LECTURE NOTES AT UTAH GEOMETRY SUMMER SCHOOL BY S.Y. ALICE CHANG

Introduction.

This is the set of lecture notes for a graduate course which the author gave at the Summer Geometry Institute at Park City, Utah in the summer of 1992. The main theme of the notes is the study of extremals of Sobolov inequalies with applications to some non-linear partial differential equations arising from problems in conformal geometry. Some effort has been made to make the notes self-contained. However, the intention is to keep the material at basic and introductory level, thus many recent research articles in the subject are not mentioned.

The notes is organized as follows: In Chapter 1, we cover some background material concerning the notion of eigenvalue, the heat kernel and Weyl's asymptotic formula. In Chapter 2, we discuss the trace of the heat kernel, the concept of log-determinant of Laplacian operator and the Ray-Singer/Polyakov formula of zeta function determinant on compact surfaces. In Chapter 3 , we discuss a limiting form of Sobolev inequality called the Trudinger-Moser-Onofri inequality, we also mention as an application of such inequality the isospectral problem on compact surfaces. In Chapter 4, we establish the existence of extremals for Moser's inequality. In Chapter 5, we discuss the higher dimensional analogue of Moser's inequality and application to the extremal and compactness problem for zeta-function determinant of 4-manifolds. In Chapter 6, we return to the "isospectral" problem on three dimensional manifolds, we discuss the relation of the problem to the "Yamabe " problem, and the role played by the first eigenvalue in the spectral compactness problem. In the last chapter Chapter 7, we discuss the role of Trudinger-Moser type inequality to the problem of " prescribing " curvature equation in conformal feometry.

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Chapter 1

Some Background Material

§1. Preliminaries.

Let M be a compact Riemannian manifold with boundary ∂M (∂M may be empty). In terms of local coordinates $(x^1, \ldots, x^n)$ the metric can be expressed as $ds^2 = \sum g_{ij} dx^i dx^j$ and the Laplace operator is defined by

$$\Delta = \frac{1}{\sqrt{g}} \sum \frac{\partial}{\partial x^i} (\sqrt{g} g^{ij} \frac{\partial}{\partial x^j})$$

where $(g^{ij}) = (g_{ij})^{-1}$, $g = \det(g_{ij})$.

Given $f \in C^\infty(M)$, define a Hilbert space norm as

$$\{\int_M | f |^2 \, dx + \int_M | \nabla f |^2 \, dx\}^{\frac{1}{2}},$$

and denote by $W^{1,2}(M)$ the Sobolev space obtained by taking the closure of $C^\infty(M)$ in the above norm, and by $W_0^{1,2}(M)$ the closure of $C_0^\infty(M)$ in the above norm.

(a). If ∂M is empty, then Δ is a self-adjoint operator on $W^{1,2}(M)$. According to the spectral theory of self-adjoint operators, Δ has discrete eigenvalues,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$$

with corresponding eigenfunctions ϕ_i satisfying

$$-\Delta \phi_i = \lambda_i \phi_i$$

where $\phi_i \in C^\infty(M) \bigcap W^{1,2}(M)$ can be chosen to form an orthonormal basis of $W^{1,2}(M)$ (Parseval Theorem).

(b). When ∂M is not empty, we should specify some boundary conditions so that Δ is self-adjoint. There are two standard conditions:

(1). Dirichlet boundary conditions, in which case $\mathrm{Dom}(\Delta) = W_0^{1,2}(M)$, and the eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \to \infty$$

and the corresponding eigenfunctions satisfy

$$-\Delta\phi_i = \lambda_i\phi_i$$

$$\phi_i|_{\partial M} = 0 \quad \phi_i \in C^\infty(M).$$

(2) Neumann boundary conditions, in which case $\mathrm{Dom}(\Delta) = W^{1,2}(M)$, and the eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty$$

and the corresponding eigenfunctions satisfy

$$-\Delta\phi_i = \lambda_i\phi_i$$

$$\frac{\partial\phi_i}{\partial\nu}|_{\partial M} = 0, \quad \phi_i \in C^\infty(M).$$

Some Basic Examples

(1) Let $M = [0,a]$. Then the eigenvalues are $\lambda_i = \frac{i^2\pi^2}{a^2}, i = 1,2,\ldots,$ and the corresponding eigenfunctions are $\sin\frac{i\pi x}{a}$ for the Dirichlet boundary conditions; and the eigenvalues are $\lambda_i = \frac{i^2\pi^2}{a^2}, i = 0,1,\ldots$, and the corresponding eigenfunctions are $\cos\frac{i\pi x}{a}$ for the Neumann boundary conditions.

(2) If M is a domain in R^n, then in Euclidean and polar coordinates,

$$\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} = r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial}{\partial r}) + \frac{1}{r^2}\Delta_{S^{n-1}}.$$

In the special case when $M = D$ the unit disk in R^2, using separation of variables , we consider $u(r,\theta) = f(r)h(\theta)$. Then $\Delta u + \lambda u = 0$ is equivalent to the following equation:

$$\frac{r^2(f''(r) + \frac{1}{r}f'(r) + \lambda f(r))}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = c = constant$$

Since $h(\theta)$ is periodic in θ with period 2π, $c = n^2$ and $h = a\cos n\theta + b\sin n\theta$. Set $f(r) = y$; we have

$$(*) \qquad r^2y'' + ry' + (r^2\lambda - n^2)y = 0$$

To find λ we solve $(*)$ with y continuous at the origin and $y(1) = 0$. Let $r\sqrt{\lambda} = \rho$ $(\lambda \neq 0)$.We set $\lambda = k^2$, then $rk = \rho$ and

$$(**) \qquad \frac{d^2y}{d\rho^2} + \frac{1}{\rho}\frac{dy}{d\rho} + (1 - \frac{n^2}{\rho^2})y = 0$$

Solutions of $(**)$ are called Bessel functions $J(\rho)$. Actually all eigenfunctions are of the form $J(\sqrt{\lambda}r)(a\cos n\theta + b\sin n\theta)$.

Remark. It is a classical result of Faber-Krahn that the ball realizes the smallest λ_1 among domains (in R^n) of the same volume. A recent result of Ashbaugh-Berguria [A-B], using special properties of the Bessel functions, proved $\frac{\lambda_2}{\lambda_1}$ (for the Dirichlet problem) is maximal if and only if the domain Ω is a ball.

(3) Let $S^n(r) = \{x \in R^{n+1}, |x| = r\}$, and $S^n = S^n(1)$. Given $x \in R^{n+1}$ write $x = r\xi$, for $r \in [0, \infty)$ and $\xi \in S^n$. It is simple to show that

$$\Delta_{R^{n+1}} F = r^{-n} \partial_r (r^n \partial_r F) + \Delta_{S^n(r)} (F|_{S^n(r)}).$$

See [C, p 34], or [S-W, p 137].

If $F(x) = r^k g(\xi)$ (F is homogeneous of degree k), then

$$\Delta_{R^{n+1}} F = r^{k-2} (\Delta_{S^n} g + k(k+n-1)g)$$

Thus F is harmonic on R^{n+1} if and only if g is an eigenfunction on S^n with eigenvalues $k(k+n-1)$. It turns out that the space of homogeneous harmonic polynomials of degree k when restricted to S^n is the eigenspace cooresponding to the eigenvalue $k(k+n-1)$ of S^n with dimension $C_k^{n+k} - C_{k-2}^{n+k-2}$.

Corollary. *$\lambda_1(S^n) = n$ with eigenfunctions*

$$\{x^1|_{S^n}, x^2|_{S^n}, \ldots, x^{n+1}|_{S^n}\}.$$

§2 Rayleigh Quotients .

We define the space H in the following way:

(1) If $\partial M = \emptyset, then H = \{f \in W^{1,2}(M) | \int_M f dx = 0\}$.

(2) If $\partial M \neq \emptyset$, then for the Dirichlet problem take $H = W_0^{1,2}(M)$, and for the Neumann problem take H as in (1).

For

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$$

we have

$$\lambda_1 = \inf_{f \in H} \frac{\int_M |\nabla f|^2 dx}{\int_M f^2 dx}$$

$$\lambda_i = \inf\{\frac{\int_M |\nabla f|^2 dx}{\int_M f^2 dx} | f \in H, \int_M f \phi_j dx = 0, \text{for } j = 1, 2, \ldots, i-1\}.$$

In particular, λ_1 is the smallest value c such that ;

$$c \int_M f^2 dx \le \int_M |\bigtriangledown f|^2 dx, \text{for any} f \in H,$$

which is called the Poincare inequality [C, p.16].

In general, one likes to estimate λ_1 from below by simple geometric quantities of the manifold, such as the isoperimetric constant, diameter of the manifold, volume of the manifold, and so on. The reader is refered to [C, Chapter 3] for some examples in this direction.

Examples.

(1) For the dumb-bell region, λ_1 is very small when the neck is either thin or long.

(2) (Hersch Theorem) for (S^2, g) with any metric g with Area$(M) = 4\pi$ we have $\lambda_1(g) \le 2$ and $\lambda_1(g) = 2$ if and only if g is isometric to the standard metric on S^2.

Exercise. If ϕ is a conformal map between compact surfaces (M, g_1) and (M, g_2), then

$$\int_M < \nabla_1 f, \nabla_1 g > dv_1 = \int_M < \nabla_2 (f \circ \phi^{-1}), \nabla_2 (g \circ \phi^{-1}) > dv_2.$$

Exercise. Verify the following:

$$(*) \qquad \sum_{k=1}^N \frac{1}{\lambda_k} = \sup \sum_{k=1}^N \frac{\int_M u_k^2 dv}{\int_M |\bigtriangledown u_k|^2 dv}$$

where the sup is taken over functions $u_1, u_2 \cdots, u_k$ satisfying the conditions $\int_M u_k dv = 0$ and $\int_M < \nabla u_k, \nabla u_l > dv = 0$ if $l \neq k$. Equality holds if and only if u_k is the $k-$th eigenfunction.

<u>Proof of the Hersch Theorem</u> We take the standard coordinate functions x_1, x_2, x_3 as the test functions in (*). However, to meet the restrictions it will be necessary to find a conformal transformation ϕ of S^2 so that $\int_{S^2} x_k \circ \phi dv = 0$ for $k = 1, 2, 3$. This can be done by a topological argument. The remaining orthogonality condition follows from conformal invariance of the Dirichlet product. Thus we find, using $\int_{S^2} |\bigtriangledown x_k|^2 dv = \frac{8\pi}{3}$ for each k,

$$\frac{3}{\lambda_1} \ge \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} \ge \frac{3}{8\pi} \int_{S^2} (\sum_{k=1}^3 x_k^2) dv = \frac{3}{2}.$$

Hence $\lambda_1 \le 2$. On the other hand, if $\lambda_1 = 2$, equality must hold everywhere in the formula above. Hence x_1, x_2, x_3 must be the eigenfunctions of the metric. This means that the metric is the standard sphere.

Remark. N.Korevaar has recently proved $\lambda_k \leq Ck$ on compact surfaces, and also an extension of this result to conformal metrics to higher dimensional manifolds.

§3 Weyl's Asymptotic Formula.

Let M be a compact Riemannian manifold (or bounded domain in R^n). Let's denote $N(\lambda) = \#$ of eigenvalues $\leq \lambda$, where we count multiplicity. Then

$$N(\lambda)J \sim \omega_n \mathrm{Vol}(M) \frac{\lambda^{\frac{n}{2}}}{(2\pi)^n}$$

as $\lambda \to \infty$, where ω_n is the volume of the unit ball in R^n. If $\lambda = \lambda_k$, then we get

$$(\lambda_k)^{\frac{n}{2}} \sim \frac{(2\pi)^n k}{\omega_n \mathrm{Vol}(M)}$$

Remarks.

(1) In the case M is a bounded smooth domain, Weyl's formula can be derived by using the max-min principle (Rayleigh quotient). [C, p 32-33].

(2) Polya conjecture: $(\lambda_k)^{\frac{n}{2}} \geq \frac{(2\pi)^n k}{\omega_n \mathrm{Vol}(M)}$ and $(\mu_k)^{\frac{n}{2}} \leq \frac{(2\pi)^n k}{\omega_n \mathrm{Vol}(M)}$ for plane domains. Polya proved his conjecture for "plane-tiling" domains in 1961 [P].

(3) The formula means that

$$\lim_{n \to \infty} \frac{N(\lambda)}{\lambda^{\frac{n}{2}}} = \frac{\omega_n}{(2\pi)^n} \mathrm{Vol}(M)$$

i.e., Vol(M) is determined by the spectral counting function $N(\lambda)$. A sharper study of the asymptotic distribution of the eigenvalues is through the study of the trace of the heat kernel.

Let's write

$$Z(t) = \sum_i e^{-\lambda_i t} = \mathrm{Trace}\, e^{-\Delta t}.$$

Theorem.([MP, MS]) *When $t \to 0^+$, $Z(t) \to (4\pi t)^{-\frac{n}{2}}$ Vol(M). Actually,*

$$Z(t) \sim (4\pi t)^{-\frac{n}{2}} (a_0 + a_{\frac{1}{2}} t^{\frac{1}{2}} + a_1 t + \cdots)$$

as $t \to 0^+$, where $a_0 =$ Vol(M) , and a_i are given by the integrals of the local invariants of g.

We will return to this theorem in Chapter 2.

If we write

$$Z(t) = \int_0^\infty e^{-t\lambda} dN(\lambda)$$

then we see that Weyl's formula implies $a_0 = \mathrm{Vol}(M)$. It turns out that $a_0 = \mathrm{Vol}(M)$ also implies Weyl's asymptotic formula through an application of Karamata's Tauberian Theorem.

Remark. One cannot mention this subject without mentioning the problem of Kac, "Can one hear the shape of a drum?" [K]. Let Ω be a domain in R^2 with smooth boundary. Is Ω determined by its spectrum $\lambda_1, \lambda_2, \cdots,$? i.e., can we hear Ω (a membrane of a drum) if we had perfect pitch? Weyl's formula shows that the area of Ω can be heard. Kac [K] proved that if Ω is convex, then $L = l(\Omega)$ can be heard. In fact, L appears in the coefficient $a_{\frac{1}{2}}$ in the asymptotic formula of $Z(t)$:

$$Z(t) = \sum_{i=1}^\infty e^{-\lambda_i t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4} \frac{1}{(2\pi t)^{\frac{1}{2}}} + \cdots$$

for $t \to 0$.

Since a disk is the only region with $L^2 = 4\pi|\Omega|$, we have

Corollary. *Circular drums can be heard.*

Remark. Kac's originally problem was answered last year in the negative by C. Gordon, D. Webb and S. Wolpert [G-W-W] based on a method of Sunada [S]. The main statement is that there is a pair of non-congruent simply connected domains in the Euclidean space which have the same spectrum for both the Dirichlet and Neumann problems .

References

The basic reference for this chapter is [C] I. Chavel, Eigenvalues in Riemannian Geometry. Chapter 1 and 2. Academic Press, 1984.

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Chapter 2

Ray-Singer-Polyakov formula on compact surfaces

§1. Heat kernel.

In chapter 1 we mentioned Weyl's formula:

$$\lambda_k \sim C(n)(\frac{k}{\mathrm{Vol}(M)})^{\frac{2}{n}}$$

as $k \longrightarrow \infty$. If we let $Z(t) = \sum_{k=0}^{\infty} e^{-\lambda_k t}$, $Z(t)$ tends to $(4\pi t)^{-\frac{n}{2}}(a_0)$ as $t \longrightarrow 0^+$, where $a_0 = \mathrm{Vol}(M)$. $Z(t)$ is called "the trace of the heat kernel". Assume $\{\phi_i\}$ is a sequence of eigenfunctions which forms an orthonormal basis for $L^2(M)$, then

$$\begin{aligned} Z(t) &= \int_M \sum_{i=0}^{\infty} e^{-\lambda_i t}\phi_i(x)\phi_i(x)dV(x) \\ &= \int_M K(x,x,t)dV(x) \end{aligned}$$

where $K(x,y,t) = \sum e^{-\lambda_i t}\phi_i(x)\phi_i(y)$ is called the heat kernel .

Proposition. *$K(x,y,t)$ is the unique fundamental solution of the heat equation on M. It is continuous in $M \times M \times (0,\infty)$, C^2 in x,y, and C^1 in t. That is, for any given bounded continuous function f on M, the function $u(x) = \int K(x,y,t)f(y)dV(y)$ satisfies the following equation:*

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t}u(x) = \Delta u(x) \\ \lim_{t \longrightarrow 0^+} u(x,t) = f(x) \text{ on } M. \end{array} \right.$$

Examples.

(1) On $S^1 = \{z \in C : |z| = 1\}$, the Laplacian has spectrum $\{n^2, e^{in\theta}\}_{n=-\infty}^{\infty}$ and $dV = \frac{d\theta}{2\pi}$. So

$$K(x,y,t) = \sum_{n=-\infty}^{\infty} e^{-n^2 t} e^{inx} e^{-iny}.$$

In particular, $K(x,x,t) = Z(t) = \sum_{-\infty}^{\infty} e^{-n^2 t}$ which tends to $\frac{1}{\sqrt{4\pi t}}$ as $t \longrightarrow 0^+$.

(2) On R^n, $K(x,y,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x-y|^2}{4t}}$.

Estimates of the heat kernel for general manifolds can be found in Davies [D], and in Li-Yau 's [L-Y] for the case of Ric ≥ 0.

§2 Asymptotic behavior of the trace of the heat kernel.

The main tool for studying the asymptotic behavior of the trace of the heat kernel is the Pseudo-differential calculus (see, e.g., P.Gilkey [G,Chapter 3]). In the following we will briefly mention this approach.

Recall the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi - z} d\xi$$

where f is analytic in a neighborhood of the closed curve C and z is inside C. We apply this formula to the elliptic operator $D : L^2(M) \longrightarrow L^2(M)$. Define

$$f(D) = \frac{1}{2\pi i} \int_C \frac{f(\xi)}{\xi I - D} d\xi$$

where C is a curve avoiding the spectrum of D. Then f is analytic outside a neighborhood of the spectrum of D.

Example.: $f_s(z) = z^s$, $s \in C$ and $D = -\frac{d^2}{d\theta^2}$ on S^1. Let $\phi = \sum a_n e^{inx}$ be a function in $L^2(S^1)$. We have

$$\begin{aligned}(D^s\phi)(x) &= \sum a_n n^{2s} e^{inx} \\ &= \sum (\int \phi(y) e^{iny} dy) n^{2s} e^{inx} \\ &= \int (\sum n^{2s} e^{in(x-y)}) \phi(y) dy.\end{aligned}$$

So kernel of D^s is $K_s(x,y) = \sum n^{2s} e^{in(x-y)}$ and in particular, $K_s(x,x) = \sum n^{2s}$ which converges when $\mathrm{Re}(s) < -\frac{1}{2}$. K_s as a function of s, which is closely related to Riemann zeta function, is meromorphic in all C with only a simple pole at $s = -\frac{1}{2}$.

Take $f(\xi) = e^{-t\xi}$. It turns out that e^{-tD} is a infinitely "smoothing" operator. Taking a "good" approximation D_ξ of $(\xi I - D)^{-1}$, and using the ψDO calculus on manifolds, we can get

Theorem . (Minakshisundaram-Pleijel[M-P], McKean-Singer[M-S]). *For any positive self-adjoint elliptic operator D of order 2, we have*

$$K(x,x,t) \sim \sum_0^\infty B_k(x) t^{\frac{(k-\dim M)}{2}} \text{ as } t \to 0^+$$

where B_k is computable in terms of the symbol σ_D of D and derivatives of σ_D. If $\partial M = \emptyset$, then $B_k \equiv 0$ when k is odd.

Let us denote $a_k = \int B_{2k}(x) dV(x)$. Applying the above theorem to $D = \Delta$ on a manifold, we get

Theorem . [M-K] *Suppose (M,g) is a compact Riemannian manifold without boundary. Then*

$$Z(t) \sim (4\pi t)^{-\frac{n}{2}} (a_0 + a_1 t + a_2 t^2 + \cdots) \text{ as } t \to 0^+, \text{ where}$$

$$\begin{aligned}a_0 &= Vol(M) \\ a_1 &= \frac{1}{3} \int_M K dV \\ a_2 &= \frac{1}{180} \int_M (10A - B + 2C) dV\end{aligned}$$

where K is the Gaussian curvature of the metric (M,g), and where A, B, C are polynomials of degree 2 in the curvature tensor R_{ijkl}.

The formula which we will use in this lecture is the following:

Corollary. When M is a compact closed surface ($\dim(M) = 2$), then

$$\int_M K = 2\chi(M) = 2\pi(2-2g)$$

hence

$$K(x,x,t) \sim \frac{1}{4\pi t} + \frac{K(x)}{12\pi} + O(t) \text{ } as \text{ } t \to 0^+$$

and

$$Z(t) \sim \frac{\text{area}(M)}{4\pi t} + \frac{\chi(M)}{6} + \frac{\pi t}{60}\int_M K^2 + O(t^2) \text{ } as \text{ } t \to 0^+.$$

Corollary. $\chi(M)$ *is audible.*

Remark. A computation of a_1 can also be found in [G, Ch 6].

§3 Polyakov and Ray-Singer's log determinant formula on a compact surface without boundary.

Recall Weyl's formula $\lambda_k^{\frac{n}{2}} \sim C_n(\frac{k}{\text{Vol}(M)})$. Then $\lambda_k^{-s} \sim C_n k^{-\frac{2}{n}s}$. In particular $\sum_0^\infty \lambda_k^{-s}$ converges if $\text{Re}(s) > \frac{\dim M}{2} = \frac{n}{2}$.

Now let $n = 2$ and define the zeta function with $\text{Re}(s) > 1$ as $\zeta(s) = \sum_{k>0} \lambda_k^{-s}$. By using $\frac{d}{ds}\lambda^{-s} = -\lambda^{-s}\log\lambda$, we can formally write

$$\zeta'(s) = -\sum \lambda_k^{-s} \log \lambda_k$$

and

$$-\zeta'(0) \text{``}=\text{''} \sum \log \lambda_k$$

where we must notice that the last sum diverges in general. Now we would like to formally define $\det\Delta = e^{-\zeta'(0)}$ "$=$" $\Pi_{k=1}^{\infty}\lambda_k$ if $\zeta'(0)$ exists. To see $\zeta'(0)$ is well-defined, with $\text{Re}(t) > 0, x > 0$ we argue as follows:

$$x^{-s} = \frac{1}{\Gamma(s)}\int_0^\infty e^{-xt}t^{s-1}dt$$

where $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$ is the gamma function . So $\lambda_k^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-\lambda_k t} t^{s-1} dt$ and hence

$$\begin{aligned} \zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty (\sum_{k=1}^\infty e^{-\lambda_k t}) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty (\mathrm{Tr} e^{-\Delta t} - 1) t^{s-1} dt \end{aligned}$$

To see the behavior as $t \longrightarrow 0^+$ of the above integral, we apply Theorem 2 and get

$$\mathrm{Tr} e^{-\Delta t} \sim \frac{A}{4\pi t} + \frac{\chi(M)}{6} + O(t) \ as \ t \longrightarrow 0^+$$

Thus for $\mathrm{Re}(s) > -1$ we have

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^1 (\frac{A}{4\pi t} + \frac{\chi(M)}{6} + O(t) - 1) t^{s-1} dt + \text{ analytic functions in } s$$

For s close to zero, we may write

$$\zeta(s) = \frac{1}{\Gamma(s)} (\frac{A}{4\pi(s-1)} + (\frac{\chi(M)}{6} - 1)\frac{1}{s} + \frac{1}{s+1}) + \text{ analytic functions in } s$$

Here we have used the property that $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt = \frac{1}{s} - \frac{1}{s+1} + \cdots = \frac{1}{s} +$ analytic function in s around $s = 0$. Thus $\zeta(s)$ is well-defined for $\mathrm{Re} s > -1$ and meromorphic with pole at $s = +1$ (it is analytic at $s = 0$). We know $\zeta(0) = \frac{\chi(M)}{6} - 1$, so we define

$$\zeta'(s)|_{s=0} = \lim_{s \longrightarrow 0} \frac{\zeta(s) - \zeta(0)}{s}$$

which is our definition of $-\log \det \Delta$.

Ray-Singer-Polyakov Formula. [P1] [P2] [R-S] *Suppose (M, g_0) is a compact closed surface, and $g_1 = e^{2u} g_0$ is a metric conformal to g_0 with $vol(M, g_1) = vol(M, g_0)$. Then we have*

$$\log \frac{\det \Delta_1}{\det \Delta_0} = -\frac{1}{12\pi} \int_M (2K_0 u + |\nabla_0 u|^2) dV_0$$

where $\Delta_i, (i = 1, 2)$ denotes the Laplacian operators with respect to g_i repectively.

Proof Since $g_1 = e^{2u} g_0$, we have $\Delta_1 = e^{-2u} \Delta_0$ with volume $A = \int_M dV_1 = \int_M e^{2u} dV_0$ and

$$\zeta_0(s) - \zeta_1(s) = \frac{1}{\Gamma(s)} \int_0^\infty (Tr(e^{-\Delta_0 t}) - Tr(e^{-\Delta_1})) t^{s-1} dt$$

Thus

$$\begin{aligned} I = log\frac{\det \Delta_1}{\det \Delta_0} &= \lim_{s\to 0}\frac{\zeta_0(s)-\zeta_1(s)}{s} \\ &= \lim_{s\to 0}\frac{1}{s\Gamma(s)}\int_0^\infty (Tr(e^{-\Delta_0 t}) - Tr(e^{-\Delta_1 t}))t^{s-1}dt \end{aligned}.$$

Consider the one parameter family of metrics $g_\alpha = e^{2u_\alpha}g_0$, where $u_\alpha = \alpha u, 0 \le \alpha \le 1$. Denote $\Delta_\alpha = e^{-2u_\alpha}\Delta_0$, then the curvature K_α of the metric $g_\alpha = \rho_\alpha g_0$ satisfies

$$\Delta_0 u_\alpha + K_\alpha e^{2u_\alpha} = K_0.$$

Observe that Δ_α and $\tilde{\Delta_\alpha} = \rho_\alpha^{-\frac{1}{2}}\Delta_0\rho_\alpha^{-\frac{1}{2}}$ have the same set of eigenvalues (hence the same trace of the heat kernel) with the advantage that the domain of $\tilde{\Delta_\alpha}$ is $L^2(dV_0)$. Hence

$$\begin{aligned} I &= \int_0^\infty (Tr(e^{-\Delta_0 t}) - Tr(e^{-\Delta_1 t}))\frac{dt}{t} \\ &= -\int_0^\infty \frac{1}{t}\int_0^1 \frac{d}{d\alpha}Tr(e^{-t\tilde{\Delta_\alpha}})d\alpha dt \\ &= \int_0^\infty\int_0^1 Tr(e^{-t\tilde{\Delta_\alpha}}\frac{d\tilde{\Delta_\alpha}}{d\alpha})d\alpha dt \\ &= -\int_0^\infty\int_0^1 Tr(\frac{\rho_\alpha'}{\rho_\alpha}e^{-t\tilde{\Delta_\alpha}}\tilde{\Delta_\alpha})d\alpha dt \;\; (\; where\ \rho_\alpha' = \frac{d}{d\alpha}\rho_\alpha\;) \\ &= \int_0^1 d\alpha\int_0^\infty Tr(\frac{d}{dt}\frac{\rho_\alpha'}{\rho_\alpha}e^{-t\tilde{\Delta_\alpha}})dt \\ &= \int_0^1 d\alpha\int_0^\infty \frac{d}{dt}Tr(\frac{\rho_\alpha'}{\rho_\alpha}e^{-t\tilde{\Delta_\alpha}})dt \\ &= \int_0^1 d\alpha\ Tr(\frac{\rho_\alpha'}{\rho_\alpha}e^{-t\tilde{\Delta_\alpha}})|_0^\infty \\ &= -\int_0^1 d\alpha(\lim_{\epsilon\to 0}\ Tr(\frac{\rho_\alpha'}{\rho_\alpha}e^{-\epsilon\tilde{\Delta_\alpha}})). \end{aligned}$$

To compute $\lim_{\epsilon\to 0}\ Tr(\frac{\rho_\alpha'}{\rho_\alpha}e^{-\epsilon\tilde{\Delta_\alpha}})$, we apply the asymptotic formula for the trace of heat-kernel for $\tilde{\Delta_\alpha}$, and get

$$\begin{aligned} I_\epsilon(\alpha) &= \lim_{\epsilon\to 0}\ Tr(\frac{\rho_\alpha'}{\rho_\alpha}e^{-\epsilon\tilde{\Delta_\alpha}}) \\ &= \lim_{\epsilon\to 0}\sum_k\int_M \frac{\rho_\alpha'}{\rho_\alpha}e^{-\epsilon\lambda_k^{\tilde{(\alpha)}}}(\phi_k^{\tilde{(\alpha)}}(x))^2 d\tilde{V_\alpha} \end{aligned}$$

where $\lambda_k^{\tilde{(\alpha)}}$, and $\phi_k^{\tilde{(\alpha)}}(x)$ are eigenvalues and eigenfunctions of $\tilde{\Delta_\alpha}$. Notice that $\lambda_k^{\tilde{(\alpha)}} = \lambda_k^{(\alpha)}$, and $\phi_k^{(\alpha)}(x) = \rho_\alpha^{-1}\phi_k^{\tilde{(\alpha)}}(x)$ are eigenvalues and eigenfunctions for $\tilde{\Delta_\alpha}$. We get

$$I_\epsilon(\alpha) = \lim_{\epsilon \longrightarrow 0} \int_M \rho'_\alpha(x) \frac{1}{4\pi\epsilon}(1 + \frac{\epsilon}{3} K_\alpha(x) + O(\epsilon^2)) dV_0(x).$$

Notice that by our assumption $\int \rho_\alpha(x) dV_0(x) = A$ is fixed, thus $\int_M \rho'_\alpha(x) dV_0(x) = 0$. Hence

$$\begin{aligned} I = -\int_0^1 d\alpha I_\epsilon(\alpha) &= -\int_0^1 \lim_{\epsilon \longrightarrow 0} \int_M \rho'_\alpha(x) \cdot \frac{1}{12\pi} K_\alpha(x) dV_0(x) \\ &= -\frac{1}{12\pi} \int_0^1 d\alpha \int_M \frac{\rho'_\alpha(x)}{\rho_\alpha(x)} (K_0 - \Delta_0 u_\alpha)(x) dV_0(x) \\ &= -\frac{1}{12\pi} \int_0^1 d\alpha \int_M 2u'_\alpha (K_0 - \Delta_0 u_\alpha) dV_0(x) \\ &= -\frac{1}{12\pi} \int_0^1 d\alpha \int_M \frac{d}{d\alpha} (2K_0 u_\alpha + |\nabla u_\alpha|^2) dV_0(x) \\ &= -\frac{1}{12\pi} \int_M (2K_0 u + |\nabla u|^2) dV_0. \end{aligned}$$

Which finishes the proof of the formula.

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Chapter 3

Moser-Onofri Inequality and Applications

Let's recall the Ray-Singer-Polyakov's formula on a compact surface. Suppose that $g_1 = e^{2u} g_0$ and g_0 are two conformal metrics on a compact surface M without boundary with $\Delta_1 = e^{-2u}\Delta_0$. Then we have

$$\log \frac{\det \Delta_1}{\det \Delta_0} = -\frac{1}{12\pi}\int_M (2K_0 u + |\nabla_0 u|^2) dV_0$$

In the special case with $M = S^2$ and g_0 the standard metric of constant Gaussian curvature $K_0 = 1$. For any u with $\int_{S^2} e^{2u} = 4\pi$ and $\Delta_u = e^{-2u}\Delta_0$,

$$\log \frac{\det \Delta_u}{\det \Delta_0} = -\frac{1}{12\pi}\int_{S^2} (2u + |\nabla u|^2) dV_0$$

Onofri's inequality.

(1)
$$\frac{1}{4\pi}\int_{S^2} e^{2u} \leq \exp\Big(\frac{1}{4\pi}\int_{S^2} (2u + |\nabla u|^2)\Big)$$

with equality iff $e^{2u} g_0$ is isometric to g_0.

Corollary. *Among all metrics on S^2, $\log\det\Delta_0$ is the maximum.*

Remark. This should be compared to result of Hersch that $\lambda_1 \leq 2$.

There are now many different proofs of Onofri's inequality. The author knows at least four: Onofri's original proof [O] and [O-V], an independent proof by Hong [H], subsequent proof by O-P-S [O-P-S-1] and a recent proof by Beckner. Here we will present the original proof by Onofri as it connects the inequality to conformal geometry on S^2. All these proofs depend on the following inequality of Moser:

Moser. ([M]) *If u is smooth on S^2 then there is some constant C_1 such that*

(2)
$$\int_{S^2} exp\left(\frac{\alpha(u-\bar{u})^2}{\int_{S^2}|\nabla u|^2 dV_0}\right) \leq C_1$$

for any $\alpha \leq 4\pi$, with 4π being the best constant, i.e., if $\alpha > 4\pi$, the integral can be made arbitrarily large by appropriate choice of u. In the above inequality $\bar{u} = \frac{1}{4\pi}\int_{S^2} u dV_0$.

Remark 1.. Trudinger first proved that there exists some constant α so that the above inequality (2) holds. (2) should be considered as the limiting case of Sobolev imbedding Theorem: $W_0^{1,q} \hookrightarrow L^p$ with $\frac{1}{p} = \frac{1}{q} - \frac{1}{n}$ for $1 < q < n$. If $q > n$, then $W_0^{1,q}$ can be identified with Hölder continuous function with Hölder exponent $1 - \frac{n}{q}$. When $q = n$, $W_0^{1,n} \not\subset L^\infty$, but $W_0^{1,n}$ is contained in functions of the exponential class: $W_0^{1,n}(D) \hookrightarrow \exp \frac{n}{n-1}$ class, i.e., for any bounded domain D in R^n, there is a constant $\alpha = \alpha(D)$ and constant $C = C(D)$ so that if u is in $W_0^{1,n}(D)$ with $\int_D | \bigtriangledown u|^n \leq 1$ then

$$\int_D \exp(\alpha |u|^{\frac{n}{n-1}}) \leq C \text{meas}(D).$$

Later Moser improved Trudinger's inequality and obtained the value of the best constant α, in the case of plane domains $\alpha = 4\pi$.

Remark 2.. Moser's proof is quite sharp and delicate. We will not include his original proof here. (An easier proof is included in D. Adams ([A]) given by A. Garsia). But the key elements in the proof are:

Step 1. Apply "symmetrization" based on "isoperimetric inequality" and reduce the function to one variable, i.e.,

Exercise. Let u^* be symmetric decreasing function with same distribution function as u. Then $\int |\nabla u^*|^2 \leq \int |\nabla u|^2$.

Step 2. The one-variable inequality after the change of variable $t = logr$ becomes the following: Given w smooth on $(-\infty, \infty)$ with $\dot{w} \geq 0$, $\int_{-\infty}^{\infty} \dot{w}^2 dt \leq 1$, $\int_{-\infty}^{\infty} w \rho dt = 0$ for some positive function ρ with $\int_{-\infty}^{\infty} \rho = 1$, $\rho(t) \leq C_0 e^{-|t|}$. Then $\int_{-\infty}^{\infty} e^{w^2(t)} \rho(t) dt$ is bounded independent of w.

We will present at the end of this lecture a proof of Moser's inequality based on step 2 above given by [C-C], the proof also indicates that an extremal function for the inequality exists.

Remark 3.. We call u defined on S^2 an even function if $u(\xi) = u(-\xi)$ for all $\xi \in S^2$. The best constant α in Moser's inequality for functions in the even class is $\alpha = 8\pi$.

To see this we have the following lemma.

Lemma.[C-Y-5] *Assume that D is a piecewise C^2 domain in R^2 with $\int_D |\nabla u|^2 = 1$ and $\int_D u = 0$. Let L_M = length of$\{x|u(x) = M\}$ and A_M = area of$\{x|u(x) \geq M\}$. If $\alpha_u = \sup_{M \longrightarrow \infty, |A_M| \longrightarrow 0} \frac{L_M^2}{A_M}$, then there exists a constant C_2 (independent of u) so that*

$$\int_D e^{\alpha_u u^2} \leq C_2 |D|.$$

In particular,

(a) when u is smooth function with compact support in D, then $\alpha_u \geq 4\pi$.

(b) When $D = S^2$, we have the isoperimetric inequality $L^2 \geq A(4\pi - A)$, hence $\alpha_u \geq 4\pi$.

(c) If u is even, for two small regions on S^2 symmetric under antipodal map, $L = L_1 + L_2$ with $L_1 = L_2$, $A = A_1 + A_2$ with $A_1 = A_2$, and $L_i \geq A_i(4\pi - A_i)$, thus $L^2 = 4L_1^2 \geq 4A_i(4\pi - A_i) = A(8\pi - A)$. So $\alpha_u \geq 8\pi$.

Corollary.

$$\frac{1}{4\pi} \int_{S^2} e^{2u} \leq C_2 \exp\left(\frac{1}{4\pi} \int_{S^2} |\nabla u|^2 + 2\frac{1}{4\pi} \int_{S^2} u\right) \tag{4}$$

with $C_2 \leq C_1$ in (2).

<u>Proof</u>

$$2(u - \bar{u}) \leq \frac{4\pi(u - \bar{u})^2}{\int_{S^2} |\nabla u|^2 dV_0} + \frac{1}{4\pi} \int_{S^2} |\nabla u|^2 dV_0$$

by using $2ab \leq a^2 + b^2$.

Notice the best constants C_1 in (2) is greater than 1, while the statement in Onofri's inequality indicates the best constant C_2 in (4) is 1. <u>Proof of Onofri's inequality</u> We will use the following fact:

Lemma 1. *Let $S[u] = \frac{1}{4\pi} \int_{S^2} |\nabla u|^2 + 2\frac{1}{4\pi} \int_{S^2} u$. Then $S[u]$ is conformally invariant in the following sense: Given ϕ a conformal transformation of S^2, let $T_\phi u = v$ denote the induced conformal factor given by $\phi^*(e^{2u} g_0) = e^{2v} g_0$, of more explicitly, $T_\phi u = (u \circ \phi) + \frac{1}{2} \log |J_\phi|$. Then $S[T_\phi u] = S[u]$.*

<u>Proof</u> Since log determinant is an intrinsic quantity of the metric it is invariant under conformal transformation, $S[T_\phi u] = S[u]$ as a consequence of Polyakov's formula.

Remark. This lemma also can be proved directly without applying Polykov's formula, but based on the following corollary.

Corollary. *If* $u = \frac{1}{2}\log|J_\phi|$ *for some conformal transformation* ϕ*, then* $S[u] = 0$.

Definition. *We say* $u \in S$ *if* $\int_{S^2} e^{2u}x_j = 0$ *for* $j = 1, 2, 3$.

Lemma 2. *Given a* C^1 *function* u *defined on* S^2*, there exists a conformal transformation* ϕ *such that* $T_\phi u \in S$.

Proof Consider the following family of conformal transformations of S^2. Given $P \in S^2, t \in [1, \infty)$, rotate the coordinates so that P corresponds to the north pole $(0, 0, 1)$. Using the stereographic projection from P mapping the sphere to the equatorial plane on which we have the complex coordinate z, we denote the transformation $\phi_{P,t}(z) = tz$. Observe that $\phi_{P,1} = id, \phi_{P,t} = \phi_{-P,t^{-1}}$, hence this set of conformal transformations can be parametrized by the unit ball $B^3 = S^2 \times [1, \infty)/S^2 \times \{1\}$. Notice that for $\phi = \phi_{P,t}$ we may write with a change of variable

$$\int e^{2T_\phi u}x_j = \int e^{2u}(x_j \circ \phi^{-1}_{P,t}).$$

Define the center of mass of the mass distribution e^{2u} by $C.M.(\phi_{P,t}) = \frac{\int e^{2u}x_j \circ \phi^{-1}_{P,t}}{\int e^{2u}}; j = 1, 2, 3.$. The center of mass map may be considered as a map from B^3 to B^3, it has a continuous extension to the boundary mapping P to $-P$. It follows from the Brouwer's fixed point theorem that there exists $P \in S^2, t \in [1, \infty)$ for which $C.M.(\phi_{P,t}) = 0$.

Lemma 3.(Aubin [Au]) *If* $u \in S$*, for all* $\epsilon > 0$ *there is a constant* C_ϵ *such that*

$$\frac{1}{4\pi}\int_{S^2} e^{2u} \le C_\epsilon \exp((\frac{1}{2} + \epsilon)\frac{1}{4\pi}\int_{S^2} |\nabla u|^2 + 2\frac{1}{4\pi}\int_{S^2} u)$$

We will now finish the proof of Onofri's inequality based on Lemmas 1, 2 and 3.

Proof Consider the functional

$$F[u] = \log \frac{1}{4\pi}\int_{S^2} e^{2u} - S[u]$$

defined for all u on $W^{1,2}(S^2)$. Apply Moser's inequality, we get

$$\log \frac{1}{4\pi}\int_{S^2} e^{2u} \le S[u] + C_1$$

so $F[u] \le C_1$. Choose $u_k \in W^{1,2}(S^2)$ so that $F[u_k] \longrightarrow M = \max F[u]$.

Apply Lemma 1, we know $F[u] = F[T_\phi u]$ for any confomal transformation. Apply Lemma 2, we may assume w.l.o.g. that $u_k \in S$ and $\int u_k = 0$. Apply Lemma 3, then

$$\log \frac{1}{4\pi} \int_{S^2} e^{2u_k} \leq C_\epsilon + (\frac{1}{2} + \epsilon) \frac{1}{4\pi} \int_{S^2} |\nabla u_k|^2 + 2 \frac{1}{4\pi} \int_{S^2} u_k$$

Hence

$$M - \epsilon \leq F[u_k] \leq C_\epsilon - (\frac{1}{2} - \epsilon) \frac{1}{4\pi} \int_{S^2} |\nabla u_k|^2$$

for all $\epsilon < \frac{1}{2}$. i.e.,

$$\frac{1}{4\pi} \int_{S^2} |\nabla u_k|^2 \leq C_\epsilon + \epsilon - M$$

is bounded. Choose $\int u_k = 0$. Thus $u_k \longrightarrow u$ weakly in $W^{1,2}$ and $\int |\nabla u|^2 \leq \int |\nabla u_k|^2$, $F[u] \geq F[u_k]$ and $\int e^{2u_k} \longrightarrow \int e^{2u}$. So $F[u]$ attains the maximum value and u satisfies

$$\Delta u + e^{2u} = 1$$

with $u \in S$. This is equivalent to $u = \frac{1}{2} \log |J_\phi|$ for some conformal transformation ϕ, but among this class of functions only the constant function $u = 0$ belongs to the symmetric class S.

Exercise. Prove that if $\psi_{p,t} = \frac{1}{2} \log |J_{\Phi_{p,t}}|$, then $\int |\nabla \psi_{p,t}|^2 \longrightarrow +\infty$ as $t \longrightarrow \infty$ and $\int \psi_{p,t} \longrightarrow -\infty$ as $t \longrightarrow \infty$. While $\int |\nabla \psi_{p,t}|^2 + 2 \int \psi_{p,t} = 0$.

As an application, we have the following result due to Osgood-Phillips-Sarnak:

Theorem. [O-P-S-2] *Isospectral family of metrics on compact surface without boundary form a compact set in the C^∞- topology.*

<u>Sketch of Proof</u> We will only prove here the genus zero case, i.e., $M = S^2$. In this case the theorem says given $\{e^{2u_k}\}$ with $\Delta_k = e^{-2u_k} \Delta_0$ having the same spectrum data, then the conformal class $[u_k]$ of u_k forms a C^∞- compact family.

Recall the heat kernel asymptotics:

$$\text{Tr} e^{-\Delta t} \longrightarrow (4\pi t)^{-1} (a_0 + a_1 t + a_2 t^2 + \cdots) \text{ as } t \to 0^+$$

with

$a_0 = \int dV = \int e^{2u} dV_0,$

$a_1 = \frac{1}{3}\int_M K dV = \frac{1}{6}\chi(M) = \frac{4\pi}{3}$,

$a_2 = \frac{\pi}{60}\int K^2 dV = \frac{\pi}{60}\int (1-\Delta_0 u)^2 dV_0$.

For $k \geq 3$, $a_k = \int_M B_{2k} dV$ where B_{2k} are universal polynomials of weight $2k$ in K and Δ with K, Δ each counting as degree 2. a_0 is fixed while a_2 gives some information of $\Delta_0 u$. Thus we need some information about the 1st derivative of u (which is missing– which a_1 does not provide); while leading term a_k provides some informations about $W^{k,2}$ norm of u.

Key idea is to use log determinant formula to bound $\int |\nabla u|^2$.

$$\log \frac{\det \Delta_u}{\det \Delta_0} = -\frac{1}{3} S[u]$$

here $\int e^{2u} = 4\pi$ say. W.l.o.g., assume that $u \in S$, then Aubin's inequality implies that $\int |\nabla u|^2 \leq$ bdd.

Remark. In the papers of [O-P-S-1], they have also treated the higher genus case, and also the class of simply connected plane domains. In the case of plane domain (via conformal map, the unit disc), the role played by Moser's inequality is replaced by Beurling's inequality, the role of Onofri's inequality is replaced by the Lebedev-Milin inequality and the analogue of the Ray-Singer-Polyakov formula is proved by Alvarez [Al].

Beurling's inequality. *Let D be the unit disc and f be an analytic function on D. If $\int\int_D |f'(z)|^2 dxdy \leq \pi$, $f(0) = 0$, then*

$$|\{\theta \in \partial D : |f(e^{i\theta})| > t\}| \leq 2\pi e^{1-t^2}$$

Lebedev-Milin's inequality.

$$\log \int_{\partial D} e^{2u} \frac{d\theta}{2\pi} \leq \int_D |\nabla u|^2 \frac{dxdy}{\pi} + 2\int_{\partial D} u \frac{d\theta}{2\pi}$$

where u is harmonic in D.

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Chapter 4

Existence of Extremal Functions for Moser Inequality

In this chapter, we will describe a proof to establish the existence of extremal functions for Moser-Trudinger inequality as stated in the previous chapter. The proof has appeared in [C-C], see also [St].

Theorem 1 Let B_n denote the unit ball in R^n, $n \geqq 2$, then

$$\sup \int_{B_n} \exp(\alpha_n |u|^{\frac{n}{n-1}})\, dx$$

is attained for functions $u \in W_0^{1,n}(B_n)$ with $\int_{B_n} |\bigtriangledown u|\, dx \leqq 1$, where $\alpha_n = n(\omega_{n-1})^{\frac{1}{n-1}}$, ω_{n-1} is the area of $(n-1)$ spheres.

We first remark that the result is surprising in the following sense: Moser's inequality is the limiting case of the Sobolev embedding Theorem $W_0^{1,q}(D) \subset L^p(D)$ with $\frac{1}{p} = \frac{1}{q}_{\frac{1}{n}}$ for $1 \leq q \leq n$ and domain D in R^n. In the limiting case for $q = n$, we have $W_0^{1,n}(D) \subset \exp \frac{n}{n-1}$ class, while the embedding with exponent α_n) is not compact. In another well-known case of Sobolev Embedding $W_0^{1,2}(D) \subset L^{\frac{2n}{n-2}}$, while the embedding is not compact, the extremal function for the embedding does not exist unless $D = R^n$. In the latter case the study of the noncompactness phenomenon leads to the solution of "Yamabe" problem ([A] [S-1]), and many other interesting problems in non-linear PDE (cf [B-N]). In contrast, in the case of the Moser functional, the extremal function is attained. We also want to point out that the proof we shall present below is an existence proof, with the explicit expression of the extremal function remains an open question.

Our second remark is that we have obtained the existence result only for $D = B_n$ the unit ball in R^n. The reason for this constrain is that only in this case, we may apply the symmetrization technique mentioned in chapter 3 to reduce the theorem to the following equivalent form via a change of variable as in [M].

Theorem 2 Let K denote the class of C^1 functions $\omega(t)$ defined on $0 \leqq t1 \leq \infty$ satisfying

$$\omega(0) = 0,\ \dot{\omega}(t) \geqq 0 \textit{ and } \int_0^\infty \dot{\omega}^n(t)\, dt \leqq 1,$$

then

$$\sup_{\omega \in K} \int_0^\infty e^{\omega^{\frac{n}{n-1}}(t) - t}\, dt \tag{1}$$

is attained by some function $\omega^* \in K$.

In a recent paper by M. Flucher [F], using ingenious method in complex analysis, he was able to establish Theorem 1 for general domain in R^2. In another related development, T.L. Soong [So] has proved the analogous results of Theorem 1 for functions u defined on S^2 with $\int_{S^2} u = 0$, and some partial result for functions v defined on S^4, with $\int_{S^4} v = 0$ and $< Pv, v > \leqq 1$ where $P = (-\Delta)^2 + 2\Delta$ is the Panitz operator on S^4.

In the remaining part of this chapter, we will prove Theorem 2 for the special case $n = 2$. The proof for general n is more complicated only in the technical sense.

Proof of Theorem 2, for $n = 2$

The existence Theorem was established via method of contradiction. This is accomplished through the following two steps:

Step 1 Assume there is no extremal function attaining the supremum in (1), then

$$\sup_{\omega \in K} \int_0^\infty e^{\omega^2(t)-t}\, dt \leqq 1 + e.$$

Step 2 There exists some $\omega \in K$ with

$$\sup_{\omega \in K} \int_0^\infty e^{\omega^2(t)-t}\, dt > 1 + e.$$

The proofs of both step 1 and step 2 are motivated by the existence of a sequence of "broken line" functions $\{\omega_a\}_{a>0}$ defined as follows: $\omega_1(t) = t$ when $0 \leqq t \leqq 1$, $\omega_1(t) = 1$ when $t \geqq 1$, and $\omega_a(t) = \sqrt{a}\omega_1(\frac{t}{a})$. This is the sequence of functions which has been used by Moser [5] in establishing that $\alpha_2 = 4\pi$ is the best exponent in the embedding $W_0^{1,2} \subset L^2$. Notice that for this sequence of functions $\{\omega_a\}$, $\omega_a \in K, \omega_a(x) \to 0$ as $a \to \infty$ uniformly in compact subsets, and $I(\omega_a) = \int_0^\infty e^{\omega_a^2(t)-t}\, dt \nrightarrow I(0) = 1$. The intuitive idea in the proof of the theorem is to prove that if the extremal of $I(\omega)$ for $\omega \in K$ does not exist, the maximal sequence would behave quite like the sequence $\{\omega_a\}$.

Proof of step 2

Define

$$\omega(t) = \begin{cases} \frac{t}{2} & \text{when } 0 \leqq t \leqq 2, \\ (t-1)^{\frac{1}{2}} & \text{when } 2 \leqq t \leqq e^2 + 1, \\ e & \text{when } t > e^2 + 1 \end{cases}$$

then

$$I(\omega) = 2 \int_0^1 e^{t^2-2t}\, dt + e$$

$$= \frac{2}{e} \int_0^1 e^{s^2}\, ds + e > \frac{2.723}{e} + e > 1 + e,$$

based on lower Riemann sum estimate of the integral $\int_0^1 e^{s^2}\, ds$.

Proof of step 1

Choose a sequence $\omega_m \in K$ with $I(\omega_m) \to M = \sup_{\omega\in K} I(\omega)$. Assume that there does not exist any $\omega \in K$ attaining supremum M, we claim that the sequence ω_m has the following properties:

(a) For each $A > 0$, $\int_0^A (\omega_m)^2\, dt \to 0$ as $m \to \infty$,

(b) Let a_m be the first in $[1,\infty)$ with $\omega_m^2(a_m) = a_m - 2\log a_m$ (if such a_m exists). Then $a_m \to \infty$ as $m \to \infty$.

(c) $\lim_{m\to\infty} \int_0^{a_m} e^{\omega_m^2(t)-t}\, dt = 1$,

(d) $\limsup_{m\to\infty} \int_{a_m}^{\infty} e^{\omega_m^2(t)-t}\, dt \leqq e$.

(a) (b) can be established via argument by contradiction and elementary calculus.

To prove (c), we notice that it follows from (a) that $\omega_m \to 0$ uniformly on compact subsets. Thus for each $\epsilon > 0$, A and m large with $\omega_m^2(t) \leqq \epsilon$ for $t \leqq A$, using the property that a_m is the first point with $\omega_m^2(t) \geqq t - 2\log^+ t$ we have

$$\int_0^{a_m} e^{\omega^2(t)-t}\, dt = \int_0^A + \int_A^{a_m}$$

$$\leqq e^{\epsilon} \int_0^A e^{-t}\, dt + \int_A^{a_m} e^{2\log^+ t}\, dt$$

$$= e^{\epsilon}(1-e^{-A}) + (\frac{1}{A} - \frac{1}{a_m}) \leqq 1,$$

as $\epsilon \to 0, A \to \infty$. On the other hand

$$\int_0^{a_m} e^{\omega_m(t)-t}\, dt \geqq \int_0^{a_m} e^{-t}\, dt = 1 - e^{-a_m},$$

which tends to one as $a_m \to \infty$, which finishes the proof of (c).

To prove (d), we first establish a lemma.

Lemma 3 Let $K_\delta = \{\phi : C^1$ *functions defined on* $0 \leqq t < \infty,\ \phi(0) = 0,\ \int_0^\infty \dot\phi^2\, dt \leqq 0\}$, then for each $c > 0$ we have

$$\sup_{\phi\in K_\delta} \int_0^\infty e^{c\phi(t)-t}\, dt < e \cdot e^{\frac{c^2}{4}\delta}. \tag{2}$$

Also when $c^2\delta \to \infty$, the inequality in (2) tends asymptotically to an equality.

Proof of Lemma 3

We first remark that since $\int_N^\infty e^{c\phi(t)-t}\,dt$ is uniformly small for all $\phi \in K_\delta$ as $N \to \infty$, it is easy to verify that the extremal function for $\sup_{\phi\in K_\delta} \int_0^\infty e^{c\phi(t)-t}\,dt$ exists.

Suppose $\phi \in K_\delta$ is such sn extremal function, then a variational method shows that it satisfies the following differential equation:

$$e^{c\phi(t)-t} = A\,\ddot{\phi}, \ \textit{for some constant } A. \tag{3}$$

Let $k(t) = c\phi(t) - t$, we may rewrite (3) into

$$e^{k(t)} = A\,\ddot{k}(t), \tag{4}$$

with $k(t)$ satisfying

$$\int_0^\infty (\dot{k}+1)^2\,dt = c^2\delta, k(0) = 0; \tag{5}$$

hence $\dot{k}(\infty) = -1, k(\infty) = \infty$. Multiply (4) by $\dot{k}$ and integrate, we get

$$e^{k(t)} = A(\frac{1}{2}(\dot{k}+1)^2(t) - (\dot{k}+1)(t) + C). \tag{6}$$

Letting $C \to \infty$ and using (5), we find $c = 0$. Compare (4) and (6) we get

$$\ddot{k}(t) = \frac{1}{2}(\dot{k}+1)^2(t) - (\dot{k}+1)(t). \tag{7}$$

After integrating (7), we have

$$1 + \dot{k}(t) = \frac{2}{1+Be^t},$$

where the constant B is determined by the equality

$$c^2\delta = \int_0^\infty (\dot{k}+1)^2\,dt = 4(\log\frac{1+B}{B} - \frac{1}{1+B}). \tag{8}$$

Evaluating (6) at $t = 0$ we get

$$1 = A(\frac{2}{(1+B)^2} - \frac{2}{1+B}). \tag{9}$$

On the other hand, integrating (4) directly we get from (9)

$$J = \int_0^\infty e^{k(t)}\,dt = Ak(t)|_0^\infty = -\frac{2A}{1+B} = \frac{1+B}{B}.$$

Since $1+\dot{k}(t) = c\phi(t) \geqq 0$ we have $1+B \geqq 0$. Thus $B \geqq 0$. It follows from (9) that $B \to 0$ when $c^2\delta \to \infty$ and $J \leqq \exp(\frac{1}{4}c^2\delta + 1)$ with the inequality approaching an equality when $c^2\delta \to \infty$ as claimed in the statement of this lemma.

We may now apply the above lemma to finish the proof of (d) as follows: Let $x = t - a_m, \phi_m = \omega_m(x+a_m) - \omega_m(a_m)$, then $\omega_m(t) = \phi_m(x) + \omega_m(a_m)$ for all $x > 0$,

$$\omega_m^2(t) = \omega_m^2(a_m) + 2\omega_m(a_m)\phi_m(x) + \phi_m^2(x).$$

Since $\phi_m(0) = 0, \int_0^\infty \dot{\phi}^2(x)\,dx = \int_{a_m}^\infty \dot{\omega}_m^2(t)\,dt = \delta_m$. Thus if we set $y = (1-\delta)x, c = 2\omega_m(a_m)$ and $\phi(y) = \phi_m(x)$ we get

$$\int_{a_m}^\infty e^{\omega_m^2(t)-t}\,dt \leqq e^{\omega_m^2(a_m)-a_m}\frac{1}{1-\delta_m}\int_0^\infty d^{c\phi(y)-y}\,dy, \tag{10}$$

applying the above lemma

$$\int_{a_m}^\infty e^{\omega_m^2(t)-t}\,dt \leqq e^{k_m}\frac{1}{\delta_m}e,$$

where $k_m = \omega_m^2(a_m) - a_m + \frac{\delta_m}{1+\delta_m}\omega_m^2(a_m)$. Since $\omega_m \in K$, $\omega_m^2(a_m) \leqq (1-\delta_m)a_m$, from this using the definition of a_m we can check $\delta_m \leqq \frac{2\log^+ a_m}{a_m} \to 0$ as $m \to \infty$ and $k_m \to 0$ as $m \to \infty$. Thus it follows from (10) we have

$$\limsup_{m\to\infty}\int_{a_m}^\infty e^{\omega_m^2(t)-t}\,dt \leqq e,$$

which establishes (d). We have finished the proof of Theorem 2 for $n = 2$ case.

Remark The author has learned from B. Beckner that Lemma 3 is actually equivalent to the Onofri's inequality on S^2 as stated in Chapter 3, which is not an obvious fact at all.

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Chapter 5

Beckner-Adams Inequalities and Extremal Log-determinants in 4-D.

In this chapter, we will discuss some attempts to generalize the Onofri and [O-P-S] results to higher-dimensional manifolds.

Recall that Polyakov's formula depends highly on the invariant property of the Laplacian operator, i.e. if $g = e^{2w} g_0$, then

$$\int_M |\nabla u|^2 dV_g = \int_M |\nabla_0 u|^2 dV_0$$

for every u defined compact closed surface M of dimension 2. Rewrite

$$\int_M |\nabla u|^2 dV_g = -\int_M (\Delta u) u dV_g$$

We see that this property follows from that $\Delta_g = e^{-2w} \Delta_{g_0}$.

Definition. *We call an operator A a conformal covariant operator if $g = e^{2w} g_0$ then $A = e^{-bw} A_0 e^{aw}$ with $b - a = 2$.*

Example 1. $A = \Delta$ then $b = 2$, $a = 0$ in the case $dim\ M = 2$.

Example 2. M is a closed manifold of dimension n and

$$A = L = -c_n \Delta + R, \qquad \text{where } \; c_n = \frac{4(n-1)}{n-2},$$

R is the scalar curvature of M. L is called the conformal Laplacian. Then L has the following conformal covariance property

$$L_\omega(\phi) = e^{-\frac{n+2}{2}\omega} L_0 (e^{\frac{n-2}{2}\omega} \phi)$$

for every $\phi \in C^\infty(M)$.

Exercise. Prove the conformal covariant property of the conformal Laplacian.

Hint:Let u denote $e^{\frac{n-2}{2}\omega}$, then u satisfies the equation $Lu = Ru^{\frac{n+2}{n-2}}$, where R is the scalar curvature of g.

In [**P-R**] (see also Branson[**B**]), Parker and Rosenberg have computed the heat kernel for conformal covariant operator, and proved that when n is an even integer, $a_{n/2}$ is an invariant quantity under conformal change of metrics, and when n is an odd integer, log determinant is an invariant quantity.

Restricting our attention to $n = 4$, for compact 4-manifold, the a_2 coefficient of the trace of the kernel for the conformal Laplacian operator is a conformal invariant of the metric. The work of [**O-P-S 1,2**] thus suggests that the role of a_2 should be replaced by $\log \det L$ in the compactness isospectral theorem. This is in fact the case. Branson-Orsted [B-O] have given a generalized version of the Polyakov formula for logdeterminants of conformally covariant operators in four dimension on compact locally symmetric Einstein manifolds. A special case is the conformal Laplacian L:

In particular, suppose $V(M, g) = \int_M e^{4\omega} dV_0 = v_0$, then

$$F(\omega) = \log \frac{\det L_\omega}{\det L_0} = -\,2\{\frac{1}{4} k(M, g_0, L) \log \frac{1}{v_0} \int_M e^{4(\omega - \bar{\omega})}$$

$$- \int_M (\Delta\omega)^2 - R_0 \int |\nabla\omega|^2\} - 4\{R_0 \int_M |\nabla\omega|^2 - \frac{1}{2} \int_M \left(\frac{\Delta e^\omega}{e^\omega}\right)^2\}$$

where $k(M, g_0, L) = -\frac{3}{2} v_0 c^2 + 16\pi^2 \chi$, and c^2 denotes the square-norm of the Weyl comformal curvature of g_0. Indeed, compactness information similar to those in two dimension can be obtained for metric $e^{2\omega} g_0$ with $F(\omega)$ bounded. This is formulated as the following result.

Theorem 1 [B-C-Y]. *If $k(M, g_0, L) < 32\pi^2$, and with normalized volume $\int_M e^{4\omega} dV_0 = v_0$, then $\|\omega\|_{2,2}$ is bounded by a constant depending only on $F(\omega)$.*

Examples of manifolds with $k(M, g_0, L) < 32\pi^2$: any manifold M which is compact and locally symmetric,Einstein but not (S^4, g_0), or a hyperbolic space form. $k(S^4, g_0, L) = 32\pi^2$. But RP^4, CP^2, $S^2 \times S^2$, T^4 and compact quotients of polydisc with standard metric all have $k(M) < 32\pi^2$.

Theorem 1'. *The conclusion of Theorem 1 holds on (S^4, g_0) with ω replaced by $T_\phi \omega$ for some ϕ a conformal transformation of S^4. ($T_\phi(\omega) = \omega_0 \phi + \frac{1}{4} \ln |J_\phi|$, $e^{2T_\phi \omega} g_0 = \phi^*(e^{2\omega} g_0)$)*

The role $32\pi^2$ plays here is the same as that of 4π in Moser's inequality. Actually there is a higher dimensional analogue of Moser's inequality when $n \geq 2$ due to Adams, which we will describe below.

Recall by the Sobolev embedding theorem, $W^{\alpha,q} \subset L^p$, when $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$ if $q > 1$ and $\alpha q < n$. The following result deals with the limiting case $\alpha q = n$.

Theorem 2 (Adams). *Suppose $m < n$ are positive integers, Ω is a bounded domain in R^n. Then there are constants $C_0 = C(m, n)$, $\beta_0 = \beta(m, n)$ such that if $u \in W^{m,q}$ of compact support in Ω with $\|\nabla^m u\|_q \leq 1$, where $qm = n$, then for $\beta \leq \beta_0$,*

$$\int_\Omega \exp(\beta |u(x)|^{q'}) \leq C_0 |\Omega|,$$

where $q' = \frac{q}{q-1}$ and β_0 is the best constant for the inequality to hold.

When $n = 4$, $q = m = q' = 2$ we have $\beta_0(2, 4) = 32\pi^2$. L. Fontana (1991) has generalized the above inequality to compact n-manifold without boundary.

In particular we have

Theorem 2'. *Suppose that (M, g) is a compact 4-manifold without boundary, then there exists a constant C_0 such that for every $u \in C^2(M)$*

$$\int_M \exp\left(\frac{32\pi^2 (u - \bar{u})^2}{\int_M (\Delta u)^2}\right) dV_g \leq C_0 V(M).$$

Corollary. *If M is a compact 4-manifold without boundary, then for every $\omega \in C^2(M)$,*

$$\log \frac{1}{Vol(M)} \int \exp(4(\omega - \bar{\omega})) \leq \log C_0 + \frac{1}{8\pi^2} \int_M (\Delta\omega)^2, \quad \textit{where } \bar{\omega} = \frac{1}{Vol(M)} \int \omega.$$

Suppose $k(M, g_0, L) < 32\pi^2$, then when $F_A(\omega)$ is bounded, we may apply the corollary above to show

$$\int_M (\Delta\omega)^2 + \int_M |\nabla\omega|^4$$

is bounded. The proof is in the same spirit as the proof of Onofri's inequality that $u \in S$ then $S[u]$ is bounded implies that $\int_M |\nabla u|^2$ is bounded. Details of the argument is contained in [B-C-Y].

Restricting our attention to $M = S^4$, we have that on (S^4, g_0), where g_0 the standard metric on S^4, then denote $v_0 = Vol(S^4)$, we have

$$F(\omega) = \text{constant} \cdot \log \frac{\det L_0}{\det L_\omega} = \left[\log \frac{1}{v_0} \int_{S^4} e^{4(\omega - \bar{\omega})} - \frac{1}{3}\frac{1}{v_0} \int_{S^4} (\Delta\omega)^2 - \frac{2}{3}\frac{1}{v_0} \int_{S^4} |\nabla\omega|^2\right]$$
$$+ \frac{2}{3}\{4\frac{1}{v_0} \int_{S^4} |\nabla\omega|^2 - \frac{1}{v_0} \int_{S^4} \left(\frac{\Delta e^\omega}{e^\omega}\right)^2\} = I + \frac{2}{3} II.$$

Theorem 3. [B-C-Y] *On (S^4, g_0), $F(\omega) \leq 0$, and $F(\omega) = 0$ if and only $e^{2\omega} g_0 = \phi^*(g_0)$ for some conformal transformation on S^4. That is, on S^4 log determinant L is extremal if and only if g is isometric to the standard metric.*

It is interesting to see that the proof of the Theorem 3 can be formulated into two extremal inequalities.

Lemma 1. *$I \leq 0$ on S^4 and $I = 0$ iff $e^{2\omega} g_0 = \phi^*(g_0)$.*

Lemma 2. *$II \leq 0$ on S^4 and $II = 0$ iff $e^{2\omega} g_0 = \phi^*(g_0)$.*

Lemma 1 is a special case of an inequality of Beckner which holds for general n. It is a linearized version of the Adams inequality just as Onofri's inequality is a linearized version of Moser's inequality.

Theorem 4. (Beckner [**B**]) quad *If $f \in C^\infty(S^n)$ has an expansion $\sum_{k=0}^\infty Y_k$ in spherical harmonic functions Y_k, then*

$$\log \frac{1}{V(S^n)} \int_{S^n} e^{(f - \bar{f})} \leq \frac{1}{2n} \sum_{k=1}^\infty B(n,k) \frac{1}{V(S^n)} \int_{S^n} |Y_k|^2$$

where $B(n,k) = \Gamma(n+k)/\Gamma(n)\Gamma(k)$ *and equality holds iff* $e^{2f/n}g_0 = \phi^*(g_0)$ *for some conformal transformation* ϕ *of* S^n.

The proof of the Beckner inequality is quite delicate and depends on the Fourier analysis method (Lieb-Young inequality).

Proof of Lemma 1 Take $n = 4$ in Beckner's inequality, in this case,

$$B(4,k) = \frac{\Gamma(3+k)}{\Gamma(3)\Gamma(k)} = \frac{k(k+1)(k+2)(k+3)}{6}$$

Recall that the k-th eigenvalue of Δ on S^4 is $\lambda_k = k(k+3)$. Thus if Y_k is a harmonic polynomial of degree k, then $\Delta Y_k + k(k+3)Y_k = 0$

Letting $f = e^{4\omega}$ we get :

$$\begin{aligned}
\log \frac{1}{V(S^4)} \int_{S^4} e^{4(\omega - \bar{\omega})} &\leq \frac{16}{8\cdot 6} \sum_{k=0}^{\infty} \int_{S^4} k(k+1)(k+2)(k+3)|Y_k|^2 \\
&= \frac{1}{3} \sum_{0}^{\infty} \langle -\Delta(-\Delta+2)Y_k, Y_k \rangle \\
&= \frac{1}{3} \int_{S^4} \omega(-\Delta(-\Delta+2)\omega), \\
&= \frac{1}{3}(\int_{S^4} (\Delta\omega)^2 + 2\int_{S^4} |\nabla\omega|^2).
\end{aligned}$$

Thus Lemma 1 is equivalent to Beckner's inequality on S^4.

Remark. Beckner's proof also indicates that

$$S_4(\omega) = \int (\Delta\omega)^2 + 2|\nabla\omega|^2 + 4\bar{\omega}$$

is a conformal invariant and $S_4(\omega) \geq 0$ if $\int e^{4\omega} = Vol(S^4)$. In particular we have

$$P_4\omega + 4!e^{4\omega} = 4! \text{ on } S^4 \iff \omega = \frac{1}{4}\log|J_\phi|$$

where the operator $P_4(\omega) = -\Delta(-\Delta+2)\omega$ has appeared in the literature previously and is called Paneitz's operator.

Proof of the Lemma 2 We will present a geometric proof here. Recall that Lemma 2 is equivalent to the inequality:

$$4\int_{S^4} |\nabla\omega|^2 \leq \int_{S^4} \left(\frac{\Delta e^\omega}{e^\omega}\right)^2$$

with "=" holding iff $e^{2\omega} g_0 = \phi^*(g_0)$.

Let $u = e^{\omega}$. Then u satisfies the Yamabe equation

$$-6\Delta u + Ru^3 = R_0 u = 12u$$

where R is the scalar curvature of the metric $e^{2\omega} g$.

Thus

$$\frac{\Delta e^{\omega}}{e^{\omega}} = \frac{\Delta u}{u} = -2 + \frac{R}{6} u^2.$$

We also observe that

$$\frac{\Delta e^{\omega}}{e^{\omega}} = \Delta \omega - |\nabla \omega|^2, \text{and hence} \int \left(\frac{\Delta e^{\omega}}{e^{\omega}} \right) = - \int |\nabla \omega|^2$$

Denote $V = Vol(S^4)$, we have

$$\begin{aligned} -II &= \frac{1}{V} \int \left(\frac{\Delta e^{\omega}}{e^{\omega}} \right)^2 - \frac{4}{V} \int |\nabla \omega|^2 \\ &= \frac{1}{V} \int (-2 + \frac{R}{6} u^2)^2 + 4 \frac{1}{V} \int (-2 + \frac{R}{6} u^2) = -4 + \frac{1}{V} \int_{S^4} \frac{R^2}{36} u^4. \end{aligned}$$

Recall the Yamabe functional (which we will give further discussion in Lecture 5) defined as:

$$\begin{aligned} Q(u) &= Q(S^4, g_0, u) \\ &= V^{1/2} \frac{\int |\nabla u|^2 + 2u^2}{(\int u^4)^{1/2}}, \end{aligned}$$

the infimum of $Q(u)$ is achieved only by conformal factors u of the form $u^2 g_0 = \phi^* g_0$ where ϕ is a conformal transformation of S^4. We observe that inequality II is scale invariant, thus we may assume w.l.o.g that $\int u^4 = V$. Hence applying Schwartz inequality and the Sobolev inequality in the Yamabe functional above, we get

$$\begin{aligned} -II &= -4 + \frac{1}{V} \int \frac{R^2}{36} u^4 \\ &\geq -4 + \frac{1}{V^2} \left[\int \frac{R}{6} u^2 \right]^2 \geq 0, \end{aligned}$$

with equality holding if and only if $u^2 g_0 = \phi^* g_0$ for some conformal transformation ϕ. This finishes the proof of lemma 2 and hence the proof of Theorem 3.

Beckner's inequality highly suggests that the results above for 4-manifolds holds for all even dimensional manifolds. Yet so far there is difficulty to compute the precise logdeterminant formula for higher dimensional manifold.

Remark added after the lecture:

More recently, results in this section have been extended substaintially in many directions. In [B-G], Polyakov-Ray-Singer type formula for functional determinant has been worked out for 4-manifolds with boundary. In [Br], Theorem 3 in this chapter has been extended to

$$(S^6, g_0)$$

. And in [C-Y], the zeta functional determinant F[w] as defined in this chapter has been studeied for general compact 4-manifolds. Under condition similar to that of Theorem 1 (i.e. $k < 32\pi^2$), existence for extremal metric for F[w] has been established and proved to satisfy some sharp Moser-Trudinger type inequality. Also in [C-Y],a different proof of Beckner's inequality (Theorem 4 in this chapter) was given,the proof relies on the conformal invariant property of the Paneitz operator amd Adam's inequality instead of the sharp Lieb-young inequality.

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Chapter 6

Isospectral Compactness on 3-manifolds and Relation to the Yamabe Problem

In Chapter 3 and 5, we have discussed the compactness results for isospectral families of conformal metrics for compact, closed manifolds of dimension 2 and 4 ([O-P-S-1] and [B-C-Y]) and the extremal metric of the log-determinant of the (conformal) Laplacian operator on S^2 and S^4 ([O] [B-C-Y]). In this chapter, we will briefly discuss some progress which has been made for these problems on 3-manifolds.

Recall that we have mentioned in the previous chapter that [P-R] when n is odd, $L_g =$ conformal Laplacian operator w.r.t. the metric g, then $\log \det L_g$ is a conformal invariant quantity when one changes the metric g in the same conformal class. Thus in particular, $F[\omega] = \log \frac{\det L_{g_\omega}}{\det L_g}$ for $g_\omega = e^{2\omega} g$ does not carry any information on ω.

Recall also that in the case of dimension $n = 2$, to achieve the C^∞ compactness for isospectral family of conformal metrics $\{g_\omega = e^{2\omega} g\}$ on compact surfaces, one of the chief strategies in [O-P-S-1] is to use the spectral information in $F[\omega] = \log \frac{\det \triangle_{g_\omega}}{\det \triangle_g}$ to control the $W^{1,2}$ norm of ω, then use the information in the k-th coeffient a_k of the heat kernel expansion $e^{-\triangle t} \sim \sum_{k=0}^{\infty} a_k t^{k-\frac{n}{2}}$ to control the $W^{k,2}$ norm of ω for $k \geqq 2$.

It turns in the case of dimension 2 when restricting metrics to a fixed conformal class, one can also give an alternative argument (using λ_1 to replace the log-determinant of the Laplacian) to show that isospectral conformal metrics on compact surfaces form a compact set in the C^∞-topology

Theorem 1 *Suppose (M, g_0) is a compact surface, $\{e^{2u_j}\}$ is a sequence of conformal* *factors on M with*

(1) $\int e^{2u_j}\ dV_0 = a_0$

(2) $\int K_j^2 e^{2u_j} dV_0 = a_2 < \infty$ *where* K_j = *Gaussian curvature of the metric* $e^{2u_j} g_0$.

And assume in addition that, the first eigenvalue λ_1 of the Laplacian w.r.t. to the metrics $e^{2u_j} g_0$ are bounded from below by $\Lambda > 0$, i.e.

(3) *For each function ϕ defined on M*

$$\int_M \phi^2 e^{2u_j}\ dV_0 \le \left(\int_M \phi e^{2u_j}\ dV_0\right)^2 \Big/ \left(\int_M e^{2u_j}\ dV_0\right) + \frac{1}{\Lambda}\int_M |\nabla_0\ phi|^2\ dV_0.$$

Then either (a) and (b): in case K_0 (Gaussian curvature of the metric g_0) is < 0, or $= 0$ respectively; $\{u_j\}$ forms a bounded family in W_2^1 (i.e. $\sup_n \int_M |\nabla u_j|^2 dV_0$ is finite) or (c): in case $K_0 = 1$ and $(M, g_0) = (S^2, g_0)$ with g_0 = surface measure on S^2, then the isometry class of u_j forms a bounded family in $W^{1,2}$. In the setting of Theorem 1 one can generalize the compactness result to compact manifolds of dimension 3 ([C-Y-2]).

Theorem 2 *Let $g_j = u_j^4 g_0$ be a sequence of conformal metrics satisfying the following conditions:*

(i) $Vol(M, g_j) = \alpha_0$ *for some positive constant* α_0.

(ii) $\int R^2(g_j) + |\rho(g_j)|^2\ dV_j \le \alpha_2$, *for some positive constant α_2 where $R(g_j)$ is the scalar curvature of g_j and ρ is the Ricci tensor of g_j, and $dV_j = u_j^6\ dV_0$,*

(iii) $\lambda_1(g_j)$, *the lowest eigenvalue of the Laplacian of the metric g_j, has a positive lower bound: $\lambda_1(g_j) \ge \Lambda > 0$; i.e. For each ϕ defined on M, we have*

$$\left(\int_M \phi^2 dV_j\right) \le \left(\int_M \phi\ dV_j\right)^2 \Big/ \left(\int_M dV_j\right) + \frac{1}{r}\int_M |\nabla_j \phi|^2\ dV_0.$$

Then there exists constants c_1, c_2 so that

(a) $\quad c_1 \le u(x) \le \dfrac{1}{c_1},$

(b) $\quad ||u_j||_{2,2} \le c_2;$

except in the case where (M, g_0) is the standard 3-sphere. In the latter case, the isometry class of u_j satisfy (a) and (b).

Theorem 2 was proved in the special case when R_0 is negative by [B-C-Y] and when $(M, g_0) = (S^3, g_0)$ in [C-Y-1] [C-Y-3], see also Theorem 3 below. Notice that in dimension

$n = 3$ (actually for $3 \leqq n \leqq 6$, cf [K], page 3), one can make a explicit computation in the firmula of McKean and Singer [M-S] in Chapter 2, and obtain on (M, g_0)

$$\begin{aligned} a_0 &= \text{volume of } g_0 \\ a_1 &= \text{constant} \int R dV_0 \\ a_2 &= A_2 \int R^2 dV_0 + B_2 \int |\rho|^2 dV_0 \;\; A_2, B_2 > 0 \end{aligned}$$

Furthermore [8] for $k \geqq 3$

$$a_k = A_k \int |\nabla^{k-2} R|^2 dV_0 + B_k \int |\nabla^{k-2}\rho|^2 dV_0 + \text{lower order terms},$$

with $A_k, B_k > 0$. Thus the condition (i) (ii) (iii) in Theorem 2 all are spectral in formation, and once one controls $W^{2,2}$-norm of the conformal factor $\{u_j\}$ for an isospectral sequence of metrics $g_j = u_j^4 g_0$, one can gain the control of $W^{k,2}$-norm of $\{u_j\}$ through the information in a_k, and we get as a corollary of Theorem 2.

Corollary 3 *An isospectral set of conformal metrics on a compact 3-manifolds is compact in the C^∞-topology.*

Since conformal transformations preserv the spectral information another immediate corollary of Theorem 2 is the following result of Obata for the special case $n = 3$.

Corollary 4 *The conformal group of a compact 3-manifold is non-compact if and only if it is comformally equivalent to the standard 3-sphere.*

Before we discuss the ideas in the proof of Theorem 2, we would like to mention a result of Gursky [G] which generalize Theorem 2 to general dimensions. First we remark that, in the statement of Theorem 1 (for compact surface), it is well-known that the "local information" (i.e. those computed using local coordinates in the metric g) like that in conditions (1) and (2) in Theorem 1 are not enough to conclude the $W^{1,2}$ compactness of $\{\omega_j\}$ for the metrics $\{e^{2\omega_j} g_0\}$. This can be seen through the famous "Dumbell" surface with the neck getting thinner and thinner. Notice that for such surfaces, using Rayleigh quotients, it is easy to see that $\lambda_1 \to 0$. Thus it comes as a surprise that when $n \geqq 3$, some "local information" (e.g. size of curvature tensor of the metric) suffice to provide the compactness of the metrics. This is the content of the following theorem of Gursky [G].

Theorem 3 *Let (M, g_0) be a compact manifold without boundary. Suppose that $g_j = u_j^{\frac{4}{n-2}} g_0$ is a sequence of metrics satisfying*

(1) $\int u_j^{\frac{2n}{n-2}} dV_0 = Vol(g_j) \leqq \alpha_0$

(2) $\int |Rm(g_j)|^p dV_j \leqq \beta$ *for some* $p > \frac{n}{2}$,

where $Rm(g_j)$ is the full curvature tensor of g_j. Then there exist $c_1, c_2 > 0$ so that $\frac{1}{c_1} \leqq u_j \leqq c_1$ and $\|u_j\|_{W^{2,p}} \leqq c_2$, unless $(M, g_0) = (S^n, g_0)$, in that case the conclusion holds in the isometry class of u_j.

One would like to point out that the exponent $p > \frac{n}{2}$ is necessary in Theorem 3. In a recent article [C-G-W], two examples have been constructed to indicate that Theorem 3 fails when $p = \frac{n}{2}$.

Gursky's proof of Theorem 3 is quite ingenious. One of the key idea is to apply condition (2) to Bochner's formula to start Nash-Moser iteration process and to establish "Harnack type" inequality for sequence $\{u_j\}$. In this lecture, instead of proving the general result [8], we give a complete proof some special cases of Theorem 2 to convey the idea how to use the condition $\lambda_1 \geqq \Lambda > 0$ in the compactness result. We start with a definition.

Definition *We say a sequence of positive function $\{u_j\}$ satisfies condition (*), if there exist $l_0, r_0 > 0$ so that for all j*

$$(*) \qquad \int_{\{u_j(x) \geqq r_0\}} dV_0 \geqq l_0 \int_M dV_0$$

Theorem 2' *Suppose $\{u_j\}$ is a sequence on (M^3, g_0) satisfying*

(1) $vol(g_j) = \int u_j^6 dV_0 = \alpha_0$

(2)' $\int_M R_j^2 u_j^6 \leqq \alpha_2$ $R_j =$ *scalar curvature w.r.t.* g_j

(3) $\lambda_1(g_j) \geqq \Lambda > 0$ *where λ_1 is the first eigenvalue of the Laplacian operator.*

And if in addition, $\{u_j\}$ satisfies condition $(*)$ then there exist some $\epsilon_0 > 0$ and a constant C_0 depending only on the data $\alpha_0, \alpha_2, \Lambda, l_0, r_0$ so that

$$(4) \qquad \int_M u_j^{6+\epsilon_0} \leqq C_0$$

Remark

1. The underlying analysis of the compactness results in Theorem 2 and Theorem 2' is the optimal Sobolev inequality:

$$Q(M)\left(\int_M u^6 \, dV_0\right)^{1/3} \leq 8\int_M |\nabla u|^2 \, dV_0 + \int_M R_0 u^2 \, dV_0.$$

The optimal constant $Q(M)$ is an invariant of the conformal class of M. For a conformal metric $g = u^4 g_0$, its scalar curvature R is given by the equation

$$8\Delta u + Ru^5 = R_0 u \qquad \text{on} \qquad M. \tag{5}$$

Thus the Sobolev x_0 quotient

$$Q[u] = \frac{\int (8|\nabla u|^2 + R_0 u^2) \, dV_0}{(\int u^6 dV_0)^{1/3}}$$

is exactly given by $\int Ru^6 dV_0$ if the volume is held to be 1, (i.e. $\int u^6 dV_0 = 1$). The celebrated recent solution of Yamabe's problem ([A], [S]) asserts that (a) $Q(M) < Q(S^3)$ unless M is conformally S^3 and (b) a minimizing sequence for $Q[u]$ is compact if $Q(M) < Q(S^3)$. Thus in our compactness assertion, we have substituted an L^2 bound for the curvature in place of the condition $Q[u_j] < Q(S^3)$, and substituted the condition $\lambda_1(g_j) \geq \Lambda > 0$ in place of the minimizing property for $Q[u]$.

2. As we have mentioned in Chapter 4, one fact which has a key role in Yamabe problem is that the embedding $W^{1,2} \subset L^{\frac{2n}{n-2}}$ is not compact. On (S^3, g_0), if we adopt coordinates on S^3 through its stereographic projection mapping with pole of S^3 to $O = (0,0,0)$ in R^3, then in this coordinates system, volume form dV_0 on S^3 is defined by

$$dV_0 = (\frac{2}{1+|x|^2})^3 \, dx$$

For each $a > 0$, consider the function

$$u_a(x) = (\frac{a(1+|x|^2}{|x|^2+|a|^2})^{\frac{1}{2}}$$

then $\{u_a\}$ is a family of function with

$$6(2\pi^2) = Q(S^3)(\int_{S^3} u_a^6 dV_0)^{\frac{1}{3}} = 8\int_{S^3} |\nabla u_a|^2 dV_0 + 6\int_{S^3} u_a^2 dV_0,$$

where $Q(S^3) = 6(2\pi^2)^{\frac{2}{3}}$. Since $u_a(x) \to 0$ uniformly on compact subset off $x = 0$, $\{u_a\}$ is a typical example of family of functions which indicate the failure of compactness of the inclusion $W^{1,2} \subset L^6$ on S^3. Notice that $\{u_a\}$ does not satisfy comdition $(*)$ in our Theorem 2', and also for all $\epsilon > 0$, $\int_{S^3} u_a^{6+\epsilon} dV_0 \to \infty$ as $a \to 0$. Thus it is convincing that a sequence $\{u_j\}$ satisfies condition (1), (2), (3), (4) in Theorem 2' is a sequence which is bounded in the sense of the conclusions (a) and (b) in Theorem 2; i.e. there exist constants c_1, c_2 so that $c_1 \leqq u_j \leqq \frac{1}{c_1}$ and $\|u_j\|_{2,2} \leqq c_2$. Indeed, using an iteration argument (Lemma 3.4 in [C-Y-1]) one can verify this latter fact. Thus Theorem 2' is a special case of Theorem 2.

3. Examples of (M, g_0) for which conditiopn $(*)$ automaticallyfollows from assumptions (1), (2) and (3).

(a) When $R_0 < 0$ (we may assume w.r.l.g. R_0 is a constant), then

$$\int_M R_j^2 u_j^6 dV_0 = \int_M \left(\frac{8 \triangle u_j - R_0 u_j}{u_j^5}\right)^2 u_j^6 dV_0$$

$$= \int_M \left(\frac{\triangle u_j}{u_j}\right)^2 dV_0 - 16 R_0 \int_M \frac{\triangle u_j}{u_j^3} dV_0 + R_0^2 \int_M u_j^{-2} dV_0$$

Since $\int_M \frac{\triangle u_j}{u_j^3} dV_0 = 3 \int_M |\nabla u_j|^2 u_j^{-4} dV_0 \leqq 0$, we have

$$\int_M R_j^2 u_j^6 dV_0 \leqq \alpha_2, \text{so } R_0^2 \int_M u_j^{-2} dV_0 \leqq \alpha_2.$$

And it is easy to check that for $\{u_j\}$ satisfies $\int_M u_j^{-2} dV_0 \leqq \beta$, $\{u_j\}$ satisfies condition $(*)$.

(b) On (S^3, g_0), given a sequence $\{u_j\}$ satisfying (1), (2), (3) we can find v_j in the same isometry class as u_j with v_j satisfying (1), (2) ,(3) and condition $(*)$. To see this, given u_j, we apply Lemma 2 in Chapter to obtain $v_j = T_{\phi_j} u_j$ for some conformal transformation ϕ_j of S^3 and with $v_j \in S$, then $\lambda_1(u_j^4 g_0) = \lambda_1(v_j g_0)$, and $\{v_j\}$ satisfies (1), (2), (3) with the same constants α_0, α_2. To see $\{v_j\}$ satisfies condition $(*)$ we apply $\phi = x_k, k = 1, 2, 3, 4$ in the Rayleigh quotient of $\lambda_1(v_j g_0)$ and get $dV_j = v_j dV_0$

$$\int_{S^3} x_k^2 dV_j \leqq \frac{1}{\lambda_1} \int_{S^3} |\nabla_j x_k|^2 dV_j = \frac{1}{\lambda_1} \int_{S^3} |\nabla_0 x_k v_j^2 dV_0 \tag{6}$$
$$\leqq dV_j = \frac{1}{\Lambda} \int_{S^3} |\nabla_0 x_k v_j^2 dV_0$$

Sum (6) over $k = 1, 2, 3, 4$ we get

$$\alpha_0 \leqq \frac{3}{\Lambda} \int_{S^3} v_j^2 dV_0$$

Thus $\{v_j\}$ satisfies condition $(*)$.

Proof of Theorem 2'

Denote $u = u_j$, we will show that u satisfies (4) with constant depending only on $\alpha_0 \alpha_2, \Lambda, l_0, r_0$.

Multiplying the equation (5) by u^β ($\beta > 1$ to be chosen later) we have (denote $\int = \int_M dV_0$) for $w = u^{\frac{1+\beta}{2}}$

$$8 \frac{4\beta}{(1+\beta)^2} \int |\nabla w|^2 + R_0 \int w^2 = \int R u^4 w^2 \tag{7}$$

We will now apply our assumptions (1), (2), (3) to estimate the term $I = \int R u^4 w^2$. Taking a suitably large number b (again to be chosen later) on the region $|R| \geqq b$ we have

$$b^2 \int_{|R| \geqq b} u^2 dV_0 \leqq \int_{|R| \geqq b} R^2 u^6 dV_0 \leqq \alpha_2$$

Thus

$$\begin{aligned} \int_{|R| \geqq b} R u^4 w^2 &\leqq \left(\int R^2 u^6 \right)^{1/2} \left(\int_{|R| \geqq b} u^6 \right)^{1/6} \left(\int w^6 \right)^{1/3} \\ &\leqq \alpha_2^{1/2} \left(\frac{\alpha_2}{b^2} \right)^{1/6} \left(\int w^6 \right)^{1/3}. \end{aligned} \tag{8}$$

For the remaining part of the proof, we will apply condition $(*)$ in the statement of Theorem 2'.

For $dV = v^6 dV_0$, we have from the Rayleigh-Ritz characterization for λ_1,

$$\int_M \Phi^2 dV \leqq \left(\int_M \Phi dV \right)^2 \Big/ \left(\int dV \right) + \frac{1}{\lambda_1} \int_M |\nabla_u \Phi|^2 dV \tag{9}$$

where $|\nabla_u \Phi|^2 dV = |\nabla \Phi|^2 u^2 dV_0$. We will denote $E_\gamma = \{x \in M, u(x) \geqq \gamma\}$ and $|E_\gamma| = \int_{E_\gamma} dV_0$. By assumption $(*)$, there exist some $\gamma_0, l_o > 0$ so that $|E_{\gamma_0}| \geqq l_0 | \int_M dV_0|$. Applying (9) and (3) to $\Phi = u^\epsilon$ with $\beta = 1 + 2\epsilon$ and ϵ small, we have

$$\int u^{6+2\epsilon} dV_0 \leqq \left(\int u^{6+\epsilon} dV_0 \right)^2 \Big/ \left(\int u^6 dV_0 \right) + \frac{1}{\Lambda} \int |\nabla u^\epsilon|^2 u^2 dV_0. \tag{10}$$

For simplicity, we will now normalize u and assume that $\alpha_0 = \int u^6 dV_0 = 1$. We may then estimate the term $\int^{6+\epsilon} dV_0$ as

$$
\begin{aligned}
\int u^{6+\epsilon} dV_0 &= \int_{E_{\gamma_0}} u^{6+\epsilon} dV_0 + \int_{E^c_{\gamma_0}} u^{6+\epsilon} dV_0 \\
&= \int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^\epsilon dV_0 + \int_{E_{\gamma_0}} \gamma_0^6 u^\epsilon dV_0 + \int_{E^c_{\gamma_0}} u^{6+\epsilon} dV_0 \\
&\leqq \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^{2\epsilon} dV_0 \right)^{1/2} \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) dV_0 \right)^{1/2} + C(\gamma_0),
\end{aligned}
$$

where $C(\gamma_0)$ is a constant depending only on γ_0 and $\int dV_0$. Thus, for each $\eta > 0$ we have

$$
\begin{aligned}
\left(\int u^{6+\epsilon} dV_0 \right)^2 &\leqq (1+\eta) \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) u^{2\epsilon} dV_0 \right) \left(\int_{E_{\gamma_0}} (u^6 - \gamma_0^6) dV_0 \right) \\
&\quad + \left(1 + \frac{1}{\eta} \right) C^2(\gamma_0) \\
&\leqq (1+\eta)(1 - \gamma_0^6 |E_{\gamma_0}|) \left(\int u^{6+2\epsilon} dV_0 \right) \\
&\quad + \left(1 + \frac{1}{\eta} C^2(\gamma_0) \right)
\end{aligned}
\tag{11}
$$

(we may assume w.l.o.g. that γ_0 is small and $\gamma_0 |E_{\gamma_0}| \ll 1$).

Since by our assumption on $|E_{\gamma_0}|$ we have $\gamma_0^6 |E_{\gamma_0}| \geqq \gamma_0^6 l_0 > 0$, we may choose η so that $(1+\eta)(1 - \gamma_0^6 |E_{\gamma_0}|) \leqq 1 - \delta$ for some positive δ, $\delta = \delta(\gamma_0, l_0)$ and obtain from (10), (11)

$$
\delta \int u^{6+2\epsilon} dV_0 \leqq C(\gamma_0, l_0) + \frac{1}{\Lambda} \int |\nabla u^\epsilon|^2 u^2 dV_0, \tag{12}
$$

where again $C(\gamma_0, l_0)$ is a constant depending only on γ_0, l_0.

¿From this point on, we may estimate the term $\int |\nabla u^\epsilon|^2 dV_0$ as follows:

$$
\int |\nabla u^\epsilon|^2 u^2 dV_0 = \frac{\epsilon^2}{(1+\epsilon)^2} \int |\nabla u^{1+\epsilon}|^2 dV_0
$$

and notice that for $\beta = 1 + 2\epsilon$, $w = u^{\frac{1+\beta}{2}} = u^{1+\epsilon}$. Thus, combining (7) and (12) we have

$$
\int u^{6+2\epsilon} dV_0 \leqq \frac{\epsilon^2}{\delta \Lambda (1+\epsilon)^2} \frac{(1+\epsilon)^2}{8(1+2\epsilon)} I + L, \tag{13}
$$

where

$$
I = \int R u^4 w^2 \text{ and } L = 0 \left(\frac{R_0}{\delta} \int u^{2+2\epsilon} dV_0 \right) + \frac{1}{\delta} C(\gamma_0, l_0).
$$

Combining (11), (13) with (8), we find

$$
\begin{aligned}
(14) \qquad I = \int R u^4 w^2 &\leqq \left(\frac{\alpha_2^2}{b}\right)^{1/3} \left(\int w^6 dV_0\right)^{1/3} + b \int u^4 w^2 dV_0 \\
&\leqq \left(\frac{\alpha_2^2}{b}\right)^{1/3} \left(\int w^6 dV_0\right)^{1/3} + \frac{b\epsilon^2}{8\Lambda} I + bL
\end{aligned}
$$

so that

$$
(15) \qquad \left(1 - \frac{b\epsilon^2}{8\Lambda}\right) I \leqq \left(\frac{\alpha_2^2}{b}\right)^{1/3} \left(\int w^6 dV_0\right)^{1/3} + bL.
$$

Now choosing b sufficiently large so that $(\frac{\alpha_2^2}{b}) < \frac{1}{2}Q$, and then choosing ϵ sufficient small, we get

$$
\begin{aligned}
\frac{3}{4} Q \left(\int w^6 dV_0\right)^{1/3} &\leq I + |R_0| \int w^2 dV_0 \\
&\leq \frac{2}{3} Q \left(\int w^6 dV_0\right)^{1/3} + \frac{4}{3}\ bL + |R_0| \int w^2\ dV_0.
\end{aligned}
$$

Recall $w = u^{1+\varepsilon}$, hence

$$
\begin{aligned}
\left(\int u^{6+6\varepsilon} dV_0\right)^{1/3} = \left(\int w^6 dV_0\right)^{1/3} &< 16\ bL + 12\ |R_0| \int u^{2+2\varepsilon}\ dV_0 \\
&\leq C(b, |R_0|) \left(\int u^6 dV_0\right)^{(2+2\varepsilon)/6} \left(\int dV_0\right)^{(4-2\varepsilon)/6} \\
&= C_0 < \infty.
\end{aligned}
$$

This proves Theorem 2' with $\varepsilon_0 = 6\varepsilon$.

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Chapter 7

Prescribing Curvature function on S^n

In this chapter we will describe as an application to the Sobolev-Trudinger-Moser type inequalities a problem in prescribing curvature functions on the n-sphere.

We will start with a question originally proposed by L. Nirenberg, what function is allowed to be the Gaussian curvature function on the 2-sphere S^2?

Applying the uniformization Theorem, the question can be decribed in analytic term as: Denote the metric with Gaussian curvature K as $g = e^{2w} g_0$, then K and w satisfy the following equation

$$\triangle w + Ke^{2w} = 1 \quad \text{on } S^2 \tag{1}$$

where $\triangle$ denote the Laplacian operator related to the standard metric. The question is: which function K can be prescribed so that (1) has a solution?

In this chapter we will discuss some basic facts, techniques and progresses made to the above question on the 2-sphere. Before we do that, wewill mention some other related questions of prescribing curvature functions. It turns out many of the techniques for the S^2 problem, which we will mention below, can be applied, but there are also other new pheonomenon, which we will not cover, to these other questions. We do not intend to give a survey of the progress made in this fast developing field, and will mention only a few relevant results related to the techniques of the 2-sphere problem discussed here. The reader is referred to [S-2] and [C-Y-G] for more complete references on the subject.

On general compact manifolds M^n $n \geqq 3$ without boundary, an analogous question to (1) is that of prescribing scalar curvature problem. In this case, when restricting metrics conformal to a fixed one g_0, denote the metric with scalar curvature R as $g = u^{\frac{4}{n-2}} g_0$, the R and u satisfy the following equation

$$c_n \triangle u + R u^{\frac{n+2}{n-2}} = R_0 u \tag{2}$$

where $c_n = \frac{4(n-1)}{n-2}$, $\triangle$ denote the Laplacian operator w.r.t. the metric g_0 and R_0 is the scalar curvature og g_0. The question is: Which function R can be prescribied so that equation (2) has a solution? When $R =$ constant, this is the Yamabe problem mentioned earlier, and the question has been solved in the affirmative way by [A] [S-1].

Another analogous question, whose precise geometric meaning is only well-fomulated on 4-manifolds at this moment, is the problem of prescribing "curvature" for Paneitz operator. Paneitz operator was introduced by Paneitz ([P], see also [S-1]), on 4-manifolds, it is a fourth order differential operator P_g defined as

$$P_g = (-\triangle)^2 + \delta(\frac{2}{3}R - 2R_{ij}g_{ij})d$$

where R_{ij} is the Ricci tensor, δ is the divergence, d is the differential w.r.t. the metric $g = (g_{ij})$. This operator enjoys the following good properties

(a) It is "conformally invariant", in the sense that: if $g_w = e^{2w}g$ is a metric conformal to g, then $P_{g_w} = e^{-4w}P_g$.

(b) Let Q denote the "curvature" function

$$Q = \frac{1}{6}(-3|\rho|^2 + R^2 - \triangle R)$$

where $|\rho|^2 =$ square norm of Ricci tensor, the P_g and Q_g are related as

$$-P_g w + Q_{g_w} e^{4w} = Q_g \quad \text{on } M. \tag{3}$$

It is in the context of equation (3) when comparing equation (1) and (2) that a natural question to propose is that of prescribing "curvature" Q as defined by (3).

n manifold M of dimension not equal to four, in general the existence and precise form of Paneitz operator with properties (a) and (b) is unknown. But in a recent article [GJML], it is proved that when n is even, operator of order n with conformal invariant property as (a) exists; also when $M = R^n$ with the Euclidean metric the operator $P_n = (-\triangle)^{\frac{n}{2}}$ has the property. Using stereographical projection pulling S^n to R^n, one can then compute the form of P_n on S^n. It turns out that this is precisely the operator which has appeared in the Beckner's [B] inequality (which we have discussed in Chapter 5). And the precise form of P_n on S^n is

$$\begin{aligned} P_n &= \prod_{k=0}^{\frac{n}{2}}(-\triangle + k(n-k-1)) \quad \text{when } n \text{ is even} \\ &= (\triangle + \frac{n-1}{2})^{\frac{1}{2}} \prod_{k=0}^{\frac{n-3}{2}}(-\triangle + k(n-k-1)) \quad \text{when } n \text{ is odd.} \end{aligned}$$

In view of (1), (2) and (3), an analogous question to ask is that of prescribing Q in the following equation

$$-P_n w + Qe^{nw} = (n-1)! \quad \text{on } S^n. \tag{4}$$

It turns out equation (4) is a more natural extension to equation (1) than that of equation (3) of the problem of prescribing scalar curvature. Some point of view of this is explained in a recent article [C-Y].

We will now return to equation (1) -the problem of prescribing Gaussian curvature on S^2. We first remark that integrating equation (1) over S^2, we get (we denote $⨍ = \frac{1}{4\pi}\int_{S^2} dV_0$)

$$⨍ Ke^{2w} = 1$$

Thus $K > 0$ somewhere is a necessary condition for (1) to be solvable. To explore other propertyt of K, we will use variational approach. Consider the variation functional F_K associated with (1):

$$F_K[w] = \log ⨍ Ke^{2w} - S[w]$$

where $S[w] = ⨍|\nabla w|^2 + 2⨍ w$ as in Chapter 2.

Basic facts of $F_K[w]$

(i)
$$\begin{aligned} F_K[w] &\leqq \log \max K + F_1[w] \\ &\leqq \log \max K, \end{aligned}$$

via Onofri's inequality in Chapter 3.

(ii)
$$\textit{When } K > 0, \quad \sup_{w \in W^{1,2}} F_K[w] \quad \textit{is attained iff } K \equiv \textit{constant.}$$

Proof When $u = \frac{1}{2}\log|J_\phi|$ for some conformal transformation ϕ of S^2, then u satisfies $\triangle u + e^{2u} = 1$ and $K \equiv 1$, thus, when $K = c > 0$, $u = \frac{1}{2}\log|J_\phi| + \log c)$ is a nontrivial solution of (1) for which $F_c[u] = \log c$.

In general, suppose u is a local maximum of the functional $F_K[u]$, a direct computation yields:

$$0 \geqq \frac{d^2}{dt^2} F_K[u + tv]\bigg|_{t=0} = 2\left[⨍ Ke^{2u}v^2 - (⨍ Fe^{2u}v)^2\right] - ⨍|\nabla v|^2$$

for all $v \in W^{1,2}$. This implies that the first non-zero eigenvalue λ_1 of the Laplacian of the metric $Ke^{2u}g_0$ satisfies $\lambda_1 \geqq 2$. While the result of Hersch in Chapter 1 says $\lambda_1 \leqq 2$ with equality if and only if the metric $Ke^{2u}g_0$ has constant curvature 1. This coupled with the assumption that K, u satisfy (1) is equivalent to $K \equiv$ constant.

Thus the only way to solve equation (1) using the variational approach is to locate saddle point of the functional $F_K[w]$.

(iii) *Kazdan-Warner condition ([K-W]]): if K, u satisfy (1) then*

$$\fint < \nabla K, \nabla x_j > e^{2u} = 0 \quad for \quad j = 1, 2, 3. \tag{5}$$

Proof Recall Lemma 1 in Chapter 3, $S[w]$ is a conformally invariant quantity in the sense that $S[T_\phi(u)] = S[u]$, where $v = T_\phi(u)$ denote the induced conformal factor given by $\phi^*(e^{2u}g_0)e^{2v}g_0$, ϕ is a conformal transformation of S^2.

Suppose u is a critical point of $F_K[u]$, hence

$$\frac{d}{dt} F_K[T_{\phi_{P,t}}]\Big|_{t=0} = 0$$

where $P \in S^2, t \geqq 1$, $\phi_{P,t}$ is the conformal transformation on S^2 as defined in Lemma 3 of Chapter 3. But

$$\begin{aligned} F_K[T_{\phi_{P,t}}(u)] &= \log \fint K e^{2T_{\phi_{P,t}}(u)} - S[T_{\phi_{P,t}}(u)] \\ &= \log \fint K e^{2T_{\phi_{P,t}}(u)} - S[u] \\ &= \log \fint K e^{2u \circ \phi_{P,t}} \det |J_{\phi_{P,t}}| - S[u] \\ &= \log \fint (K \circ \phi_{P,t}^{-1}) e^{2u} - S[u]. \end{aligned}$$

Thus

$$\frac{d}{dt} F[T_{\phi_{P,t}}(u)]\Big|_{t=1} = \fint \frac{d}{dt}(K \circ \phi_{P,t}^{-1})\Big|_{t=1} e^{2u}.$$

Now a simple calculation shows that

$$\frac{d}{dt} K \circ \phi_{P,t}^{-1} = < \nabla K, \nabla x_3 >, \quad \text{if} \quad x_3 = x \cdot P.$$

This gives the desired condition

(iv) *K, u satisfy equation (1) if and only if $K \circ \phi, T_\phi(u)$ satisfy equation (1) for any conformal transformation ϕ of S^2.*

¿From the geometric interpretation of K, this is an obvious fact. It can also be checked directly.

(v) *(Moser [M-2]) When K is an even function (i.e. $K(\xi) = K(-\xi)$ fro all $\xi \in S^2$), (1) has an even solution.*

Proof Recall that for even function u defined on S^2, the best constant in Moser's inequality is $\alpha = 8\pi$ (Remark 3 in Chapter 3). Thus following the argument as in Chapter 3 after Remark 3, we have

$$\text{for } u \text{ even,} \quad \log \fint e^{2u} \leqq \frac{1}{2} \fint |\nabla u|^2 + 2 \fint u + C_2, \tag{6}$$

for some constant C_2.

¿From (6), the same argument as the derivation of Onofri's inequality as in Chapter 3 gives an even solution for equation (1) in case K is an even function.

We now describe an a priori estimate for solution of (1), which we think is a key step to the prescribing curvature problem.

Theorem 1 *Suppose K, w satisfy equation (1).*
(a) Assume that $0 < m \leqq K \leqq M$, then

$$S[w] \leqq C(m, M).$$

(b) Assume further that K is a nondegenerate function in the following sense

$$\triangle K(Q) \neq 0 \quad \textit{whenever} \quad \nabla K(Q) = 0 \tag{nd}$$

then there exists a constant $C = C(m, M, (nd)) > 0$ so that

$$\frac{1}{C} \leqq w \leqq C, \quad \textit{and} \quad \|w\|_{W^{1,2}} \leqq C.$$

Remarks

1. Considering fact (iv), bounds for $S[w]$ is the best one can hope for (instead of the energy bound $\int |\nabla w|^2$) for a given K. For the sequence $w_t = \frac{1}{2} \log |J_{\phi_{P,t}}|$ for some fixed $P \in S^2$, $K \equiv 1, S[w_t] \equiv 0$ while $\int |\nabla w|^2 \to \infty$ as $t \to 0$.
2. Part (a) of Theoerm 1 has appeared in [C-Y-6], part (b) is a special case of a result in [C-G-Y], where similar a priori estimate for the prescribing scalar curvature equation on S^3 was also obtained. One should also mention, using a different approach (i.e. local estimate), result analogous to Theorem 1 on S^3 proved in [D].
3. Result [C-Y] with similar statement as in part (a) also holds for the Paneitz equation (4) on S^n, with $S[w]$ replaced by a suitable defined conformal invariant term $S_n[w]$.

Proof of part (a) :

What we actually will establish is:

(a)' If $w \in S$ and satisfies the equation (1), then there exists a constant $C = C(m, M)$ with

$$\frac{1}{C} \leqq w \leqq C, \quad \text{and} \quad \fint |\nabla w|^2 \leqq C.$$

In view of Lemma 2 in Chapter 3 and (iv) above, statements (a) and (a)' are equivalent. To proceed the proof of (a)' we will first state the following result, which is a modified form of Aubin's result Lemma 3 in Chapter 3.

Lemma 2 *Given $w \in S$, there exists a constant $C(K)$ such that*

$$\fint e^{cw} \leqq C(K) \exp\left(\frac{c^2}{8} \fint |\nabla w|^2 + c \fint w\right) \tag{7}$$

where c is a real number, (w, K) satisfies equation (1).

Proof of (a)': Aubin's Lemma shows that for all $\varepsilon > 0$

$$\fint e^{cw} \leq C(\varepsilon) \exp\left(\left(\frac{c^2}{8} + \varepsilon\right) \fint |\nabla w|^2 + c \fint w \right) \tag{8}$$

for some constant $C(\varepsilon)$. But when one examines the proof of Lemma 3 in Chapter 3 ([A]), one notices that the reason the ε-term $\varepsilon \fint |\nabla w|^2$ appeared in (8) is to absorbe the term $\fint |\nabla w|$. Thus (7) is a consequence of (8) with the observation that when w, K satisfies (1) one can easily verify $\fint |\nabla w| \leq C(K)$.

We multiply equation (1) by $2w$ and integrate and apply Jensen's inequality (recall $\fint K e^{2w} = 1$)

$$\begin{aligned} 2\fint |\nabla w|^2 + 2\fint w &= 2\fint K e^{2w} w \\ &\leq \log \fint K e^{2w}\ e^{2w} \\ \text{(by (7) with } C = 4)) &\leq \log \max K + \log C(K) + 2\fint |\nabla w|^2 + 4\fint w \end{aligned} \tag{9}$$

hence $-\fint w \leq C(K)$. From this we conclude w is bounded below by using the inequality

$$\begin{aligned} -w(y) + \fint w &= \fint_{S^2} (\Delta w)(x) G(y,x) d\mu(x) \\ &= \fint (1 - Ke^{2w})(x)(G(y,x) + \text{ constant}) d\mu(x) \\ (\text{since } K > 0) &\leq \fint G(y,x) d\mu(x) + \text{ constant}. \end{aligned}$$

But once w is bounded from below, we may modify (9) as

$$\begin{aligned} 2\fint |\nabla w|^2 &= 2\fint Ke^{2w}(w - \overline{w}) \\ &= 2\fint (Ke^{2w} - \delta)(w - \overline{w}) \\ &= (1-\delta) \log \fint \frac{(Ke^{2w} - \delta)}{1-\delta} e^{2(w-\overline{w})} \\ &\leq (1-\delta)\left[\log \max K + 2\fint |\nabla w|^2 + 2\fint w - \log(1-\delta) + \log C(K)\right] \end{aligned}$$

where $\delta = \min Ke^{2w}$. From this, we conclude that $\fint |\nabla w|^2 \leq C(K)$.

To get hold of pointwise bound of w, we notice that

$$\exp\left(\fint w\right) \leqq \fint e^{2w} \leqq \frac{1}{m} \fint Ke^{2w} = \frac{1}{m}$$

hence

$$\left|\fint w\right| \leqq C(m, M) \tag{10}$$

Notice that for any $p > 1$, we may then apply Onofri's inequality to conclude

$$\fint e^{pw} \leq \exp\left(\frac{p^2}{4}\fint |\nabla w|^2 + p\fint w\right) \leq c(m, M, p),$$

hence

$$\begin{aligned} |-w(Q) + \fint w| &= |\int_{S^2} G(Q,P) \triangle w(P) dV_0(P)| \\ &\leq \left(\int |G(Q,P)|^2 dV_0(P)\right)^{\frac{1}{2}} \left(\int (Ke^{2w} - 1)^2\right)^{\frac{1}{2}} \\ &\leq c(m, M), \end{aligned} \tag{11}$$

where $G(\xi, \cdot)$ is the Green's function on S^2 with pole at ξ. Combining the estimates in (10), (11), we get the estimates in (a)'.

Proof of (b): We will prove the result by contradiction. Given $K > 0$ satisfying the non-degeneracy condition (nd), suppose the statement of part (b) does not hold. Then there exists a sequence w_k satisfying

$$\Delta w_k + K e^{2w_k} = 1 \text{ on } S^2 \tag{1}$$

with $\max_{S^2} w_k \to +\infty$. Applying Lemma 2 in Chapter 3, we get a sequence of conformal transformations $\phi_k = \phi_{P_k,t_k}$, with $e^{2v_k} g_0 = \phi_k^*(e^{2w_k} g_0), v_k \in \mathcal{S}$ satisfying

$$\Delta v_k + K \circ \phi_{P_k,t_k} \; e^{2v_k} = 1 \text{ on } S^2. \tag{12}$$

Applying Lemma 2, we have $\|v_k\|_\infty \leq c(m, M)$ and $⨍ |\nabla v_k|^2 \leq c(m, M)$. We may then conclude that some subsequence of $t_k \to +\infty$. For if not, i.e., $t_k \leq t_0$ for all k, for some t_0, then $w_k \circ \phi_k = v_k - \frac{1}{2} \log \; det \; d\phi_k$ is uniformly bounded, which contradicts our assumption that $\max_{S^2} w_k \to +\infty$. Thus, after passing to a subsequence we may assume that $t_k \to +\infty, P_k \to P \in S^2$, and $v_k \to v_\infty$ in $C^{1,\alpha}$ for some $\alpha \in (0, 1)$; the last fact following from the pointwise estimates on v_k and (12) along with the Sobolev imbedding. Notice that $K \circ \phi_K \to K(P)$ uniformly on compact subsets of $S^2 - \{-P\}$, hence v_∞ satisfies

$$\Delta v_\infty + K(P) e^{2v_\infty} = 1, \tag{13}$$

at least weakly on $S^2 - \{-P\}$. But after applying standard arguments from elliptic theory one sees that in fact v_∞ satisfies (13) on all of S^2. By the uniqueness of solutions of (13) belonging to $\mathcal{S}$ we conclude that $v_0 \equiv -\frac{1}{2} \log K(P)$. Normalizing v_k (by rotating P_k to P and adding a suitable constant), we may assume that $K(P) = 1$ and that v_k satisfies

$$\Delta v_k + (K \circ \phi_k) \; e^{2v_k} = 1 \tag{14}$$

with $\phi_k = \phi_{P,t_k}$. Also, by our work above we have

$$\|v_k\|_\infty = o(1) \quad \text{as } k \to \infty \tag{15}$$

$$\|\nabla v_k\|_\infty = o(1) \quad \text{as } k \to \infty. \tag{16}$$

Applying the Kazdan-Warner condition to (14) we have

$$\int \langle \nabla K \circ \phi_k, \nabla x_j \rangle e^{2v_k} = 0, \;\; j = 1, 2, 3 \tag{17}$$

where $\phi_k = \phi_{P,t_k}$ and $t_k \to \infty$. We now claim that (17) implies

$$\nabla K(P) = 0 \text{ and } \Delta K(P) = 0, \tag{18}$$

thus contradicting our assumption that K is a non-degenerate function and finishing the proof of part (b).

To verify (18) we will begin to compare the expression in (17) to $\int \langle \nabla K \circ \phi_k, \nabla x_j \rangle$, and then compute the asymptotic behavior of the latter term as $t_k \to \infty$. To this end we define

$$\begin{aligned} A_k &= \int < \nabla(K \circ \phi_k), \nabla \overrightarrow{x} > e^{2v_k}, \\ B_k &= \int < \nabla K \circ \phi_k, \nabla \overrightarrow{x} > = 2 \int (K \circ \phi_k - 1) \overrightarrow{x}. \end{aligned}$$

Lemma 3 *Under the assumptions (15), (16) with $\phi_k = \phi_{P,t_k}, t_k \to \infty, K(P) = 1$ we have*

$$A_k = B_k + C_k \quad \textit{with } |C_k| = \begin{cases} o(\frac{1}{t_k}) \;\; \textit{if } \nabla K(P) \neq 0, \\ o(\frac{1}{t_k^2} \log \frac{1}{t_k}) \;\; \textit{if } \nabla K(P) = 0. \end{cases} \tag{19}$$

Let $B_k^{(i)}$ denote the i-th component of B_k, $1 \le i \le 3$, then

$$\begin{cases} B_k^{(i)} = c_1 a_i \dfrac{1}{t_k} + O(\dfrac{1}{t_k^2}), \;\; i = 1, 2; \\ \\ B_k^{(3)} = c_2 (b_{11} + b_{22}) \dfrac{1}{t_k^2} \log t_k + O(\dfrac{1}{t_k^2}) \end{cases} \tag{20}$$

where c_1, c_2 are dimensional constants.

Clearly (18) is a consequence of (17), (19) and (20).

Proof of Lemma 3: To prove (2.12) we first write

$$\begin{aligned}
A_k &= \int < \nabla K \circ \phi_k, \nabla \overrightarrow{x} > e^{2v_k} \\
&= \int < \nabla (K \circ \phi_k - 1), \nabla \overrightarrow{x} > e^{2v_k} \\
&= -\int (K \circ \phi_k - 1) \Delta \overrightarrow{x} e^{2v_k} - \int (K \circ \phi_k - 1) < \nabla \overrightarrow{x}, \nabla e^{2v_k} > \\
&= 2 \int (K \circ \phi_k - 1) \overrightarrow{x} + 2 \int (K \circ \phi_k - 1) \overrightarrow{x} (e^{2v_k} - 1) \\
&\quad - 2 \int (K \circ \phi_k - 1) < \nabla \overrightarrow{x}, \nabla v_k > e^{2v_k}.
\end{aligned}$$

Thus $A_k = B_k + C_k$ where

$$C_k = 2 \int (K \circ \phi_k - 1) \overrightarrow{x} (e^{2v_k} - 1) - 2 \int (K \circ \phi_k - 1) < \nabla \overrightarrow{x}, \nabla v_k > e^{2v_k}.$$

We will see that the desired estimate of C_k follows from the same asymptotic computation of B_k as in (20) below together with the assumption that $\|v_k\|_\infty = o(1), \|\nabla v_k\|_\infty = o(1)$.

To verify (20), we will use the stereographic projection coordinates of S^2 to compute B_k in terms of the Taylor series expansion of K. To do this, denote $Q = (x_1, x_2, x_3) \in S^2$ and let $y = (y_1, y_2)$ be the stereographic projection from S^2 to the equatorial plane R^2 sending the north pole $N_0 = (0, 0, 1)$ to ∞. We can also w.l.o.g. identify the point P as the north pole N. Thus $x_i = \frac{2y_i}{1+|y|^2}$ for $1 \leq i \leq 2$, and $x_3 = \frac{|y|^2-1}{|y|^2+1}$. We assume the Taylor series expansion of K around N is given by

$$K(x_1, x_2, x_3) = K(x_1, x_2) = K(N) + \sum_{i=1}^{2} a_i x_i + \sum_{i,j=1}^{2} b_{ij} x_i x_j + o(\sum_{i=1}^{2} x_i^2) \tag{21}$$

and (2.14) holds in a neighborhood $\tilde{D} = \{y \in R^2, |y| \geq M\}$ of N, for some $M > 0$ large. Notice in this notation $\phi_k(y) = t_k y$. Denote $D_k = \{y \in R^2, |y| \geq \frac{M}{t_k}\}$, then $\phi_k(D_k) = \tilde{D}$. To estimate B_k, let $dA(y) = \frac{1}{\pi} \frac{d|y|^2}{(1+|y|^2)^2} d\theta$ denote the area form, then

$$\int_{D_k^c} dA(y) = 2 \int_0^{\frac{M}{t_k}} \frac{d|y|^2}{(1+|y|^2)^2} = \frac{M^2}{t_k^2 + M^2} = O(\frac{1}{t_k^2}) \text{ as } t_k \to \infty. \tag{22}$$

Thus

$$B_k = 2\int (K\circ\phi_k - 1)\overrightarrow{x}$$

$$= 2\int_{D_k} (K\circ\phi_k - 1)\overrightarrow{x} + O(\frac{1}{t_k^2}).$$

Next we notice that by circular symmetry,

$$\int_{D_k} x_i(t_k y)x_j(y)dA(y) = 0 \quad \text{if } i \neq j;\ \ 1 \leq i,j \leq 3.$$

Hence,

$$B_k^{(i)} = \int_{D_k} a_i x_i(t_k y)x_i(y)dA(y) + 2\int_{D_k}\left(\sum_{j,\ell=1}^{2} b_{j\ell}x_j(t_k y)x_\ell(t_k y)\right)x_i(y)dA(y)$$
$$+ E_k^i + O(\frac{1}{t_k^2}) \quad \text{for } i = 1,2,3,$$

where

$$E_k^{(i)} = O\left(\int_{D_k}\left(\frac{|t_k y|}{1+|t_k y|^2}\right)^3 |x_i(y)|dA(y)\right),\ i = 1,2,3,$$

and

$$B_k^{(3)} = 2\int\left(\sum_{j,\ell=1}^{2} b_{j\ell}x_j(t_k y)x_\ell(t_k y)\right)x_3(y)dA(y) + E_k^{(3)} + O(\frac{1}{t_k^2}).$$

Similarly we define $C_k^{(i)}$ for $i = 1,2,3$ to be the components of C_k, and $B_k^{(i)}$ for $i = 1,2,3$ to be the components of B_k. (19) and (20) will be consequences of the following lemma which can be verified by direct computation.

Lemma 4.

(23) $$\int_{D_k} x_i(t_k y)x_i(y)dA(y) \sim \frac{1}{t_k} \ \ as\ k \to \infty$$

(24) $$\int_{D_k} x_j(t_k y)x_\ell(t_k y)x_i(y)dA(y) = \begin{cases} 0, & 1 \leq i,j,\ell \leq 2; \\ 0, & i = 3,\ \ j \neq \ell; \\ \frac{1}{t_k^2}\log t_k, & i = 3,\ \ 1 \leq j = \ell \leq 2. \end{cases}$$

$$|E_k^i| = O(\frac{1}{t_k^2}), \quad i = 1, 2, 3.$$

We conclude from (23), (24) that

$$\tag{25} \begin{cases} B_k^{(i)} &= c_1 a_i \frac{1}{t_k} + O(\frac{1}{t_k^2}), \quad i = 1, 2, \\ \\ B_k^{(3)} &= c_2 (b_{11} + b_{22}) \frac{1}{t_k^2} \log t_k + O(\frac{1}{t_k^2}), \end{cases}$$

with $c_1 > 0, c_2 > 0$ dimensional constants.

By a similar argument we can prove

$$\tag{26} C_k^{(i)} = o(|a|)\frac{1}{t_k} + o(|b|)\frac{\log t_k}{t_k^2} + O(\frac{1}{t_k^2})$$

where $|a| = \sum |a_i|, |b| = \sum_{i,j} |b_{ij}|$. Then (19), (20) are direct consequences of (25), (26). We have thus finished the proof of part (b) of Theorem 1.

In the rest of this chapter, we will describe some sufficient conditions on K $(n = 2)$ or R $(n \geqq 3$ for equations (1) and (2) to have solutions.

Recall that in the proof of Kazdan-Warner condition (iii) above, we have, for a solution w 0f (1), $\frac{d}{dt}\Big|_{t=1} F_K[T_{\phi_{P,t}}(w)] = 0$, for any $P \in S$., and

$$\frac{d}{dt}\Big|_{t=1} F_K[T_{\phi_{P,t}}(w)] = \fint < \nabla K, \nabla(\overrightarrow{x} \cdot P) > e^{2w} = 0$$

A relevant observation here is that if e^{2w} is close to the constant 1, then the integral is comparable to

$$2 \fint K \overrightarrow{x} \cdot \overrightarrow{p} \, dA.$$

A parallel argument works for the higher dimensional setting. Thus given a potential curvature function R (or K) we form the following map $G : B \to R^{n+1}$ given by

$$\tag{27} G\left(\frac{t-1}{t} P\right) = \int_{S^n} (R \circ \phi_{P,t}) \cdot \overrightarrow{x}.$$

It is natural to investigate the asymptotic behavior of the map $G : (P,t) \mapsto \fint (R \circ \phi_{P,t}) \overrightarrow{x}$ as the parameter $t \to \infty$. We choose coordinates $x_1, \ldots, x_{n+1}$ so that P is the South pole $P = (0, \ldots 0, -1)$ and let $y_1 \ldots y_n$ be the stereographic coordinates

$$\begin{cases} x_i = \frac{2y_i}{1+|y|^2} & 1 \leq i \leq n \ , \\ x_n = \frac{|y|^2-1}{|y|^2+1} \ . \end{cases}$$

(In the coordinate system $y, \phi_t(y) = \frac{1}{t} y$.)

Expand R in Taylor series in y coordinates around P:

$$R(y) = R(P) + \sum_{k=1}^{\infty} R_k(y)$$

where R_n is a homogenous polynomial in $y_1, \ldots, y_n$ of degree k, so that

$$(R \circ \phi_{P,t})(y) = R(P) + \sum_{k=1}^{\infty} R_k(y) t^{-k}.$$

We denote by $R^{(\alpha)}(y) = R(P) + \sum_{k=1}^{\alpha} R_k(y)$, the truncated Taylor polynomial of order α.

Definition. *We say R is non-degenerate at P of order α if*

(i) the truncated Taylor polynomial of order α has the form

$$R^{(\alpha)}(y) = R(P) + R_\alpha(y),$$

(ii) for t sufficiently large, the map

$$G(P,t) = \fint (R^{(\alpha)} \circ \phi_{P,t}) \overrightarrow{x}$$

satisfies a lower bound:

$$|G(P,t)| \geq \begin{cases} c \frac{1}{t^\alpha} & \alpha < n \\ c \frac{1}{t^n} \log t & \alpha = n \ . \end{cases}$$

At any point $P \in S^n$, one can examine this notion of non-degeneracy with some simple computations (as in introduction of [C-Y-6]), we can see for example:

(1) At a point P where $\nabla R(P) \neq 0$, it is non-degenerate of order 1.

(2) At a point P where $\nabla R(P) = 0$ but $\triangle R(P) \neq 0$, it is non-degenerate of order 2.

We observe that for a function R uniformly non-degenerate of order $\alpha \leqq n$, the map $G(P,t)$ is non-zero for $t \geqq t_0$, hence the restriction of G to the sphere $\{(P,t), P \in S^n\}$ are mutually homotopic for $t \geqq t_0$, and the degree $\deg(G|_{t=t_0}0)$ is well defined.

Theorem 2 *There exists constants $\epsilon(n)$ such that if K or R is a smooth function satisfying:*

(i) $||K-1||_\infty \leq \epsilon(2)$ or $||R-R_0||_\infty \leq \epsilon(n)$;

(ii) K or R is a uniformly nondegenerate function of order α, where $\alpha \leq n$ when n is even and $\alpha \leq n-1$ when n is odd, i.e.

$$(*) \qquad |G(P,t)| \geq \begin{cases} \frac{C}{t^\alpha}, & \textit{when } \alpha < n, \\ \frac{C}{t^n \log t}, & \textit{if } \alpha = n \end{cases}$$

for $t \geq t_0$, uniformly in P; for all P in S^n.

(iii) deg$(G|_{t=t_0}, 0) \neq 0$;

then the equation (1) has a solution.

Remarks

(1) In the case $\alpha = 2$, uniformity in $(*)$ is a consequence of the nondegenerate assumption on the critical points of the function K or R. In the general case, the uniformity requirement does not follow from the nondegenerate requirement on the critical points of K or R alone. In principle, it is possible to reduce this requirement to algebraic criteria on the Taylor coefficients of the function R at its critical points. For example we determine below necessary and sufficient conditions on the Taylor coefficients of R at its critical points when $\alpha = 3$ in [C-Y-6].

(2) In dimension 2 (and 3), result in Theorem 2 is optimal in the sense that if the differential equation (1) admits a solution then it is necessarily captured by the variational scheme used in the proof. This amounts to an a priori estimate for solutions of (1) when $\|K-1\|_\infty$ is sufficiently small

It turns out in the case of $n = 2$ and 3, one can drop the small assumptionin the statement of Theorem 2.

Theorem 3 *On S^2 (S^3), suppose $K > 0$ ($R > 0$ is a smooth function satisying the non-degeneracy condition that $\nabla K(Q) = 0$ implies $\triangle K(Q) \neq 0$ (respectively for R) and* $\deg(G|_{t=t_0} 0) \neq 0$, *then the equation (1) (respectively (2)) has a solution.*

Theorems 2 generalize previous existence results of Chang-Yang ([C-Y-4] [C-Y-5] and [H]) on S^2 and Bahri-Coron ([B-C]) and Schoen-Zhang ([S-Z]) on S^3 where K (respectively R) is assumed positive, having only isolated non-degenerate critical points and in addition satisfying $\Delta K(Q) \neq 0$ at critical points, and the index count condition:

$$\sum_{Q \text{ critical}, \Delta K(Q) < 0} (-1)^{ind(Q)} \neq (-1)^n. \tag{28}$$

In the appendix of [C-G-Y], it was shown that the index count condition (28) implies $deg(G, 0) \neq 0$. In the other hand, the nondegeneracy condition (nd) allows K (or R) to have non-isolated critical point sets. In dimension 4, there is an interesting example of Bianchi-Egnell ([B-E]) which indicates that further complications arise and it will be necessary to understand the interactions of more than one concentrated masses. The validity of Theorem 3 remains open for $n \geqq 5$.

Theorem 3 is a consequence of a topological degree argument. The earlier perturbation result (Theorem 2) provides the initial step of a continuity argument. One then verify that $deg(G, 0)$ gives in fact the Leray Schauder degree of a non-linear map whose zeroes correspond to solutions of the differential equations when R satisfies the close to constant condition. The the a priori estimates as in Theorem 1 then provide the continuity argument needed to verify the invariance of the Leray Schauder degree as one moves along the parameter in the continuity scheme. We refer interested reader to [C-G-Y] for details.

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