Section 7.6: Special Techniques

- Remember, office hours are now 2:30 Tues and 10:00 Thurs.
- On the notation tan^{-1} for $arctan$.

0.1 Trigonometric identities

Sometimes we can use our good old trig identities in order to help us compute integrals, by writing the integrand in a different form. Remark that section 2.6 of the book is a refresher on trigonometry.

Recall the main guys:

$$\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\tan^2 x + 1 &= \sec^2 x \\
1 + \cot^2 x &= csc^2 x \\
\sin(x+y) &= \sin x \cos y + \cos x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
\sin(-x) &= -\sin x \\
\cos(-x) &= \cos x
\end{aligned}$$

Note how you can just remember the first one and the next two follow. Lots of other identities follow from these, in particular this one. Note

$$\begin{aligned}
\cos(x-y) &= \cos x \cos y + \sin x \sin y \\
\cos(x+y) &= \cos x \cos y - \sin x \sin y \\
(1/2)(\cos(x+y) + \cos(x-y)) &= \cos x \cos y \\
(1/2)(\cos(x+y) - \cos(x-y)) &= \sin x \sin y
\end{aligned}$$

This "turns a product into a sum," so we like it.

Ex: Compute

$$\int \cos x \cos(x + \pi/6) dx.$$

Now we could certainly expand the latter cosine of a sum, but faster to rewrite the integral as

$$
\begin{aligned}
\int [(1/2)\cos(2x+\pi/6)+\cos(-pi/6)]dx & \\
&= (1/2)\int (1/2)\cos u du + (1/2)\int (\sqrt{3}/2) x dx \\
&= (1/4)\sin(2x+\pi/6) + (\sqrt{3}/4)x.
\end{aligned}
$$

Note that by the same method we can attack guys of the form $\sin x \cos y$, see book.

Note that specializing to $x = y$ gives

$$\cos^2 x = (1/2)(\cos 0 + \cos 2x)$$

and

$$\cos^2 x = (1 + \cos 2x)/2.$$

Likewise,

$$\sin^2 x = (1 - \cos 2x)/2.$$

Ex: Compute

$$\int_0^{\pi/2} \sin^4 x dx.$$

Ask for suggestions. Comment that we could do this by parts; indeed we already have. How can we attack $\sin^4$? Can we write this in terms of $\sin^2$? OK, write

$$
\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 \\
&= [(1-\cos 2x)/2]^2 \\
&= (1/4)(1 - 2\cos 2x + \cos^2 2x) \\
&= (1/4)(1 - 2\cos 2x + (1/2)(1+\cos 4x)) \\
&= (1/4)(3/2 - 2\cos 2x + (1/2)\cos 4x).
\end{aligned}
$$

So the original integral becomes

$$\int_0^\pi 3/8 - (1/2)\cos 2x + (1/3)\cos 4x dx$$

$$\begin{aligned} &= (3x/8) - (1/4)\sin 2x + (1/12)\sin 4x|_0^{\pi/2} \\ &= 3\pi/16. \end{aligned}$$

Indeed, this equals $(1/2)(3/4)(\pi/2)$ as worked out in the homework, and this is easier than integrating by parts three times. Note that this example is not unlike question 35 on this week's homework.

As for the other identities, here is a representative example. **Ex:**Compute

$$\int_{\sqrt{2}}^2 \sqrt{x^2-1}dx.$$

Note that this is a hard example and we're going to use lots of techniques!

Now one approach would be to substitute $u = \sqrt{x^2-1}$. Whence we'd have $x = \sqrt{u^2+1}$, and dx is $udu/\sqrt{u^2+1}$, so we don't really make progress.

Instead, we observe that

$$\sec^2 x - 1 = \tan^2 x,$$

which leads us to consider the substitution $x = \sec\theta$, so that $dx = \sec\theta\tan\theta$ and we are left with

$$\int \sec\theta\tan\theta\tan\theta d\theta = \int \sec\theta\tan^2\theta d\theta.$$

This looks a little nicer, but still hard. How to attack? Solicit questions.

Yes, let's try integration by parts. With $u = \tan\theta$ and $dv = \sec\theta\tan\theta d\theta$. So then we get $du = \sec^2\theta d\theta$ and $v = \sec\theta$, and we have

$$\begin{aligned} \int \sec\theta\tan^2\theta d\theta &= \sec\theta\tan\theta - \int \sec^3\theta d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta(\tan^2\theta + 1)d\theta \\ &= \sec\theta\tan\theta - \int \sec\theta d\theta - \int \sec\theta\tan^2\theta d\theta. \end{aligned}$$

We can look up the integral of the secant in our tables (or in the example in the book) and conclude:

$$\begin{aligned} \int \sec\theta\tan^2\theta d\theta &= (1/2)\sec\theta\tan\theta - \log|\sec\theta + \tan\theta| \\ &= (1/2)x\sqrt{x^2-1} - \log|x + \sqrt{x^2-1}|. \end{aligned}$$

Whew! Well, what is the moral? How could you have been expected to think of all this?

Moral: If it is possible to use a trigonometric identity, try using it. Question: what if I had $\sqrt{1-x^2}$? What substitution might I think of?

Moral 2: Persevere.

Warning: I never went back and computed the definite integral in this case. But NOTICE YOU WELL that I stipulated that x was always positive, so can choose θ always to lie in the first quadrant–if I couldn't do that, there'd be real problems with saying $\tan\theta = \sqrt{1-x^2}$. It just wouldn't necessarily be true.

"A great truth is a truth whose opposite is also a great truth," Niels Bohr.

0.2 Expressions of the form $\sqrt[n]{ax+b}$

When we have an expression like that in an integrand, what to substitute? Consider

$$\int x\sqrt{3x+1}dx.$$

Our instinct might say "we see the composition of two functions, so substitute the interior function," which is to say $u = 3x+1$. And that would work. But we recall we also used the substitution $u = \sqrt{3x+1}$ and that worked too.

Compare on the other hand

$$\int \frac{\sqrt{3x+1}}{x+1}dx..$$

Now suppose we try $u = 3x+1$. Then as before $x = (u-1)/3$ and $dx = (1/3)du$, so we get

$$\int \frac{\sqrt{u}}{u+2}du$$

which we don't know how to do. Instead, use $v = \sqrt{3x+1}$. Then $x = (v^2-1)/3$ and $dx = 2vdv/3$ and we get

$$\int \frac{v}{(v^2-1)/3+1}(1/3)2vdv = \int \frac{2v^2}{v^2+2}dv$$

which we now know how to do by partial fractions.

Moral: In this case, substituting $u = \sqrt[n]{ax+b}$ is a little wiser.

1 Section 7.7: What to do in the face of an integral.

If there's time, talk about Fourier analysis, as follows.

Consider

$$\int_0^{2\pi} \cos(mx)\cos(nx)dx.$$

Now remember that by our technique of last time, we can write this as

$$\int_0^{2\pi} [\cos(m+n)x + \cos(m-n)x]dx = \frac{1}{m+n}\sin(m+n)x + \frac{1}{m-n}\sin(m-n)x|_0^{2\pi}$$

and this is going to be zero unless $m = n$, in which case it is 2π.

Why do we care? Draw a picture of a squiggly, staticy wave, say, this is going to be some combination of cosine waves. The ones we want, and the high-frequency, low-amplitude ones which come from static. Say we're the computer inside your car radio and we want to remove the static. How do we do it? We take this function $f(x)$ defining the wave, where now

$$f(x) = a_1 cos v_1 x + a_2 cos v_2 x + \ldots + a_{100} cos v_{100} x + \ldots$$

and we want to get rid of the "bad stuff" at the top. We could do that if we knew the a_i, but we don't, we just know the shape of the function.

But observe that

$$\int f(x) cos ix dx = 2\pi a_i$$

So we can now compute the contribution of the static, and eliminate it!

So today I want to talk about tactics in the face of an integral. Let's sum up the tactics we know so far, and describe when we use them. Also solicit comments on use. Keep track of this on the board and try to get as much as possible from the class, not from me.

- **Substitution.** Use when we have something in the integrand that looks like $f(g(x))$. Substitute $u = g(x)$. (But remember the thing about roots of linear functions.) Comment: *Linear substitutions are the safest ones.* Because then du is just a multiple of dx, and we don't get added complication from that angle.

- **Integration by parts.** Use when the integrand looks like a product of two disparate things. Comment: Choose dv to be something you can integrate. Comment: Choose u to be something that gets “simpler” or at least no worse when we take its derivative. Comment: This is a good one to try when you have no idea. Comment: Persevere. The first choice of u and v you make might not be the most useful one.

- **Partial fractions.** Use when integrating a rational function.

- **Trigonometric substitution.** Use when you see an expression that can be simplified via trig identity, like $\sqrt{4-x^2}$.

Now split them into groups, say, come up with the hardest integral you can, which they'll have to use a technique to do. See if you can “conceal” the technique! Take 7 minutes to do this, then switch groups and give $7-10$ more minutes to think about each other's problems while walking around.

y