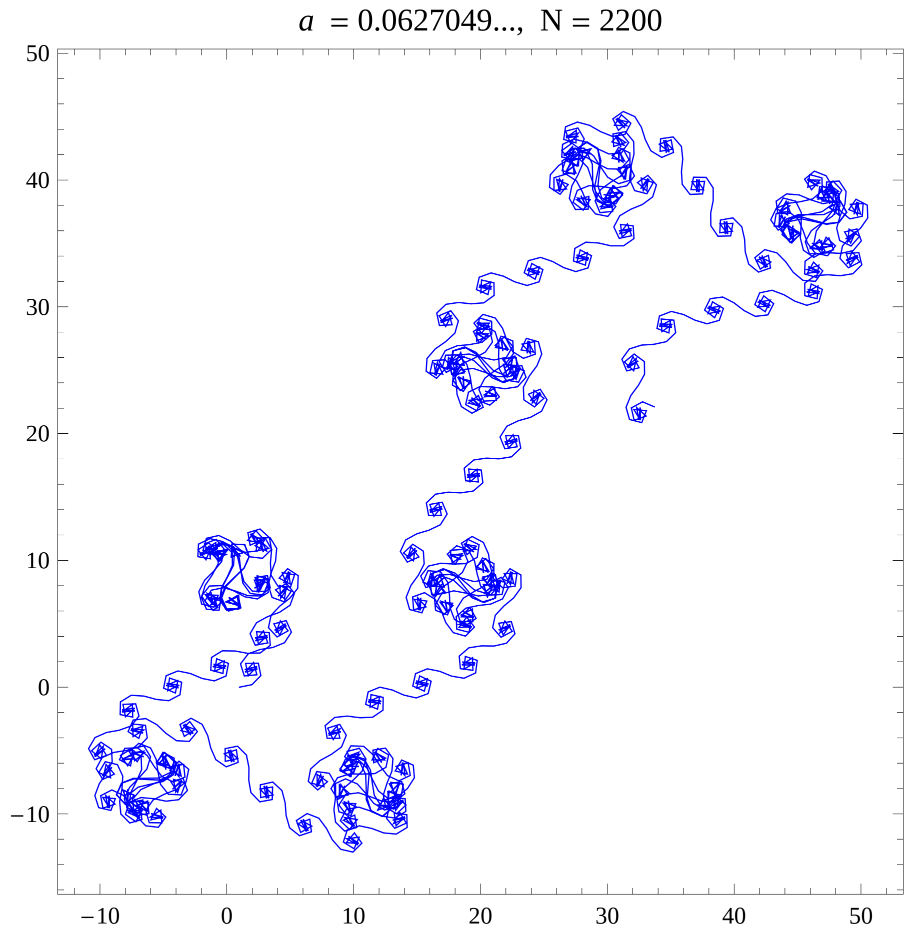
$$\left|\mathcal{S}_a(N) - e^{\frac{\pi}{4}i}\,|a|^{-\frac{1}{2}}\,\mathcal{S}_{a_1}(N_1)\right| \le C_1\,|a|^{-\frac{1}{2}} + C_2,$$

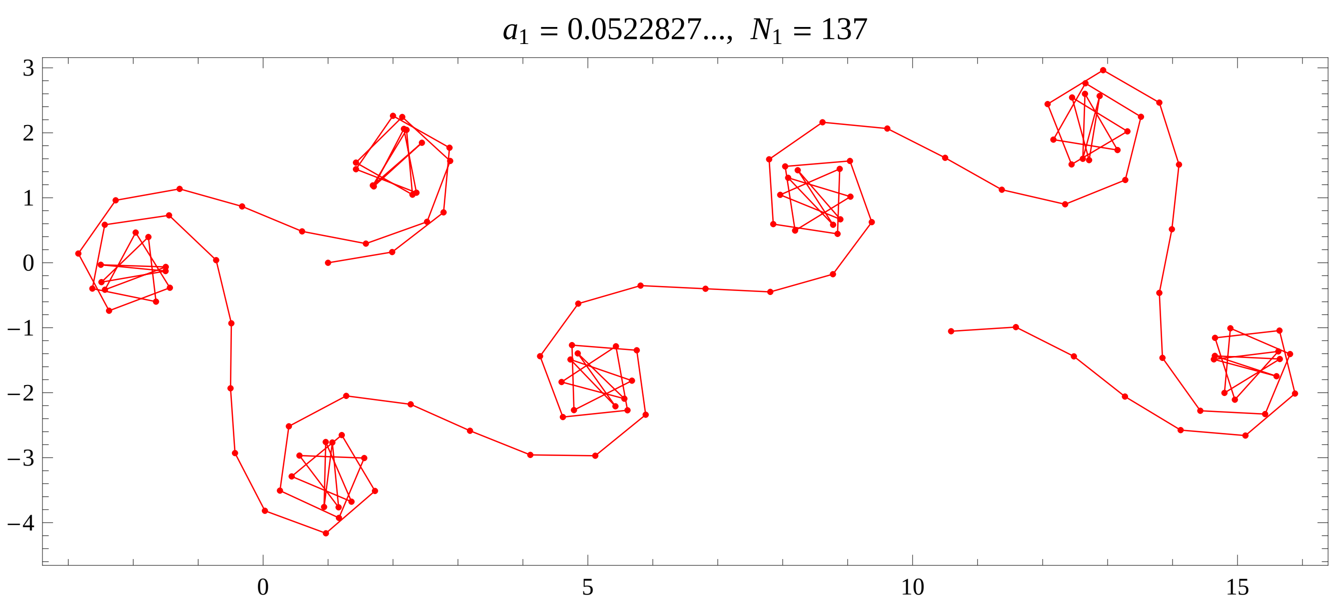
where $a_1 = U(a)$ and $N_1 = \lfloor |a|\,N \rfloor$. Notice that this estimate is uniform in N.

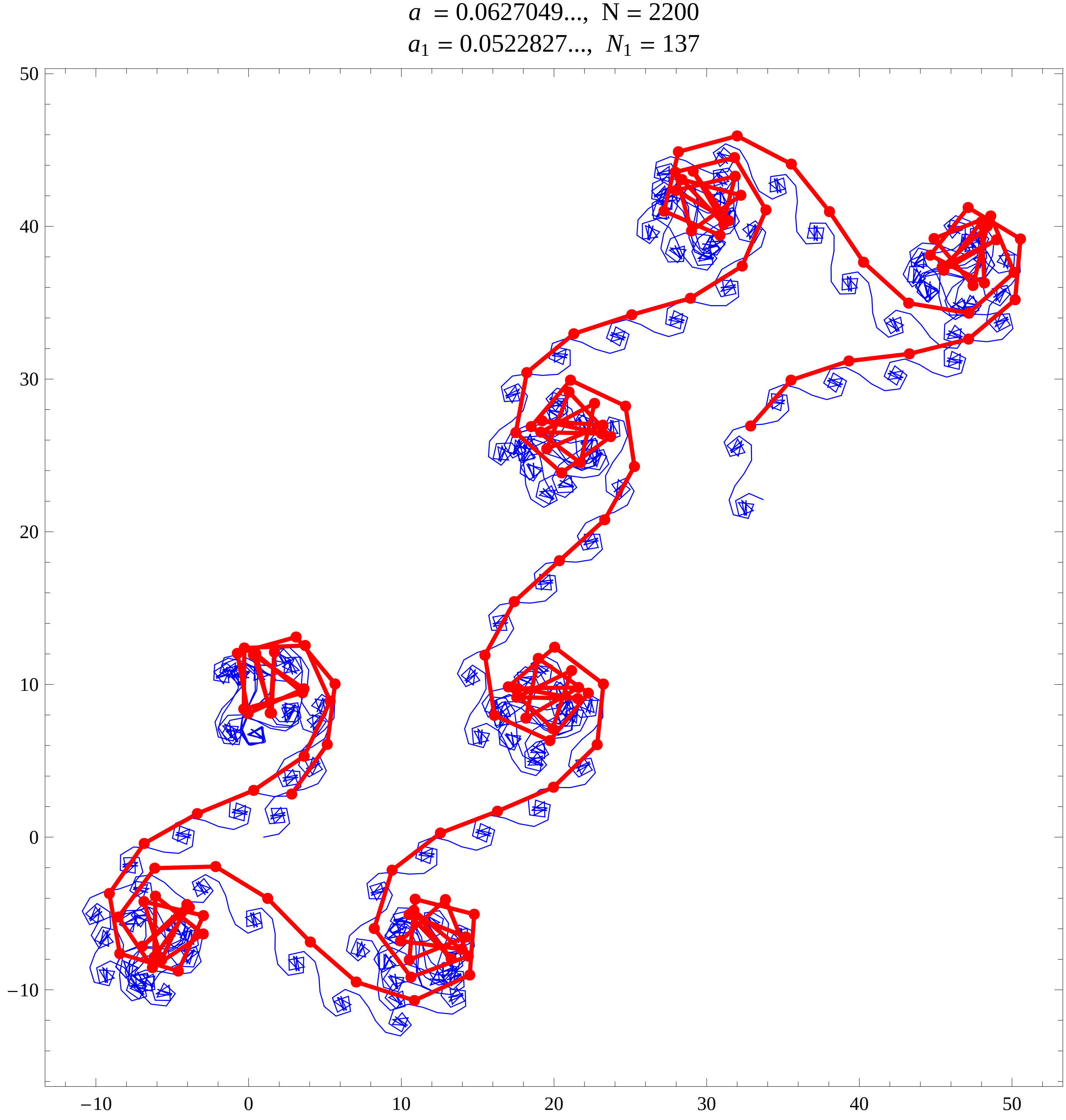
The curve $\{\mathcal{S}_a(n)\}_{n=1}^{N}$ contains approximately $|a|N$ $(= N_1 < N)$ blocks at level 0.

By (ARF), $\{\mathcal{S}_a(n)\}_{n=1}^{N}$ can be approximated by $\{\mathcal{S}_{a_1}(n)\}_{n=1}^{N_1}$ (up to scaling by $|a|^{-\frac{1}{2}}$ and rotating by $\frac{\pi}{4}$), i.e. we replace each block in $\{\mathcal{S}_a(n)\}_{n=1}^{N}$ by a point in $\{\mathcal{S}_{a_1}(n)\}_{n=1}^{N_1}$. The details which we erase are of size $\mathcal{O}(|a|^{-\frac{1}{2}})$.

The geometric structure at **level 0** for $\{\mathcal{S}_{a_1}(n)\}_{n=1}^{N_1}$ corresponds to the structure at **level 1** for $\{\mathcal{S}_a(n)\}_{n=1}^{N}$.


a = 0.0627049..., N = 2200
50
40
30
20
10
0
−10
−10
0
10
20
30
40
50


a_1 = 0.0522827..., N_1 = 137
3
2
1
0
−1
−2
−3
−4
0
5
10
15



a = 0.0627049..., N = 2200
a_1 = 0.0522827..., N_1 = 137
50
40
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0
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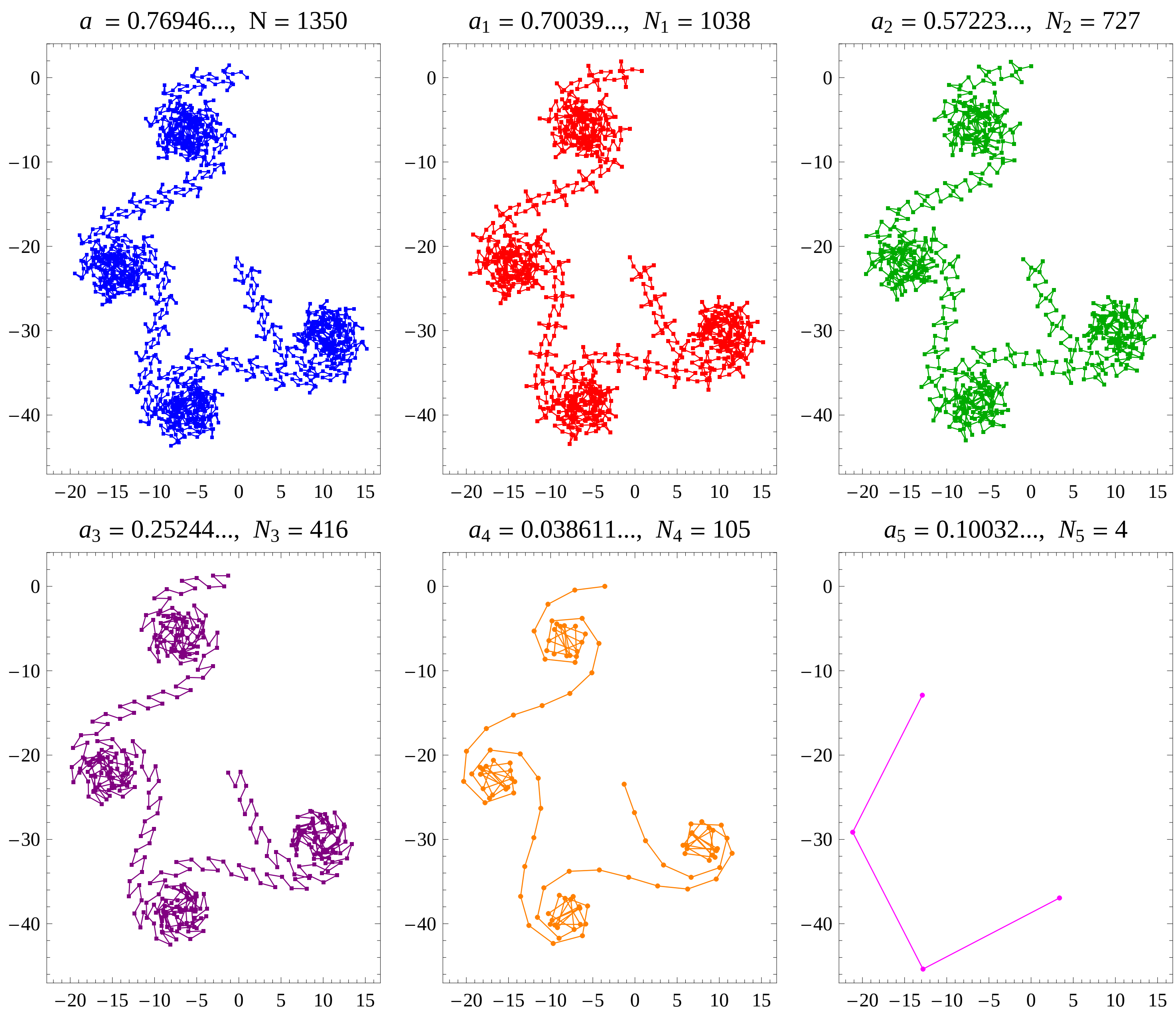
50

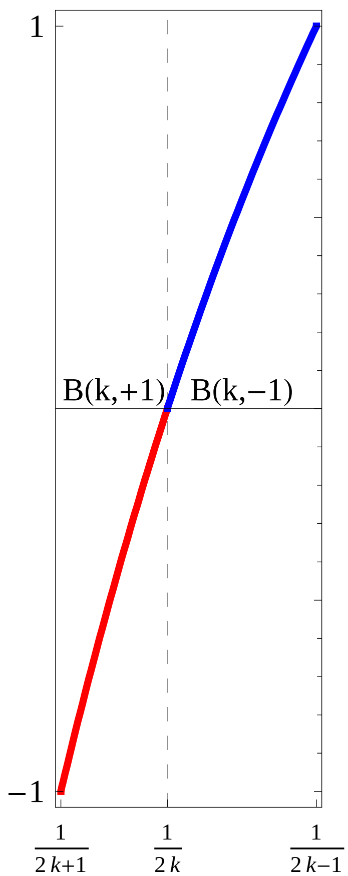
Of course, we want to iterate the renormalization process...



We want to study the map $U : t \mapsto -\frac{1}{t}\,(\mathrm{mod} 2)$.
Its k-th positive branch is

$$U_k^+ : \left(\frac{1}{2k+1}, \frac{1}{2k-1}\right] \longrightarrow (-1, 1].$$

Define
$B(k,-1) = \left(\frac{1}{2k}, \frac{1}{2k-1}\right]$ and $B(k,+1) = \left(\frac{1}{2k+1}, \frac{1}{2k}\right]$.



For $\alpha = |a| \in (0,1]$ set
$\eta(\alpha) := \mathrm{sgn}(U(\alpha))$ and $\xi(\alpha) = -\eta(\alpha)$
and denote

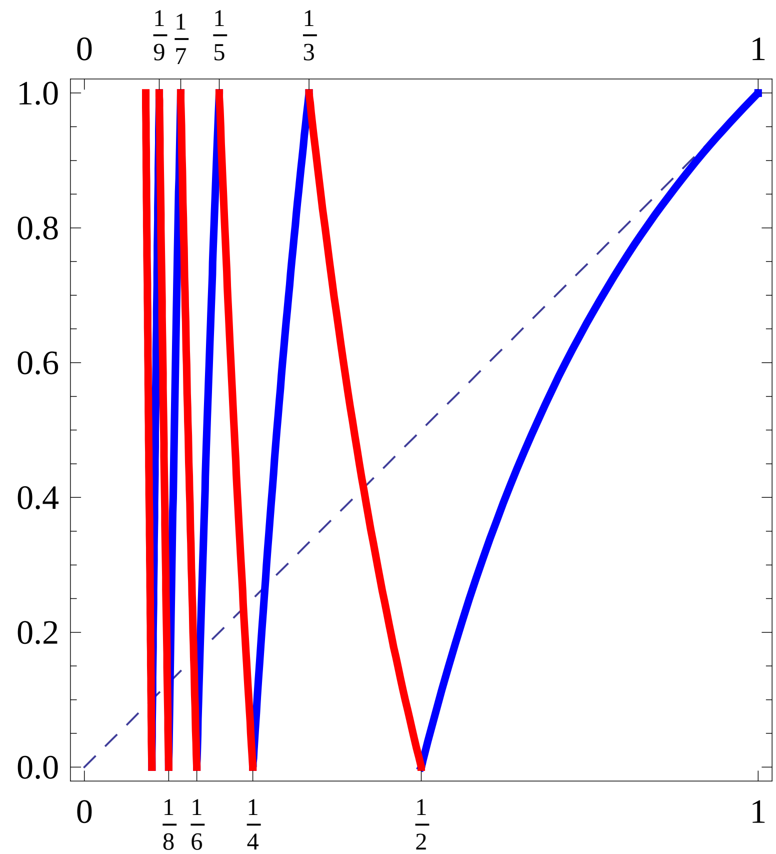
$$\mathcal{S}_\alpha^{(\eta)}(N) = \begin{cases} \mathcal{S}_\alpha(N) & \text{if } \eta = +1 \;, \\ \overline{\mathcal{S}_\alpha(N)} & \text{if } \eta = -1. \end{cases}$$

We want to keep track of $|U(a)|$ and $\mathrm{sgn}(U(a))$ separately: we introduce a new map $T : (0,1] \to (0,1]$,

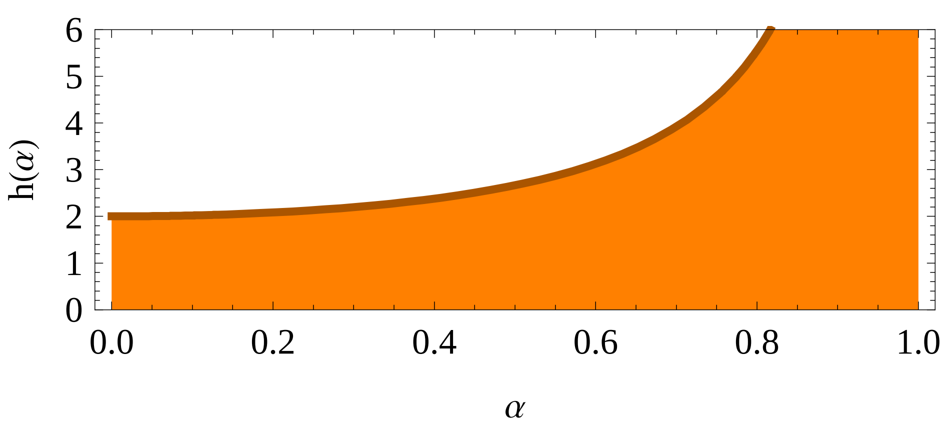
$$T(\alpha) = \xi \cdot \left(\frac{1}{\alpha} - 2k\right)$$

for $\alpha \in B(k,\xi)$, $k \in \mathbb{N}$, $\xi \in \{-1, +1\}$.

$$T(\alpha) = \xi \cdot \left(\frac{1}{\alpha} - 2k\right),\ \alpha \in B(k,\xi),$$
$$k \in \mathbb{N},\ \xi \in \{-1, +1\}.$$



Theorem A (Schweiger, 1982). The map T has a σ-finite, infinite, ergodic, invariant measure μ. Its density is $h(\alpha) = \frac{1}{\alpha+1} - \frac{1}{\alpha-1}$.



Using the map T and the new notation $\mathcal{S}_{\cdot}^{(\cdot)}(\cdot)$, the (ARF) can be rewritten as

$$\left|\mathcal{S}_{\alpha}(N) - e^{\frac{\pi}{4}i}\,\alpha^{-\frac{1}{2}}\,\mathcal{S}_{\alpha_1}^{(\eta_1)}(N_1)\right| \le C_1\alpha^{-\frac{1}{2}} + C_2,$$

$$\alpha_1 = T(\alpha),\ \eta_1 = \eta(\alpha),\ N_1 = \lfloor \alpha N \rfloor$$

Fedotov and Klopp (2006) provided an explicit formula for the error term $\Lambda(\alpha, N) = \mathcal{S}_{\alpha}(N) - e^{\frac{\pi}{4}i}\,\alpha^{-\frac{1}{2}}\,\mathcal{S}_{\alpha_1}^{(\eta_1)}(N_1)$ and obtained the formula

$$\Lambda(\alpha, N) = \alpha^{-\frac{1}{2}}\left(e^{\pi\, i\left(\alpha\, N^2 - \frac{\zeta^2}{\alpha}\right)}\int_{-\infty}^{\zeta/\sqrt{\alpha}} e^{\pi\, i\tau^2}\mathrm{d}\tau - \frac{e^{\frac{\pi}{4}i}}{2}\right) + \mathcal{O}(1),$$

where $\zeta = \{\alpha N\}$, $-\frac{1}{2} < \zeta \le \frac{1}{2}$.

Iterating the renormalization process r times:

$$\alpha = \alpha_0 \overset{T}{\mapsto} \alpha_1 \overset{T}{\mapsto} \alpha_2 \overset{T}{\mapsto} \cdots \overset{T}{\mapsto} \alpha_r, \qquad \alpha_j = T^j(\alpha_0),$$

$$N = N_0 \mapsto N_1 \mapsto N_2 \mapsto \cdots \mapsto N_r, \qquad N_{j+1} = \lfloor \alpha_j N_j \rfloor,$$

$$N_0 > N_1 > N_2 > \ldots > N_r, \qquad \eta_{j+1} = \eta(\alpha_j),$$

$$\eta_0 = 1.$$

Continued Fractions with Even Partial Quotients

For $\alpha \in B(k,\xi)$ we have $\alpha = \frac{1}{2k+\xi\, T(\alpha)}$.
Thus we get the *ECF-expansion* of $\alpha \in (0,1]$,

$$\alpha = \cfrac{1}{2k_1 + \cfrac{\xi_1}{2k_2 + \cfrac{\xi_2}{2k_3 + \cfrac{\xi_3}{\ddots}}}} = \\ =: [[(k_1,\xi_1),(k_2,\xi_2),(k_3,\xi_3),\ldots]]\,,$$

with $(k_n,\xi_n) \in \mathbb{N} \times \{\pm 1\} =: \Omega$, and T acts as a shift over the space $\Omega^{\mathbb{N}}$:

If $\alpha = [[(k_1,\xi_1),(k_2,\xi_2),\ldots]]\,,$
then $T^n(\alpha) = [[(k_{n+1},\xi_{n+1}),(k_{n+2},\xi_{n+2}),\ldots]]$.

<u>References</u>: Schweiger (1982, '84), Kraaikamp-Lopes (1996).

The multi-scale geometry of $\{\mathcal{S}_\alpha(n)\}_{n=1}^N$ is encoded in the ECF-expansion of α: the length of blocks at the l-th scale is $\sim 2k_{l+1}$.

We define de the *ECF-convergents* of α as

$$\begin{aligned}\frac{p_n}{q_n} &= \cfrac{1}{2k_1 + \cfrac{\xi_1}{2k_2 + \ldots + \cfrac{\xi_{n-2}}{2k_{n-1} + \frac{\xi_{n-1}}{2k_n}}}} = \\ &= [[(k_1, \xi_1), (k_2, \xi_2), \ldots, (k_n, \pm 1)]],\end{aligned}$$

$\text{GCD}(p_n, q_n) = 1$.

The convergents satisfy a **recurrence relation**

$$\begin{aligned}&p_n = 2k_n\, p_{n-1} + \xi_{n-1}\, p_{n-2},\\ &q_n = 2k_n\, q_{n-1} + \xi_{n-1}\, q_{n-2},\\ &q_{-1} = p_0 = 0,\ \ p_{-1} = q_0 = \xi_0 = 1.\end{aligned}$$

Consider $L > 0$ and define the **renewal time** $n_L = n_L(\alpha) = \min\{n \in \mathbb{N} :\ q_n > L\}$.

Theorem (C.). The ratio $\frac{q_{n_L}}{L}$ has a limiting probability distribution as $L \to \infty$ w.r.t. some a.c. measure λ on $(0, 1]$.

In other words: there exists a probability measure P_{lim} on $(1, \infty)$ such that for all $a, b > 1$

$$\lambda\left(\left\{\alpha :\ a < \frac{q_{n_L}}{L} < b\right\}\right) \xrightarrow{L\to\infty} \mathsf{P}_{\text{lim}}\Big((a, b)\Big).$$

Iterated Renormalization Formula

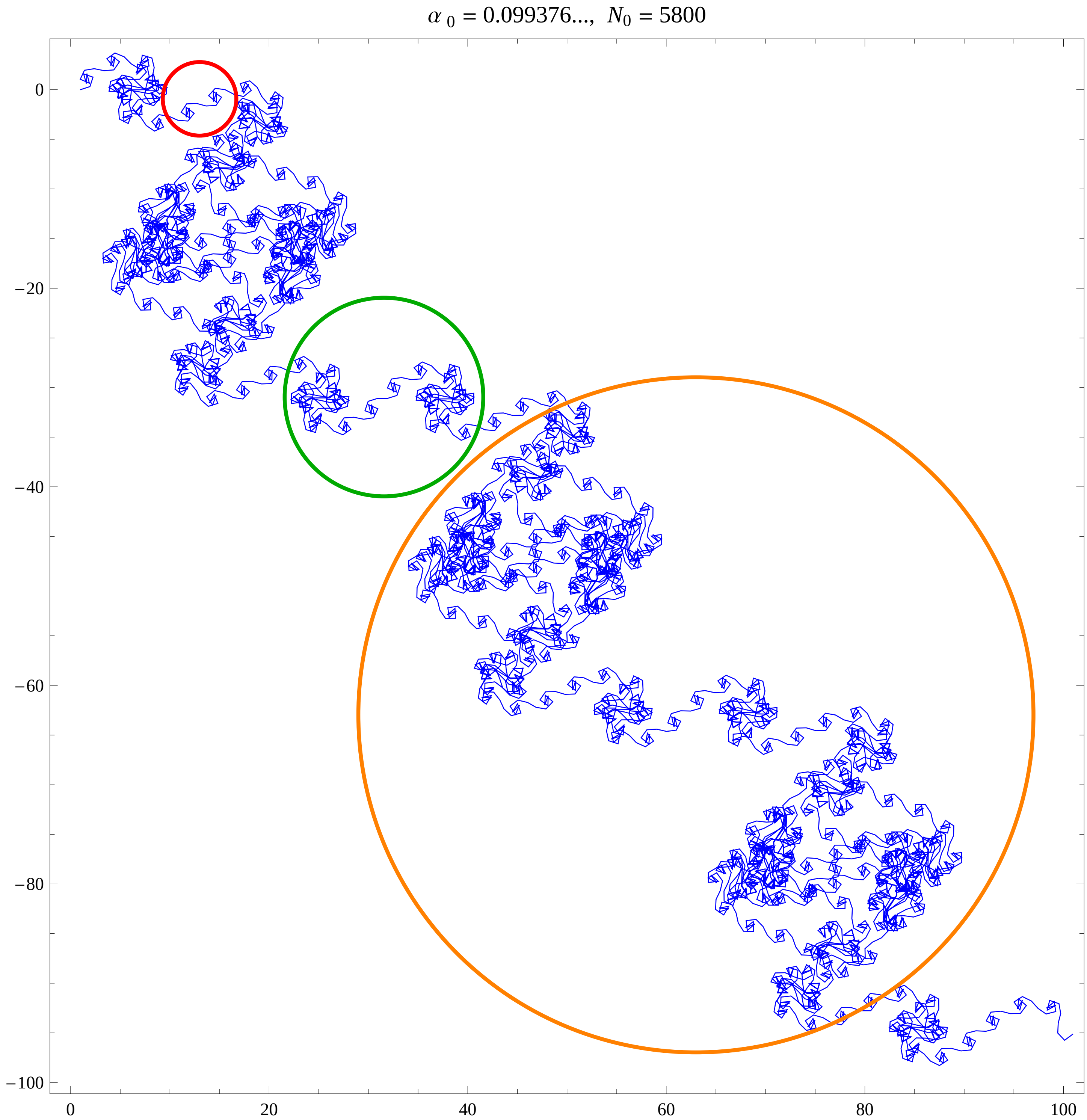
We define the function $\kappa(\alpha, s) = \sum_{t=0}^{s-1} \prod_{m=0}^{t} \eta_m$ and l-th error term $\tilde{\Lambda}_l = \tilde{\Lambda}_l(\alpha, N)$,

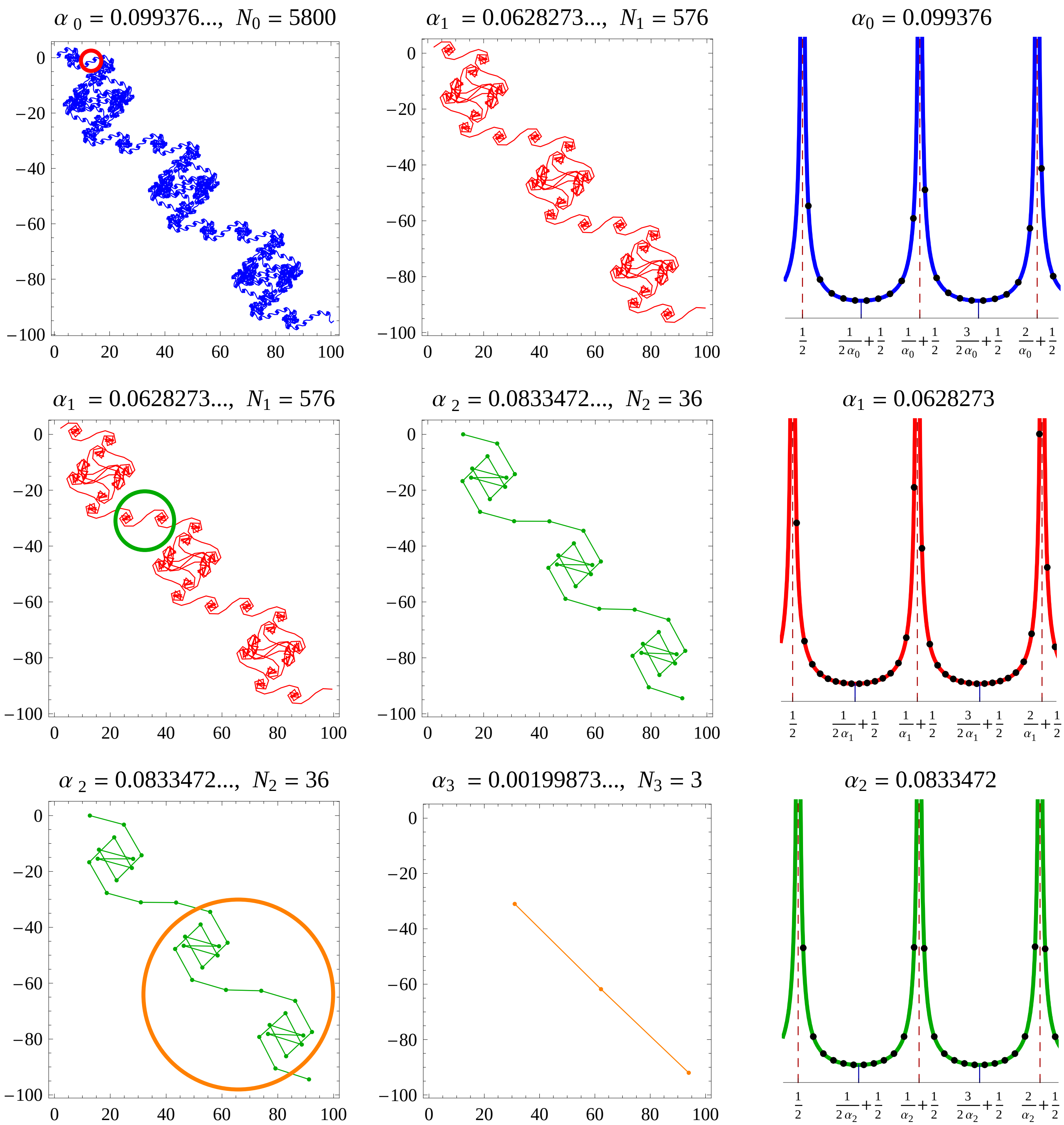
$$\tilde{\Lambda}_l := \exp\{\kappa(\alpha, l)\tfrac{\pi}{4}i\}\, \Lambda^{(\eta_1 \cdots \eta_l)}(\alpha_l, N_l),$$

which satisfies $|\tilde{\Lambda}_j(\alpha, N)| \leq C_1 \alpha_j^{-\frac{1}{2}} + C_2$. We get the **iterated (ARF)**:

$$\begin{aligned}
\mathcal{S}_\alpha(N) \;=\; & e^{\kappa(\alpha,r)\frac{\pi}{4}i}\,(\alpha_0 \cdots \alpha_{r-1})^{-\frac{1}{2}}\, \mathcal{S}_{\alpha_r}^{(\eta_1 \cdots \eta_r)}(N_r) + \\
& + (\alpha_0 \cdots \alpha_{r-2})^{-\frac{1}{2}}\, \tilde{\Lambda}_{r-1} + \ldots + \\
& + (\alpha_0 \cdots \alpha_{r-j})^{-\frac{1}{2}}\, \tilde{\Lambda}_{r-j+1} + \ldots + \\
& + \alpha_0^{-\frac{1}{2}}\, \tilde{\Lambda}_1 + \tilde{\Lambda}_0.
\end{aligned}$$

Remark: $\left|(\alpha_0 \cdots \alpha_{l-1})^{-\frac{1}{2}}\, \tilde{\Lambda}_l\right| \leq C_3\, (\alpha_0 \cdots \alpha_l)^{-\frac{1}{2}}$, i.e. the patterns at the l-th scale have magnitude $(\alpha_0 \cdots \alpha_l)^{-\frac{1}{2}}$ and are described as integer sampling of the function $\nu \mapsto \rho_{\alpha_l}(\nu)$.


$\alpha_0 = 0.099376...,\ N_0 = 5800$
0
−20
−40
−60
−80
−100
0
20
40
60
80
100


$\alpha_0 = 0.099376...,\ N_0 = 5800$
$\alpha_1 = 0.0628273...,\ N_1 = 576$
$\alpha_0 = 0.099376$
$\alpha_1 = 0.0628273...,\ N_1 = 576$
$\alpha_2 = 0.0833472...,\ N_2 = 36$
$\alpha_1 = 0.0628273$
$\alpha_2 = 0.0833472...,\ N_2 = 36$
$\alpha_3 = 0.00199873...,\ N_3 = 3$
$\alpha_2 = 0.0833472$
$\frac{1}{2}$ $\frac{1}{2\alpha_0}+\frac{1}{2}$ $\frac{1}{\alpha_0}+\frac{1}{2}$ $\frac{3}{2\alpha_0}+\frac{1}{2}$ $\frac{2}{\alpha_0}+\frac{1}{2}$
$\frac{1}{2}$ $\frac{1}{2\alpha_1}+\frac{1}{2}$ $\frac{1}{\alpha_1}+\frac{1}{2}$ $\frac{3}{2\alpha_1}+\frac{1}{2}$ $\frac{2}{\alpha_1}+\frac{1}{2}$
$\frac{1}{2}$ $\frac{1}{2\alpha_2}+\frac{1}{2}$ $\frac{1}{\alpha_2}+\frac{1}{2}$ $\frac{3}{2\alpha_2}+\frac{1}{2}$ $\frac{2}{\alpha_2}+\frac{1}{2}$

Question: How fast does the size of the patterns at different scales grow? I.e. how does the product $\alpha_0 \cdots \alpha_{r-1}$ decay to 0 as $r \to \infty$? In particular, how many scales $r(N)$ are there in the picture $\{\mathcal{S}_\alpha(n)\}_{n=1}^{N}$?

In terms of Birkhoff sums:

$$\log(\alpha_0 \cdots \alpha_{r-1}) = -\sum_{j=0}^{r-1} \Big(-\log T^j(\alpha)\Big) = -\mathsf{S}_r(f)(\alpha),$$

where $\mathsf{S}_n(f)(\alpha) = \sum_{j=0}^{n-1} f(T^j(\alpha))$ and $f(\alpha) = -\log(\alpha)$. Notice that $f \in L^1(\mu)$ and $\mu(f) = \int_0^1 f(\alpha)\, \mathrm{d}\alpha = \frac{\pi^2}{4}$.

Therefore the question is about the growth of Birkhoff sums of an integrable function over a system preserving an infinite measure.

Set $\mathcal{P}_\mu := \{$P prob. meas. on $[0,1]$, $P \ll \mu\}$.

Theorem 1 (Weak LLN). For every $f \in L^1(\mu)$, for every probability measure P $\in \mathcal{P}_\mu$ and for every $\varepsilon > 0$ we have

$$\mathsf{P}\left(\left|\frac{\mathsf{S}_n(f)}{\frac{n}{\log n}} - \mu(f)\right| > \varepsilon\right) \to 0 \quad \text{as } n \to \infty.$$

(Open) Questions:

1. How fast $e_n := \frac{\mathsf{S}_n(f)}{\frac{n}{\log n}} - \mu(f) \xrightarrow{\mathsf{P}} 0$?
2. CLT: Does there exist a sequence of constants $d_n \uparrow \infty$ and a nontrivial random variable $\mathcal{V}$ s.t.

$$\frac{\mathsf{S}_n(f) - \mu(f)\frac{n}{\log n}}{d_n} \longrightarrow \mathcal{V} \quad \text{in distribution?}$$

Remark. $2 \Rightarrow 1$, giving $e_n \sim d_n \frac{\log n}{n}$ as $n \to \infty$.

Theorem 1 implies that $\alpha_0 \cdots \alpha_{r-1}$ decays as $e^{-\frac{\pi^2}{4}\frac{r}{\log r}(1+\mathcal{O}(1))}$ in probability as $r \to \infty$.
In particular, the magnitude of the pattern at the l-th scale grows as $e^{\frac{\pi^2}{8}\frac{l}{\log l}(1+\mathcal{O}(1))}$ as $l \to \infty$ and the asymptotic growth of the number of scales in $\{\mathcal{S}_\alpha(n)\}_{n=1}^N$ as $N \to \infty$ is, in probability, $r(N) \sim \frac{4}{\pi^2} \log N \cdot \mathrm{loglog}\, N(1 + \mathcal{O}(1))$

For $r = r(N)$ we get $(\alpha_0 \cdots \alpha_{r-1})^{-\frac{1}{2}} = \mathcal{O}(\sqrt{N})$.
Denote

$$\Delta_l = \frac{(\alpha_0 \cdots \alpha_{r-1})^{-\frac{1}{2}}}{\sqrt{N}} \cdot \alpha_l^{\frac{1}{2}}\, \tilde{\Lambda}_l = \mathcal{O}(1).$$

The iterated (ARF) becomes

$$\begin{aligned}
\frac{\mathcal{S}_\alpha(N)}{\sqrt{N}} \;&=\; e^{\kappa(\alpha,r)\frac{\pi}{4}i}\,\frac{(\alpha_0 \cdots \alpha_{r-1})^{-\frac{1}{2}}}{\sqrt{N}}\,\mathcal{S}_{\alpha_r}^{(\eta_1 \cdots \eta_r)}(N_r) + \\
&\quad + \Delta_{r-1} + \alpha_{r-1}^{\frac{1}{2}}\,\Delta_{r-2} + \ldots + \\
&\quad + (\alpha_{r-j+1} \cdots \alpha_{r-1})^{\frac{1}{2}}\,\Delta_{r-j} + \ldots + \\
&\quad + (\alpha_2 \cdots \alpha_{r-1})^{\frac{1}{2}}\,\Delta_1 + \\
&\quad + (\alpha_1 \cdots \alpha_{r-1})^{\frac{1}{2}}\,\Delta_0.
\end{aligned}$$

$$
\begin{aligned}
\frac{\mathcal{S}_\alpha(N)}{\sqrt{N}} \;=\;& e^{\kappa(\alpha,r)\frac{\pi}{4}i}\,\frac{(\alpha_0\cdots\alpha_{r-1})^{-\frac{1}{2}}}{\sqrt{N}}\,\mathcal{S}_{\alpha_r}^{(\eta_1\cdots\eta_r)}(N_r)+\\
&+\Delta_{r-1}+\alpha_{r-1}^{\frac{1}{2}}\,\Delta_{r-2}+\ldots+\\
&+(\alpha_{r-j+1}\cdots\alpha_{r-1})^{\frac{1}{2}}\,\Delta_{r-j}+\ldots+\\
&+(\alpha_2\cdots\alpha_{r-1})^{\frac{1}{2}}\,\Delta_1+\\
&+(\alpha_1\cdots\alpha_{r-1})^{\frac{1}{2}}\,\Delta_0.
\end{aligned}
$$

The first two terms are $\mathcal{O}(1)$ and the others are of decreasing magnitude, the j-th being of order $(\alpha_{r-j+1}\cdots\alpha_{r-1})^{\frac{1}{2}} \sim e^{-\frac{\pi^2}{8}\frac{j}{\log j}(1+\mathcal{O}(1))}$ in probability as $j\to\infty$.

The formula above can be written for every point on curve $\frac{\mathcal{S}_\alpha(tN)}{\sqrt{N}}$, $0\le t\le 1$. As $N\to\infty$ one would like to show the convergence to a limiting random curve with infinitely many decreasing scales.

Thank you!

Appendix: Birkhoff sums for maps preserving an infinite measure

Let T be a conservative, ergodic, measure preserving transformation (c.e.m.p.t.) on the σ-finite measure space $(X, \mathcal{A}, \mu)$. Let $f \in L^1(\mu)_+$.

If $\mu(X) < \infty$, then by Birkhoff Ergodic Theorem $\frac{\mathsf{S}_n(f)}{n} \to \frac{1}{\mu(X)}\mu(f)$ a.e. as $n \to \infty$.

If $\mu(X) = \infty$, then $\frac{\mathsf{S}_n(f)}{n} \to 0$ a.e. as $n \to \infty$.

By Hopf's Ergodic Theorem there exist measurable functions $a_n(x)$ s.t. $\frac{\mathsf{S}_n(f)(x)}{a_n(x)} \to \mu(f)$ for a.e. $x \in X$ as $n \to \infty$.

Theorem (Aaronson). Suppose $\mu(X) = \infty$. Consider a sequence of constants $a_n > 0$, then

1. either $\liminf_{n\to\infty} \frac{\mathsf{S}_n(f)}{a_n} = 0$ a.e.,

2. or $\exists n_k \uparrow \infty$ s.t. $\frac{\mathsf{S}_{n_k}}{a_{n_k}} \to \infty$ a.e. as $k \to \infty$.

1-dim Maps Preserving an Infinite Measure and Proof of Theorem 1

Let J be a countable set and $\mathcal{I} = \{I_j,\, j \in J\}$ be a collection of disjoint open intervals in $[0,1]$ s.t. $\mathcal{U} := \cup_{I\in\mathcal{I}} I = [0,1]$ mod 0. Let $T : \mathcal{U} \to [0,1]$ s.t. $T|_{I_j} : I_j \to (0,1)$ is onto and $\mathcal{C}^2$ (and $\mathcal{C}^1$-extends to the endpoints). Let v_j denote the inverse of $T|_{\overline{I_j}} : \overline{I_j} \to [0,1]$ and let x_j be the unique T-fixed point in $\overline{I_j}$.

Thaler's assumptions

(T1) $\forall j \in J$, $\forall x \in \overline{I_j}$, $\left|\left(T|_{\overline{I_j}}\right)'(x)\right| \geq 1$, with equality only for $x = x_j$ and $T'(x_j) = 1$.

(T2) The set $J_1 := \{j \in J : T'(x_j) = 1\}$ is finite.

(T3) $\exists M < \infty$ s.t. $\left|\frac{T''(x)}{T'(x)^2}\right| \leq M \;\forall x \in \mathcal{U}$.

(T4) $\forall \varepsilon > 0$, $\exists \delta(\varepsilon) > 1$ s.t. $|T'(x)| \geq \delta(\varepsilon) \;\forall x \in \mathcal{U} \setminus \cup_{j\in J_1} I_j \cap (x_j - \varepsilon, x_j + \varepsilon)$.

(T5) $\exists \varepsilon > 0$ s.t. $\forall j \in J_1$, v_j' increases on $I_j \cap (x_j - \varepsilon)$ and decreases on $I_j \cap (x_j, x_j + \varepsilon)$.

Definition (Rational ergodicity) A c.e.m.p.t. T on $(X, \mathcal{A}, \mu)$ is *rationally ergodic* if $\exists A \in \mathcal{A}$, $0 < \mu(A) < \infty$ satisfying a *Rényi inequality* (RI): $\exists M > 0$ s.t.

$$\int_A (\mathsf{S}_n(\mathbf{1}_A))^2 \mathrm{d}\mu \le M \left(\int_A \mathsf{S}_n(\mathbf{1}_A) \mathrm{d}\mu \right)^2 \quad \forall n \in \mathbb{N}.$$

Theorem B. If T is rationally ergodic, then $\exists$ constants $a_n \uparrow \infty$, unique up to asymptotic equality, s.t. $\forall A \in \mathcal{A}$ satisfying a (RI), $\forall B, C \in \mathcal{A}$, $B, C \subset A$,

$$\frac{1}{a_n} \sum_{k=0}^{n-1} \mu(B \cap T^{-k} C) \to \mu(B)\mu(C) \text{ as } n \to \infty.$$

Definition (Return sequence) The sequence $a_n = a_n(T)$ is called a *return sequence* of T.

Definition (Pointwise dual ergodicity).
A c.e.m.p.t. T on $(X, \mathcal{A}, \mu)$ is *pointwise dual ergodic* if $\exists$ constants a_n s.t. $\frac{1}{a_n} \sum_{k=0}^{n-1} \widehat{T}^k f \to \mu(f)$ a.e. as $n \to \infty$ $\forall f \in L^1(\mu)$.

Definition (Regularly varying functions).
$f : \mathbb{R}_+ \to \mathbb{R}_+$ is *regularly varying at* ∞ with index γ if for every $y > 0$, $\exists \lim_{x \to \infty} \frac{f(xy)}{f(x)} = y^\gamma$. Denote this by $f \in \mathcal{R}_\gamma$. Consider sequences a_n as functions $a(t) = a_{\lfloor t \rfloor}$.

Theorem C (Thaler,Aaronson).
Assume (T1)-(T5). Then:

(a) $\exists$ σ-finite, infinite, T-invariant measure μ a.c. w.r.t. Lebesgue and $\frac{\mathrm{d}\mu}{\mathrm{dLeb}}(x) = h(x) \cdot \prod_{j\in J_1} \frac{x-x_j}{x-v_j(x)}$ for Leb-a.e. x, where $h : [0,1] \to \mathbb{R}_{\geq 0}$ is continuous.

(b) T is conservative, rationally ergodic and exact endomorphism.

(c) T has *minimal wandering rates*, i.e. $\exists L(n) \uparrow \infty$ s.t. $L_A(n) := \mu\left(\cup_{k=0}^{n-1} T^{-k}A\right) \sim L(n)$ as $n \to \infty$ for any measurable A bounded away from $\{x_j : j \in J_1\}$.

(d) $L(n) \sim \sum_{j\in J_1} c_j \sum_{k=1}^{n-1} \left(v_j^k(1) - v_j^k(0)\right)$, where $c_j = h(x_j) \prod_{l\in J_1\setminus\{j\}} \frac{x_j-x_l}{x_j-v_l(x_j)}$.

(e) T is pointwise dual ergodic (with seq. $(a_n)_n$).

(f) $(a_n)_n = (a_n(T))_n$ is a return sequence for T.

(g) If $(L(n)) \in \mathcal{R}_\gamma$ as $n \to \infty$, then

$$a_n \sim \frac{1}{\Gamma(2-\gamma)\,\Gamma(1+\gamma)} \cdot \frac{n}{L(n)}.$$

Definition (Strong distributional convergence). A seq. of measurable functions $\Phi_n : (X, \mathcal{A}, \mu) \to \mathbb{R}$ converges *strongly in distribution* to the random variable $\Phi : X' \to \mathbb{R}$ with distribution p_Φ as $n \to \infty$ if $\mathsf{P} \circ \Phi_n^{-1} \xrightarrow{w} p_\Phi$ as $n \to \infty$ for every $\mathsf{P} \in \mathcal{P}_\mu :=$ {prob. measures on X, a.c. wrt μ}. Write $\Phi_n \xrightarrow{\mathfrak{L}} \Phi$.

Remark C. If Φ is constant, than $\Phi_n \xrightarrow{\mathfrak{L}} \Phi$ iff $\mathsf{P}(\{|\Phi_n - \Phi| > \varepsilon\}) \to 0$ as $n \to \infty$ $\forall \varepsilon > 0$, $\forall \mathsf{P} \in \mathcal{P}_\mu$. Equivalently, Φ_n converges to Φ in measure on subsets of finite measure.

Theorem D (Aaronson). If T is pointwise dual ergodic and $(a_n(T))_n \in \mathcal{R}_\gamma$ as $n \to \infty$, then

$$\frac{\mathsf{S}_n(f)}{a_n(T)} \xrightarrow{\mathfrak{L}} \mu(f) \cdot \mathcal{M}_\gamma,$$

where $\mathcal{M}_\gamma$ has the *Mittag-Leffler distribution of order* γ, which is characterized by its moments: $\mathbb{E}[\mathcal{M}_\gamma^r] = r! \frac{(\Gamma(1+\gamma))^r}{\Gamma(1+r\,\gamma)}$, $r \in \mathbb{N}$.

Remark E. $\mathbb{E}[\mathcal{M}_\gamma]$=1 $\forall \gamma \in (0,1)$ and $\mathcal{M}_1 \equiv 1$.

Proof of Theorem 1. T satisfies (T1)-(T5) with $J = \mathbb{N} \times \{\pm 1\}$, $I_{(k,\xi)} = \text{int}(B(k,\xi))$, $J_1 = \{(1,-1)\}$, $M = 4$ and $v_{(1,-1)}(x) = \frac{1}{2-x}$. By Theorems A and B, $h(x) = \frac{-2}{(x-2)(x+1)}$, $c_{(1,-1)} = h(1) = 1$ and

$$L(n) \sim c_{(1,-1)} \sum_{k=1}^{n-1} \left(1 - \frac{k}{k+1}\right) \sim \log n.$$

Therefore $(L(n))_n \in \mathcal{R}_1$ and $a_n(T) \sim \frac{n}{\log n}$.
By Theorem D and Remarks C and E,

$$\mathsf{P}\left(\left|\frac{\mathsf{S}_n(f)}{\frac{n}{\log n}} - \mu(f)\right| > \varepsilon\right) \to 0 \text{ as } n \to \infty \ \forall \mathsf{P} \in \mathcal{P}_\mu.$$

□