LECTURE 1

This week, we're going to talk about notions of positivity in linear algebra. We'll see that these ideas have important applications in the theory of *non-linear* phenomena.

MOTIVATING PROBLEM. Given a function $f(x_1, \dots, x_n)$ and a *critical point*; that is, a point where

$$(\partial/\partial x_1)f = \ldots = (\partial/\partial x_n)f = 0.$$

We ask: is this point a local minimum for f? A local maximum? Or neither?

In the one-variable case: suppose $f'(x) = 0$. Then x is a local maximum if $f''(x) < 0$, a local minimum if $f''(x) > 0$. If $f''(x) = 0$, it's impossible to tell–an analysis of the *third* derivative is necessary, and that's farther than we can go.

So we see already that "positivity of the second derivative" is the issue.

Let's do an example. Let $f(x, y) = (\sin x + \cos x) \cos y$. We have

$$(\partial/\partial x)f = (\cos x - \sin x) \cos y, (\partial/\partial y)f = -(\sin x + \cos x) \sin y.$$

There's a critical point at $x = \pi/4, y = 0$.

Simplification 1: We can assume that the critical point is at the origin. If the critical point is at $(t_1, \dots, t_n)$, we can replace $f(x_1, \dots, x_n)$ with $f(x_1 - t_1, \dots, x_n - t_n)$.

In our example, we can replace $f(x, y)$ with

$$f(x + \pi/4, y) = (\sin(x + \pi/4) + \cos(x + \pi/4)) \cos y = \sqrt(2) \cos x \cos y$$

which has a critical point at the origin.

Simplification 2: We can replace f with a *quadratic* function! To do this, we need to use the Taylor expansion.

One-variable version: $f(x) = f(0) + f'(0)x + (1/2)f''(0)x^2 + \ldots$ where the $\ldots$ are terms of degree 3 and higher.

Two-variable version:

$$f(x, y) = f(0) + (\partial/\partial x)f(0)x + (\partial/\partial y)f(0)y + (1/2)(\partial^2/\partial x^2)f(0)x^2 + (\partial^2)/(\partial x \partial y)f(0)xy + (1/2)($$

Now as x and y get very close to the origin, the higher-order terms (i.e. the "dots") become negligible compared to the constant, first-order, and second-order terms. In this case,

$$(\partial^2/\partial x^2)f = -\sqrt(2) \cos x \cos y, (\partial^2/\partial x \partial y)f = \sqrt(2) \sin x \sin y, (\partial^2/\partial y^2)f = -\sqrt(2) \cos x \cos y$$

and substituting in 0 we find

$$f(x, y) = \sqrt{2} - (\sqrt{2}/2)x^2 - (\sqrt{2}/2)y^2 + \dots$$

Ask yourself: Is the origin a local minimum for the function xy? A maximum?

For a general function $g(x, y)$ we'll have

$$g(x, y) = g(0) + 0 + ax^2 + 2bxy + cy^2 + \dots$$

Why the $2b$? We'll see..

The linear term is 0 because 0 is a critical point for g. And the constant term $g(0)$ is irrelevant from the point of view of testing for a local minimum or maximum.

Definition. A *quadratic form* q in n variables $x_1, \dots, x_n$ is a degree-2 polynomial in $x_1, \dots, x_n$ with no linear or constant part. (We also say q is a *homogeneous* polynomial of degree 2.)

QUESTION (Two-variable version.) Write

$$q(x, y) = ax^2 + 2bxy + cy^2$$

What is the nature of the critical point at the origin?

(Be aware that this may have been covered in Math 203.)

Some examples: $2x^2 + 5x^2, xy, x^2 - y^2, x^2 + 4xy + y^2$. We'll discuss these together.

What does any of this have to do with linear algebra? Well, I like to write

$$q(x, y) = [xy] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}^T A \vec{x}$$

where A is a certain 2×2 symmetric matrix.

In fact, *any* quadratic form can be written as $\vec{x}^T A \vec{x}$ for some symmetric A. For instance, write $x_1^2 + x_1x_2 + x_1x_3 + x_3^2$ as

$$\vec{x}^T \begin{bmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \vec{x}.$$

Groupwork. In each of the four 2×2 cases above, write down the matrix A and compute its eigenvalues. What, if anything, does this computation have to do with the behavior of the quadratic form near the origin?

OK, so after that we should have some idea that positivity of the *eigenvalues* is the key thing.

So let's make a definition.

Definition. We say a symmetric matrix A is *positive definite* if its eigenvalues are all positive. We say a quadratic form

$$q(\vec{x}) = \vec{x}^T A \vec{x}$$

is positive definite if A is positive definite. We say A (resp. q) is *negative definite* if $-A$ is positive definite.

Now we should be pretty ready to believe the following theorem:

Theorem. Let q be a quadratic form in n variables. The critical point at the origin is a local minimum of q if and only if q is positive definite. The origin is a local maximum if and only if q is negative definite.

For instance, the form xy has neither a local maximum nor a local minimum, but something called a "saddle point." Draw this, if I dare.

Let f be a function with a critical point at the origin. Write the Taylor expansion of f at the origin:

$$f(\vec{x}) = f(\vec{0}) + (1/2) \sum_{i,j} (\partial^2/\partial x_i \partial x_j) x_i x_j + \dots$$

The quadratic form $(1/2) \sum_{i,j} (\partial^2/\partial x_i \partial x_j) x_i x_j$ is called the *Hessian* of f; it is this quadratic form that controls the behavior at the critical point.

LECTURE 2

1. Main theorem: a positive definite quadratic form has a minimum at 0.

Now let's prove the theorem above: we'll in fact prove something a little more general.

Theorem. Let A be a (real) symmetric matrix. The following are equivalent:

1. A is positive definite;
2. $\vec{x}^T A \vec{x} > 0$ for all $\vec{x} \neq \vec{0}$;
3. $A = R^T R$ for some real-valued matrix R with linearly independent columns (not necessarily square);
4. All the "upper left submatrices" of A have positive determinant.

Proof. First, suppose **2**. Then **1** follows; because if $\vec{v}$ is an eigenvector with eigenvector λ, we have

$$0 < \vec{v}^T A\vec{v} = \lambda\vec{v}^T\vec{v} = \lambda\|\vec{v}\|^2$$

and, since $\|\vec{v}\|^2$ is positive, this makes λ positive. Note that I've used the spectral theorem–otherwise I wouldn't know that $\vec{v}$ was real, and the whole argument would be ruined, ruined, ruined!

To see that **1** implies **2**, take an arbitrary $\vec{x}$ and write

$$\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n.$$

Now we're going to compute $\vec{x}^T A\vec{x}$. But note that the $\vec{v}_i$ are mutually orthogonal; again by the wonderful spectral theorem! So we can compute

$$\vec{x}^T A\vec{x} = [c_1\vec{v}_1^T + \ldots + c_n\vec{v}_n^T][c_1\lambda_1\vec{v}_1 + \ldots + c_n\lambda_n\vec{v}_n],$$

And now you note that all the $\vec{v}_i^T\vec{v}_j^T$ vanish when $i \neq j$, leaving you with

$$\vec{x}^T A\vec{x} = \lambda_1\vec{v}_1^T\vec{v}_1 + \ldots + \lambda_n\vec{v}_n^T\vec{v}_n = \lambda_1\|\vec{v}_1\|^2 + \ldots + \lambda_n\|\vec{v}_n\|^2$$

which is positive since all the λ_n are positive.

Now, show **3** implies **1**. This is easiest of all. Because

$$\vec{x}^T R^T R\vec{x} = \|R\vec{x}\|^2$$

Now $\|R\vec{x}\|^2 \geq 0$, and it *equals* 0 only if $R\vec{x} = 0$. Since R has linearly independent columns, it has no nullspace (remember the Rank Theorem....?), so we find that $\vec{x}^T A\vec{x} = 0$ only if $\vec{x} = 0$, just as we desired.

Now to show **1** implies **3.** Well, once *again* dragging out the by-now exhausted and drooling spectral theorem, we can write

$$A = Q\Lambda Q^{-1} = Q\Lambda Q^T$$

(note that $Q^{-1} = Q^T$ because Q is orthogonal.) Now Λ is a diagonal matrix

$$\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

and all the entries are positive; this means we can define

$$\sqrt{\Lambda} = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \sqrt{\lambda_n} \end{bmatrix}$$

and note that $\sqrt{\Lambda} = \sqrt{(\Lambda)^T}$. So we can write

$$Q\Lambda Q^T = Q\sqrt{\Lambda}\sqrt{\Lambda}^T Q^T$$

and now setting

$$R = (Q\sqrt{\Lambda})^T = \sqrt{\Lambda}^T Q^T$$

gives the desired result.

As for **4**, it's a very nice computational tool, and Strang gives a good proof–I'll leave it to him.

Remark. We've shown that a positive definite quadratic form q has a *global* minimum at $\vec{0}$. Likewise, a negative definite quadratic form has a globla maximum at $\vec{0}$. If the form is neither positive definite nor negative definite, it has neither a maximum nor a minimum at 0. In fact, I think I'll ask you to do this on the homework.

2. Level sets.

Now I want to talk about the meaning and use of the equation $A = R^T R$ written above. Why do we care that we can write A in that form? Well, for one thing, it makes it rather clear how to understand the solutions to quadratic equations in many variables.

Q: Describe the solution set of

$$q(\vec{x}) = x_1^2 + (0.1)x_1x_2 + 2x_2^2 = 1.$$

Ask: ask them to think about it. Yeah, it's probably an ellipse of some kind. But how is it oriented? Which way do its axes point? Not at all easy to see. But from our new viewpoint it's a piece of cake.

We can write

$$q(\vec{x}) = \vec{x}^T \begin{bmatrix} 1 & 0.05 \\ 0.05 & 2 \end{bmatrix} \vec{x}$$

Now we can compute the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0.05 \\ 0.05 & 2 \end{bmatrix}$ to be

$$\lambda_1 = .9975, \vec{v}_1 = \begin{bmatrix} 0.999 \\ -0.05 \end{bmatrix}, \lambda_2 = 2.002, \vec{v}_2 = \begin{bmatrix} 0.05 \\ 0.999 \end{bmatrix}$$

Indeed, these are orthogonal. Now we can write

$$\sqrt{\Lambda} = \begin{bmatrix} .999 & 0 \\ 0 & 1.415 \end{bmatrix}$$

and

$$R = Q\sqrt{\Lambda} = \begin{bmatrix} .998 & -.05 \\ .0705 & 1.413 \end{bmatrix}$$

So by our argument above, $R^T R = A$. So the heck what? Well. We're trying to solve

$$\vec{x}^T A\vec{x} = 1$$

which is the same as

$$\vec{x}^T R^T R\vec{x} = \|R\vec{x}\|^2 = 1.$$

So we're really just trying to find all $\vec{x}$ such that $R\vec{x}$ lies on the unit circle! In other words, we're looking at R^{-1} times the unit circle.

Does this really tell us how to see the axes? Well, maybe it's better to go back a step and say: what are the solutions to

$$\vec{y}^T \Lambda\vec{y} = 1?$$

Stop and let them think about this. After a while, rejoin. It's an ellipse with a major axis along the y-axis and a minor axis along the z-axis. Now the solutions to the original question can be written as

$$\vec{x}^T Q\Lambda Q^T\vec{x} = 1$$

which is to say, $Q\vec{x}$ is on the ellipse described above. Since Q is an orthogonal matrix, that is to say that our solution set differs from the above ellipse by some *rigid* motion of the plane. In other words, if $\vec{y} = \vec{e}_1$ was a minor axis before, $\vec{x} = Q\vec{e}_1 = \vec{v}_1$ is a minor axis now. So we see that *the axes of the ellipse are the eigenvectors of A.*

So really I guess we didn't need to factor A into $R^T R$ to do this; we only needed the spectral theorem. That, and the fact that because A is positive definite, all the entries in Λ are positive.

Comment that we can use the same approach to sketch the level set $f(\vec{x}) = C$ of a function in several variables, at least near a critical point; just use the Taylor series to write

$$f(\vec{x}) = f(\vec{0}) + 0 + \vec{x}^T A\vec{x}$$

where A is the Hessian matrix, and then solve the equation

$$\vec{x}^T A\vec{x} = C - f(\vec{0}).$$

You'll do this on the homework.

LECTURE 3

With whatever time we have left, talk about a few additional topics.

0. Positive semidefinite

Quickly say what this means, cite Strang 6E.

1. Sums of squares

We've observed over the past few days that sometimes we can just look at form and say, "squares are positive", and know instantly that the form is positive definite–e.g. $x^2 + y^2$.

In a sense, this argument is really the only arrow in our quiver in the subject. For instance, one way to verify that $x^2+2xy+2y^2$ is positive definite is to compute eigenvalues, but another way is to write it as $(x + y)^2 + y^2$.

Fact. Suppose q is a positive semidefinite quadratic form. Then q can be written as a sum of squares of linear forms.

(A *linear form* is a degree 1 polynomial with no constant term.)

We prove this by writing $q(\vec{x}) = \vec{x}^T A\vec{x}$, then writing $A = R^T R$, so that we end up with

$$q(\vec{x}) = ||R\vec{x}||^2$$

and the coordinates of $R\vec{x}$ are linear forms. Do an example of this. For instance, the above $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ with $R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Now warn that it may be a real pain to calculate R.

2. Law of inertia.

I just want to add one to my list of equivalent definitions of positive definiteness.

Let me first observe that you can always put a symmetric matrix in row-reduced form without doing any row exchanges. So let's take it as given that when we speak of row-reduced form for a symmetric matrix, no row exchanges are involved. So

Theorem (Law of inertia) Let A be a symmetric matrix, C an arbitrary matrix. The number of positive eigenvalues of A is the same as the number of positive eigenvalues of $C^T AC$. (Resp. number of negative eigenvalues, zero eigenvalues.)

Give the "non-proof" in Strang, via the homotopy argument. Explain why there's a crucial step missing.

Application Let A be a symmetric matrix. The number of positive eigenvalues of A is the same as the number of positive pivots of A. (Resp.

number of negative eigenvalues, zero eigenvalues.) In particular, A is positive definite if and only if all the pivots of A are positive.

Follows from writing A as LDL^T. Comment that to have this decomposition requires us to go all the way back to Strang's Thm 1I, uniqueness of the LDU decomposition.

This is the most effective way of computing positive definiteness in practice (although not the most theoretically interesting criterion.)

3. Unequal weights.

I really like this example Strang does. If I have time, I'll talk about it. If not, I'll leave it to Strang, who does a good job.

Draw the two coupled springs, whose positions are v and w. Get

$$m_v v'' = -5v + w m_w w'' = w + -4v$$

or

$$\vec{v}'' = \begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix} \vec{v}$$

So we're interested in finding the natural frequencies of oscillation. We expect to find solutions of the form $\vec{v} = e^{i\omega t}\vec{x}$ for physical reasons, and plugging this in we find

$$\begin{bmatrix} 5 & 1 \\ 1 & 4 \end{bmatrix} \vec{x} = \lambda \begin{bmatrix} m_v & 0 \\ 0 & m_w \end{bmatrix} x$$

where $\lambda = (i\omega)^2$ is a (presumably negative real) number. So to solve

$$(A - \lambda M)\vec{x} = 0$$

means finding values of λ where

$$A - \lambda M$$

is singular. It's much like the eigenvalue problem, with M replacing I.