# ON ZETA FUNCTIONAL DETERMINANT
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## Introduction

This is the set of lecture notes the author gave in the PDE workshop held in the University of Toronto, sponsored by the Fields Institute, in June of 1995. The notes was taken and the first draft was written up by Jie Qing, who is also a coauthor of some of the material presented in the lectures. The main theme of the lecture notes is to discuss the zeta functional determinant, with emphasize on the connection of the functional to conformal geometry, and to the sharp borderline Sobolev inequalities which governs the functional.

There are three chapters in the notes, each chapter corresponds roughly to the material covered in one lecture. In the first chapter, we introduce the notion of the zeta functional determinant as originally defined by Ray and Singer in 1971, we then covered the Polyakov formula for the quotient of the log-determinant of the zeta functional on two conformally related metrics defined on compact surfaces. We also discussed the main analytic tool–namely some sharp Sobolev inequality called Moser-Trudinger-Onofri's inequality used to study the functional and gave some applications. In chapter two, we discuss some generalization of the zeta functional determinant to manifolds of dimension 4 and related compactness and isospectral problems. Finally in chapter 3, we discuss generalization of Polyakov formula to compact manifolds with boundary, and report on some recent ongoing research work with Jie Qing in this direction. The lectures may be considered a continuation report of the author's earlier lecture notes [C] in the summer geometry institute in Utah, 1993 on related subjects.

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## Chapter 1. Zeta functional determinant on compact surfaces

### A. Preliminary about heat kernel

Suppose $(M, g)$ be a compact Riemannian manifold. We denote $\{\lambda_k\}$ the eigenvalues of the Laplacian-Beltrami operator:

$$0 \leq \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_k \leq \cdots , \tag{1}$$

and denote $\{\phi_k\}$ the corresponding eigenfunctions which form an orthonormal basis for $L^2(M)$. Then the heat kernel $K(x, y, t)$ is defined as

$$K(x, y, t) = \sum_k e^{-\lambda_k t} \phi_k(x) \phi_k(y) \,. \tag{2}$$

$K(x, y, t)$ is the fundamental solution of the heat equation in the sense that for any given $f \in C^i nfty(M)$, we have

$$\begin{cases} \frac{\partial}{\partial t} u - \Delta u = 0 \\ u(x, 0) = f(x) \,. \end{cases}$$

The trace of the heat kernel $Z(t)$ is defined as

$$Z(t) = \int_M K(x, x, t) dV(x) = \sum_k e^{-\lambda_k t} = Tr(e^{-t\Delta}) \,. \tag{3}$$

One of the fundamental result about the heat kernel is the following theorem for the asymptotic expansion of the kernel as t tends to zero.

**Theorem.** *([MP],[MS]) Suppose $(M, g)$ is a compact Riemannian manifold without boundary. Then*

$$Z(t) \sim (4\pi t)^{-\frac{n}{2}} (a_0 + a_2 t + a_4 t^2 + \cdots) \quad as \quad t \to 0^+$$

*where*

$$\begin{aligned} a_0 &= vol(M, g) \\ a_2 &= \frac{1}{3} \int_M R \; dV \\ a_4 &= \frac{1}{180} \int_M (10A - B + 2C) dV \end{aligned} \tag{4}$$

*where $R$ is the scalar curvature of $(M, g)$ and $A$, $B$, $C$ are polynomials of degree 2 in the curvature tensor $R_{ijkl}$, and $dV$ is the volume form with respect to the metric $g$.*

In fact, the expansions of type (4) have been worked out for a quite large class of formally self-adjoint elliptic operators. We will discuss more of this in later chapters.

## B. Zeta functional determinant

In this section, we will first explain the notion of log-determinant of the Laplacian operator as introduced by Ray and Singer [RS], we will then state and prove a formula of Polyakov [Po] about the log-determinant on compact surfaces.

For a compact Riemannian manifold, we define the zeta function

$$\zeta(s) = \sum_{\lambda_k \neq 0} {\lambda_k}^{-s}. \tag{5}$$

Then by Weyl's asymptotic formula for the eigenvalues

$$(\lambda_k)^{\frac{n}{2}} \sim \frac{(2n)^n k}{w_n \mathrm{vol}(M)}$$

we know that $\zeta(s)$ is well-defined when $\mathrm{Re}(s) > \frac{n}{2}$. Using the expansion (4) and the Milin transform

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1} dt\,.$$

where $\Gamma(s)$ denote the value of Gamma function at $s$, we obtain the regularized zeta function

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} (\mathrm{Tr} e^{-t\Delta} - 1) dt\,. \tag{6}$$

Formerly we have

$$e^{-\zeta'}(0) = \prod_k \lambda_k\,.$$

Therefore through the regularization of the zeta function, Ray and Singer [RS] defined the logarithm of the determinant of the Laplace as:

$$\log\det\Delta = -\zeta'(0)\,. \tag{7}$$

*Remark:*

1.The reader is referred to [RS], [OPS-1] and Chapter 1 in [C] for more detailed justification of the "regularized process" used in the definition above.

2. The notion of log-determinant of the Laplacian was introduced in [RS] to define the notion of analytic torsion,

$$\log T = 1/2 \sum_{q=0}^{n} (-1)^q q {\zeta_q}'(0)$$

where ${\zeta_q}'(0)$ denotes the negative of the log-determinant of the Laplacian on q-forms. It was later established by Cheeger [Ch] and Muller [Mu] that the notion of analytic torsion coincides with the Rademeister torsion, hence is a topological quantity.

A generalization of the expansion (4) above is the following formula:

$$\mathrm{Tr}(f e^{t\Delta}) \sim \textstyle\sum_m a_m(f,\Delta) t^{\frac{m-n}{2}} \quad \text{as} \quad t \to 0^+ \tag{8}$$

where $a_m(f,\Delta) = \int_M f U_m(\Delta) dV$ and $U_m(\Delta)$ is a local invariant of Riemannian geometry of order $m$. In particular, $a_m(1,\Delta) = a_m(\Delta)$, where $a_m(\Delta)$ is the coefficient of the heat kernel as appeared in (4).

**Ray-Singer-Polyakov formula.** *[Po] Suppose* $(M, g_0)$ *is a closed compact surface, and* $\mathrm{vol}(M, g_0) = \mathrm{vol}(M, g_w)$ *for* $g_w = e^{2w} g_0$. *Then*

$$\log \frac{\det \Delta_w}{\det \Delta_0} = -\frac{1}{12\pi} \int_M \{|\nabla w|^2 + 2K_0 w\} dV_0 \tag{9}$$

*where* $K_0$ *is the Gaussian curvature of* $(M, g_0)$.

To prove the formula, we will first establish a lemma.

**Lemma.** *Suppose* $(M, g_0)$ *is a closed compact surface. Then*

$$\frac{d}{d\varepsilon}|_{\varepsilon=0} \zeta'_{\Delta_{u+\varepsilon f}}(0) = 2a_2(f, \Delta_u) \tag{10}$$

*where* $\Delta_w$ *denotes the Laplacian of the metric* $g_w = e^{2w} g$.

*Proof of Lemma.* We will sketch the proof by some formal computations which can all be justified rigorously. We first notice that

$$a_2(f, \Delta) = \frac{1}{12\pi} \int_M f K dV. \tag{11}$$

and

$$\begin{aligned} &\frac{d}{d\varepsilon} |_{\varepsilon=0}\, Tr(e^{-t\Delta_{u+\varepsilon f}}) = \frac{d}{d\varepsilon} |_{\varepsilon=0}\, Tr(e^{-te^{-2\varepsilon f}\Delta_u}) \\ &= 2tTr(f\Delta_u e^{-t\Delta_u}) = -2t\frac{d}{dt} Tr(fe^{-t\Delta_u}). \end{aligned} \tag{12}$$

Here we have used a crucial fact that when $n = 2$, $\Delta_w = e^{-2w}\Delta$ for any metric $g_w = e^{2w} g$. Therefore

$$\begin{aligned} &\frac{d}{d\varepsilon} |_{\varepsilon=0}\, \frac{d}{ds} |_{s=0}\, \zeta_{\Delta_{\bar{u}+\varepsilon f}}(s) \\ &= \frac{d}{ds} |_{s=0}\, \frac{d}{d\varepsilon} |_{\varepsilon=0}\, \zeta_{\Delta_{u+\varepsilon f}}(s) \\ &= \frac{d}{ds} |_{s=0} \left\{ \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{d}{d\varepsilon} |_{\varepsilon=0}\, Tr(e^{-t\Delta_{u+\varepsilon f}}) dt \right\} \\ &= -2\frac{d}{ds} |_{s=0} \left\{ \frac{1}{\Gamma(s)} \int_0^\infty t^s \frac{d}{dt} Tr(fe^{-t\Delta_u}) dt \right\} \\ &= 2\frac{d}{ds} |_{s=0} \left\{ \frac{s}{\Gamma(s)} \int_0^\infty t^{s-1} Tr(f\Delta^{-t\Delta_u}) dt \right\} \\ &= \quad 2a_2(f, \Delta_u). \end{aligned} \tag{13}$$

In the last step above, we have used the expansion of $\Gamma$ function near $s = 0$ as:

$$\Gamma(s) = \frac{1}{s} - \frac{1}{s+1} + \cdots .$$

This fact together with the asymptotic formula for $Tr(f\Delta^{-t\Delta_u})$ as $t \to 0^+$ and the identity (11) justify the last step in the formula (13) above.

*Proof of the formula.* By (10) and (11) we have

$$\begin{aligned}\zeta'_{\Delta_w}(0) - \zeta'_{\Delta_0}(0) &= 2\int_0^1 a_2(w, \Delta_{tw})dt \\ &= \frac{1}{6\pi}\int_0^1 \left(\int_M wK_{tw}dV_{tw}\right)dt \\ &= \frac{1}{6\pi}\int_0^1 \left(\int_M w(-\Delta(tw) + K_0)dV_0\right)dt.\end{aligned}$$

Since $-\Delta u + K_0 = K_u e^{2u}$. Thus

$$\zeta'_{\Delta_w}(0) - \zeta'_{\Delta_0}(0) = -\frac{1}{12\pi}\int_M \{|\nabla w|^2 + 2K_0 w\}dV_0.$$

The formula thus follows by definition of the log-determinant of the Laplacian.

*Remark:* Recently there has been a lot of research work in the computation of the log-determinant formulas for general elliptic and pseudo-differential operators. The reader is referred to articles for example by Kontseivich-Vishik and Okikiolu [Ok]. In particular in the article by Okikiolu, she has applied some general formula for the log-determinant for the product of two operators to give an alternative proof of Polyakov's formula (9) above.

## C. Extremal metrics and compactness of isospectral metrics

We will now indicate some applications of the Polyakov formula. We first notice that by the uniformization theorem we may choose a metric with the constant Gaussian curvature as the background metric for a closed surface of a given conformal structure, thus we may assume without loss of generality that $K_0 =$ constant. The following result, first pointed out in [OPS-1] is an easy consequence of the Polyakov formula.

**Theorem.** *Suppose $K_0$ is an negative constant, then $F[w] = \log \frac{\det\Delta_w}{\det\Delta_0}$ attains its maximum at and only at the constant curvature metrics, when the volume of the metric $g_w$ is fixed.*

*Proof.* Since $\int_M e^{2w}dV_0 = \int_M dV_w = \int_M dV_0$ by assumption, we have $e^{2\bar{w}} \le \overline{e^{2w}} = 1$, where $\bar{w} = \frac{1}{\mathrm{Vol}(M,g_0)}\int_M w dV_0$ and $\overline{e^{2w}} = \frac{1}{\mathrm{Vol}(M,g_0)}\int_M e^{2w}dV_0$. Thus $\bar{w} \le 0$. Recall the formula

$$F[w] = -\frac{1}{12\pi}\left(\int_M |\nabla w|^2 + 2K_0\int_M w\right).$$

Thus for $K_0$ a negative constant, we have clearly that $F[w] \le 0$ and $F[w] = 0 \Longleftrightarrow w = 0$.

In the case when $K_0$ is a positive constant, as in the case when $M = S^2$, similar result as in the statement of theorem above still holds, but the proof becomes analytically much more interesting. This result is first established by Onofri [On] in connection with the string theory, see also [OPS-1] for an alternative proof.

**Theorem.** *(Onofri)*

$$\log \frac{1}{4\pi}\int_{S^2} e^{2w} \le \frac{1}{4\pi}\left\{\int_{S^2} |\nabla w|^2 dV_0 + 2\int_{S^2} w dV_0\right\}. \tag{14}$$

*Moreover the equality holds if and only if the metric $g_w = e^{2w}g_0$ is isometric to the standard metric $g_0$ of $S^2$.*

Onofri inequality (14) is a geometric form of a sharp Sobolev inequality. Recall that in the classical Sobolev embedding Theorem, we have

$$W_0^{\alpha,q}(\Omega) \hookrightarrow L^p(\Omega)$$

for $\Omega \subset \mathbb{R}^n$, and $\frac{1}{p} = \frac{1}{q} - \frac{\alpha}{n}$, $q\alpha < n$. In particular, $W_0^{1,2}(\Omega) \hookrightarrow L^{\frac{n+2}{n-2}}(\Omega)$ for $n \ge 3$. It turns out that when $n = 2$, it is not difficult to see that, for domains $\Omega$ in $R^2$, there exist unbounded functions in $W_0^{1,2}(\Omega)$ with fixed $W_0^{1,2}(\Omega)$ inorm. Nevertheless by a result of Trudinger [T], we have that for all $p \ge 1$:

$$\|w\|_{L^p(\Omega)} \le C\sqrt{p}\|\nabla w\|_{L^2(\Omega)}$$

and thus

$$\int_\Omega e^{\beta \frac{w^2}{\int_\Omega |\nabla w|^2}} \le C|\Omega|$$

for some constant $\beta$ and $C$ independent of $w \in W_0^{1,2}(\Omega)$. Later Moser[M-1] found out that the best possible constant $\beta$ in the inequality above is $4\pi$ and is attained. This is the so-called Moser-Trudinger inequality

$$\int_\Omega e^{4\pi \frac{w^2}{\int_\Omega |\nabla w|^2}} \le C|\Omega| \tag{15}$$

for an universal constant $C$.

*Remarks:*

1. It was pointed out in [CY-2] that the best constant $4\pi$ in (15) above has a geometric meaning and is actually the isoperimetric constant. i.e.

$$L^2 \ge 4\pi A.$$

where $L$ is the arc length of the curve $\{x \in \Omega | |w(x)| = t\}$, and $A$ is the area of the set $\{x \in \Omega | |w(x)| \ge t\}$ for function $w \in W_0^{1,2}(\Omega)$.

2. For the sphere $S^2$, Moser has also established in [M-1] that

$$\frac{1}{4\pi}\int_{S^2} e^{4\pi \frac{|w-\bar{w}|^2}{\int_{S^2} |\nabla w|^2}} \le C_0. \tag{16}$$

Observe that

$$2(w - \bar{w}) \le 4\pi \frac{(w-\bar{w})^2}{\int_{S^2} |\nabla w|^2} + \frac{1}{4\pi}\int_{S^2} |\nabla w|^2.$$

we obtain

$$\frac{1}{4\pi}\int_{S^2} e^{2(w-\bar{w})} \leq C_0 e^{\frac{1}{4\pi}\int_{S^2}|\nabla w|^2}. \tag{17}$$

It is clear that Onofri's inequality (14) follows from the inequality (17) if one can prove that the best constant $C_0$ in (17) is one. (Notice that the best constant $C_0$ in (16) is not one.) We will now sketch the original proof of Onofri of inequality (14). Consider the functional

$$J[w] = \log \frac{1}{4\pi}\int_{S^2} e^{2w} dV_0 - \frac{1}{4\pi}\int_{S^2}(|\nabla w|^2 + 2w) dV_0.$$

Apply (17), we have that $J[w] \leq \log C_0$, thus the functional $J[w]$ is bounded from above. To prove (14), it suffices to establish that $\sup J[w] = 0$. To do so, we will first, within the class of metric which are isometric to $g_w$, improve the inequality (17) to gain some some "compactness". One way to do so is via consideration of functions in a special class §:

$$\S = \left\{ w : \int_{S^2} e^{2w} x_j dV_0 = 0, for \quad j = 1,2,3 \right\}.$$

We now notice that for any given metric $g_w = e^{2w} g_0$, there exists a conformal transformation $\phi$ of $S^2$ such that

a. $w_\phi \in \S$, where $e^{2w_\phi} g_0 = \phi^*(g_w)$

b. $J[w_\phi] = J[w]$.

The key step that one can gain some "compactness" within the class § is a consequence of the following theorem due to Aubin.

**Theorem.** *([A-2]) For $w \in \S$ , and for any $\varepsilon \geq 0$, there exists a constant $C_\varepsilon$ so that*

$$\log \frac{1}{4\pi}\int_{S^2} e^{2(w-\bar{w})} \leq \left(\frac{1}{2} + \varepsilon\right) \frac{1}{4\pi}\int_{S^2} |\nabla w|^2 dV_0 + \log C_\varepsilon. \tag{18}$$

As a immediate consequence of (18), we have

**Corollary.** *For all $w \in \S$ , we have*

$$\int_{S^2} |\nabla w|^2 dV_0 \leq C_1(J[w] + C_2), \text{ for some constants } C_1 \text{ and } C_2. \tag{19}$$

Apply properties (a) and (b) and (19) above, we may assume, without loss of generality that the maximizing sequence $w_k$ for $J[w]$ all exist in the class § and that they are uniformly bounded in $W^{1,2}$. Thus a weak limit $w_0$ of $w_k$ exists; $w_0 \in \S$, and satisfies the equation

$$\Delta w_0 + 1 = e^{2w_0}$$

from this we conclude that $w_0$ is induced by some conformal transformation $\phi$ of $S^2$, i.e. $e^{2w_0} = \phi^*(g_0)$. Since among all conformal transform $\phi$, the only one which induces a function in § is the identity transformation, thus $w_0 = 0$ and $\mathrm{Sup} J[w] = J[w_0] = 0$. We have hence established (14).

Another interesting and beautiful application of the Polyakov formula is the following result of Osgood-Phillips-Sarnak [OPS-2].

**Theorem.** *Isospectral metrics on a closed surface are $C^\infty$-compact module the isometry class.*

In the special case of $S^2$, apply Polyakov formula and analysis similar to the proof of the Onofri inequality as indicated above, we obtain that for a given $w \in W^{1,2}(S^2)$

$$\int_{S^2} |\nabla u|^2 \lesssim CF[u] + C \tag{20}$$

where $u = w_\phi$ (in the sense of (a), (b) above) is a "good" representative of metric in the sense that $u \in S$ and the metrics $e^{2w_\phi}g_0$ and $e^{2w}g_0$ are isometric to each other. Since $F[w]$ is a spectral information which is invariant under the isometry, $F[u] = F[w]$. (20) together with the volume bound of the metric $g_u$ give a $W^{1,2}$-norm bound for $u$. After that, one can iteratively obtain $W^{k,2}$-norm bound of $u$ using the bound for the k-th heat coefficient $\{a_k(\Delta)\}$ in the expansion (4) for $k \geq 3$. One thus gets the $C^\infty$-compactness of isospectral metrics module the isometry class.

**D. Problem of prescribing Gaussian curvature**

Another closely related problem to the analysis in section C above is the following problem first proposed by Nirenberg :

Given a function $K$ on $S^2$, when can it be the Gaussian curvature of a metric on $S^2$?

To solve this problem is equivalent to ask for what function $K$ does there exist a solution $w$ to the following non-linear PDE.

$$-\Delta w + 1 = Ke^{2w} \text{ on } S^2 \tag{21}$$

for the given function $K$. Using the variational approach, solution of (21) are critical points of the functional

$$F_K[w] = \log \frac{1}{4\pi}\int_{S^2} Ke^{2w}dV_0 - \frac{1}{4\pi}\int_{S^2}(|\nabla w|^2 + 2w)dV_0.$$

It turns out that the functional $F_K[w]$ never attains its supremum unless $K$ is a constant function. Also via the Gauss-Bonnet formula, one has

$$\int_{S^2} Ke^{2w}dV_0 = 4\pi,$$

which in particular implies that $K$ must be positive somewhere. One of the original motivation to derive the inequality (16) was to study the equation [21]. For example, inequality (16) implies that for a function $K$ which is positive somewhere on $S^2$, the functional $F_K[w]$ is bounded from above. In [M-2], Moser has further applied the inequality (16) to prove that if $K$ is an even function defined on $S^2$, i.e. $K(x) = K(-x)$ for all $x \in S^2$, then equation (21) has an even solution. In recent years there has been a lot of study of the Nirenberg problem, e.g. [ChD], [CY-1], [CY-2], [CGY], [ChL], [ChLi], and [XY]. Here we will mention some results from [CY-1], [CY-2] and [CGY].

**Definition.** *We say the function $K$ satisfies a non-degenerate (nd) condition if $\Delta K(x_0) \neq 0$ whenever $\nabla K(x_0) = 0$.*

**Theorem.** *( [CY-1], [CY-2]) Suppose $K > 0$ satisfies (nd), and*

$$\sum_{\substack{\nabla K(x_i)=0,\\ \Delta K(x_i)<0}} (-1)^{\mathrm{index}_K(x_i)} \neq 1,$$

*where* $\mathrm{index}_K(x_i)$ *denotes the Morse index of $K$ at the point $x_i$, Then $K(x)$ can be realized as the Gaussian curvature of a metric on $S^2$.*

**Theorem.** *[CGY] Suppose $K > 0$ satisfies (nd), then all solutions $w$ of the equation (21) satisfy the a prior bound:*

$$\int_{S^2} (|\nabla w|^2 + 2w) dV_0 \leq C(maxK, MinK, (nd)).$$

Currently there is active research going on for the related problem of prescribing scalar curvature on $S^n$, $n \geq 3$. We will not go into the details here.

## Chapter 2. Generalization of Polyakov formula on 4-manifolds

In the last chapter we have discussed the Ray-Singer-Polyakov formula for the Laplacian operator on closed, compact surfaces. We have also mentioned some applications of the formula and the underlying analysis. We will now discuss some generalization of the formula for more general differential operators in higher dimensional closed, compact manifolds.

### A. Isospectral problem in 3-dimensional manifolds

We will first briefly mention some results which partially generalizes the isospectral compactness result of Osgood-Phillips-Sarnak on compact surfaces mentioned at then end of the last chapter.

Recall the heat kernel trace expansion on a compact, closed n-dimensional Riemannian manifold (M, g),

$$\mathrm{Tr}(e^{-t\Delta}) \sim (4\pi)^{-\frac{n}{2}} \sum_{k}^{\infty} a_k t^{\frac{k-n}{2}} \; ast \to 0^+, \tag{1}$$

where

$$\begin{aligned}
a_0 &= \int_M dV \\
a_2 &= \frac{1}{3}\int_M RdV \\
a_4 &= \frac{1}{360}\int_M \left(5R^2 - 2\sum |R_{ij}|^2 + \sum |R_{ijkl}|^2\right) dV,
\end{aligned}$$

and

$$a_{2k+1} = 0, \; fork = 0, 1, 2, ....$$

In particular when $n = 3$, $a_3 = 0$, so there is no additional information from this heat coefficient alone. On the other hand we have that when $n = 3$,

$$a_4 = \frac{1}{360} \int_M \left( 3R^2 + 6 \sum |R_{ij}|^2 \right) dV. \tag{2}$$

Therefore we have

$$\int_M R^2, \; \int_M \sum |R_{ij}|^2 \lesssim |a_4|. \tag{3}$$

We will now see that we may derive some spectral compactness result from the information from (3).

**Theorem 1.** *([CY-3], [CY-4]) On a compact, closed 3 dimensional manifold, let $g = u^{\frac{4}{n-2}} g_0$ be a metric conformal to a background metric $g_0$. Assume that*
*a. vol$(M, g)$ is bounded,*
*b. $\int_M |R_{ij}|^2 dV_g$ is bounded,*
*c. $\lambda_1(M, g) \geq \Lambda > 0$.*
*Then*

$$\begin{cases} \|u\|_{W^{2,2}} \leq C \\ \frac{1}{C} \leq u \leq C \end{cases} \tag{4}$$

*for some positive $C$ depending on the information on (a), (b) and (c) above for all manifolds unless $(M, g_0)$ is isometric to $(S^3, g_c)$; with $g_c$ being the canonical metric on $S^3$. In the latter case (4) holds in the isometry class of $g$.*

Following the same pattern of proof as in [OPS-2] as indicated in chapter one, we can also conclude that:

**Corollary 1.** *Isospectral set of conformal metrics on a closed 3-manifold is $C^\infty$-compact.*

The theorem above has been generalized by M.Gursky to higher dimensional manifold with some $L^p$, $p > \frac{n}{2}$, growth assumption on the norm of the Ricci curvature tensor.

**Theorem 2.** *[Gu] On $(M^n, g_0)$, let $g = u^{\frac{4}{u-2}} g_0$ be a metric conformal to $g_0$. Assume that*
*a. vol$(M, g)$ is bounded*
*b. $\int_M |R_{ijkl}|^p dV_g$ is bounded, for some $p > \frac{n}{2}$.*
*Then*

$$\begin{cases} \|u\|_{W^{2,p}} \leq C \\ \frac{1}{C} \leq u \leq C \end{cases} \tag{5}$$

*for some positive $C$ depending on the information on (a), (b) for all compact, closed manifolds $(M, g_0)$ unless $(M, g_0)$ is isometric to $(S^n, g_c)$. In the latter case, (5) still holds in the isometry class of $g$.*

**Remark.** *The assumption that $p > \frac{n}{2}$ is necessary in the statement of Theorem 2 above. Indeed when $p = \frac{n}{2}$ some counterexamples have been constructed in ([CGW]) which indicates that Theorem 2 fails in this case.*

We will explain some idea in the proof of Theorem 1 above by comparing it to the solution of the Yamabe problem; that is the problem of finding a metric with constant scalar curvature within a given conformal class of metrics on a closed, compact manifold. Denote $L = -\Delta + \frac{n-2}{4(n-1)}R$ the conformal Laplacian operator; then a equivalent way of stating the Yamabe problem is to solve the following PDE on a compact manifold $M$.

$$Lu = cu^{\frac{n+2}{n-2}} \tag{6}$$

for some constant $c$. Solutions of (6) can be found by minimizing $\int_M RdV$ under the constraint of fixing the volume of the metrics. Yet since the imbedding of $W^{1,2} \hookrightarrow L^p$ is compact for $p \leq \frac{2n}{n-2}$, but not compact for $p = \frac{2n}{n-2}$, the minimum may not be attained. In [A-1], T.Aubin defined the Yamabe quotient $Q(M, g)$ as:

$$Q(M, g) = \frac{\int_M R_g dV_g}{\mathrm{vol}(M, g)^{\frac{n-2}{n}}}.$$

He then proved that if

$$Y(M, g_0) =: \inf_{g \text{ conformal to } g_0} Q(M, g) < Y(S^n, g_c). \tag{7}$$

then a minimizing sequences for $Y(M, g_0)$ converges in $L^{\frac{2n}{n-2}}$, hence minimum is attained. Yamabe problem was eventually solved by Aubin [A-1] and Schoen [Sc-1] by establishing that (7) always hold for any manifold $(M, g_0)$ not isometric to $(S^n, g_c)$.

The underlying analysis in the proof of Theorem 1, as is in the case of the Yamabe problem, is to study the extremal functions of the optimal Sobolev inequality

$$Y(M, g_0) \left( \int_M u^6 dV_0 \right)^{\frac{1}{3}} \leq \int_M |\nabla u|^2 + \frac{1}{8} \int_M R_0 u^2. \tag{8}$$

In the special case when $(M, g_0) = (S^3, g_c)$, extremal functions for (8) are

$$u_\varepsilon = \left( \frac{\varepsilon(1 + |x|^2)}{\varepsilon^2 + |x|^2} \right)^{\frac{1}{2}} \text{ for } \varepsilon > 0.$$

We observed that

$$\int_{S^3} u_\varepsilon^{6+\delta} dV_0 \longrightarrow +\infty \text{ as } \varepsilon \longrightarrow 0$$

for any $\delta > 0$. Thus it is convincing that any sequence of functions $\{u_k\}$ with $\int_M u_k^{6+\delta} dV_0$ bounded for some $\delta > 0$ would converge strongly in $L^6$ on compact 3-manifolds. Indeed, the key point in the proof of Theorem 1 above is to indicate that for any sequence of metrics $g_k = (u_k)^{\frac{4}{n-2}}$ satisfying the assumptions of the theorem, there exists some $\Delta > 0$ so that $\int_M u_k^{6+\delta} dV_0$ is uniformly bounded.

**B. Functional determinants on 4-manifolds**

**Definition.** *Let $A$ be a geometric differential operator of $(M,g)$. If*

$$A_{g_w}(\phi) = e^{-bw}A_g(e^{aw}\phi)$$

*under the conformal change of metric $g_w = e^{2w}g$, then we say $A$ is conformally covariant of bidegree $(a,b)$.*

**Example.** *1. $L = -\Delta + \frac{n-2}{4(n-1)}R$, the conformal Laplacian, then $L$ is conformally covariant of bidegree $(\frac{n-2}{n}, \frac{n+2}{n})$.*
*2. An interesting fourth order differential operator defined on 4-manifolds was given by Paneitz [P]:*

$$P_4 = (-\Delta)^2 + div\left(\frac{2}{3}R\ Id - 2R_{ij}\right)d,$$

*which is conformal covariant of bidegree $(0,4)$.*
*3. [GJML] For any $k$ $\left(k \leq \frac{n}{2},\right.$ when $\left.\dim M = n \text{ is even}\right)$, there exist conformally covariant operators $\mathbb{P}_{2k}$ of bidegree $\left(\frac{n}{2}-k,\ \frac{n}{2}+k\right)$ with leading term $(-\Delta)^k$. The explicit expression for $\mathbb{P}_n$ for even dimensional manifolds of dimension $n$ is only explicit known for the Euclidean space $R^n$ and hence for the spheres $S^n$. The explicit formula for $\mathbb{P}_n$ on $S^n$, as we will discuss at the end of this chapter, has been studied in both Branson [B-1] and Beckner [Be-1], [Be-2].*

Assume that $A$ is a formally self-adjoint differential operator of positive order and has positive definite leading symbol, then $A$ also has the heat kernel trace expansion as follows:

$$\mathrm{Tr}(fe^{-tA}) \sim \sum_0^\infty a_k(f,A)t^{\frac{k-n}{2\ell}} \quad t \to 0^+ \tag{9}$$

where $a_k(f,A) = \int_M fU_k(A)dV$ and $U_k(A)$ is a local invariant of Riemannian geometry of order $k$, and where $n = \dim M$ and $2\ell$ is the order of $A$.

**Theorem.** *(Branson and Gilkey [BG]) Under the above analytical assumption , for any conformally covariant operator $A$, one has*

$$\begin{aligned}
&\frac{d}{d\varepsilon}|_{\varepsilon=0}a_m(A_{g_{w+\varepsilon f}}) = (m-n)a_m(f, A_{g_w})\\
&\frac{d}{d\varepsilon}|_{\varepsilon=0}\zeta_{A_{g_{w+\varepsilon f}}}(0) = 2\ell a_n(f, A_{g_w}).
\end{aligned}$$

Now we may proceed as in chapter one to compute the zeta functional determinant of $A$. By Weyl's invariants theory and the conformal transformation of local invariants, one can derive the formula for the normalized zeta functional determinant $F[w] = logdet\frac{detA_{g_w}}{detA_{g_0}}$. In the special case when $b-a=2$ and $vol(M,g_w) = vol(M,g_0)$, explicit expression of $F[w]$ was given by Branson -Orsted [BO] as:

$$F[w] = \gamma_1 I[w] + \gamma_2 II[w] + \gamma_3 III[w] \tag{10}$$

where $\gamma_1, \gamma_2, \gamma_3$ are constants depending only on $A$ and

$$I[w] = 4\int |C|^2 w dV - \left(\int |C|^2 dV\right) \log \fint e^{4w} dV$$

$$II[w] = \langle Pw, w\rangle + 4\int QwdV - \left(\int QdV\right) \log \fint e^{4w} dV,$$

$$III[w] = 12\left(Y(w) - \frac{1}{3}\int (\triangle R)\, wdV\right),$$

where $C$ is the Weyl tensor, $12Q = -\Delta R + R^2 - 3|\mathrm{Ric}|^2$ and

$$Y(w) = \int \left(\frac{\Delta(e^w)}{e^w}\right)^2 - \frac{1}{3}\int R\,|\nabla w|^2.$$

Both the quantities $Q$ and $|C|^2$ are terms which appear in the Gauss Bonnet integrand for compact. closed 4-manifolds:

$$4\pi^2\chi(M) = \int \left(\frac{|C|^2}{8} + Q\right) dV, \tag{11}$$

where $\chi(M)$ denotes the Euler-characteristic of the manifold M. It is also important to remark that the functional $III[w]$ may be written as

$$III[w] = \frac{1}{3}\left[\int R_w^2\, dV_w - \int R^2\, dV\right]$$

so that when the background metric is assumed to be the Yamabe metric in a positive conformal class, the functional $III$ is non-negative. Another important point is the relationship between the *Paneitz* operator and the $Q$ curvature:

$$-Pw + 2Q_w e^{4w} = 2Q, \tag{12}$$

Where $Q_w$ is the $Q$ curvature with respect to the metric $g_w = e^{2w}g_0$. Notice the comparison between the equation (12) to that of the Gaussian curvature equation on compact surfaces:

$$\Delta w + K_w e^{2w} = K.$$

In [BCY] and [CY-5], we continue the study of the log-determinant formula of Polyakov (9) as given in chapter 1 to the setting of general compact 4-manifolds. We first establish the compactness criteria for the functional determinant to a more general class of 4-manifolds; then we isolate the functional $II[w]$ and provide certain compactness criteria. We also provide certain a-priori estimates for solutions of the Euler equations for the functional $II$. To state the results more explicitly, we define

$$\begin{aligned} k_d &= -\gamma_1\int |C|^2 dV - \gamma_2\int QdV \\ &= (-\gamma_2)\, 4\pi^2\chi(M) + \left(\frac{\gamma_2}{8} - \gamma_1\right)\int |C|^2 dV \end{aligned} \tag{13}$$

and

$$\begin{aligned} k_p &= \int Q dV \\ &= 4\pi^2\chi(M) - \frac{1}{8}\int |C|^2. \end{aligned} \tag{14}$$

We have the following existence results ([CY-5]):

**Theorem.** *If the functional $F$ satisfies $\gamma_2 < 0$, $\gamma_3 < 0$, and $k_d < (-\gamma_2)8\pi^2$, then $\sup_{w \in W^{2,2}} F[w]$ is attained by some function $w_d$ and the metric $g_d = e^{2w_d} g_0$ satisfies the equation*

$$\gamma_1 |C_d|^2 + \gamma_2 Q_d - \gamma_3 \triangle_d R_d = -k_d \cdot Vol(g_d)^{-1}. \tag{15}$$

*Further, all functions $\varphi \in W^{2,2}$ satisfy the inequality:*

$$k_d \log \fint e^{4(\varphi - \tilde{\varphi})} dV_d \leq (-\gamma_2) \langle P\varphi, \varphi \rangle - 12\gamma_3 Y_d(\varphi).$$

*where $\tilde{\varphi}$ denotes the mean value of $\varphi$ with respect to the metric $g_d$, and $\fint$ denotes $\frac{1}{vol(M,g_d)} \int_M dV_d$.*

For the functional $II$, we have a similar existence result:

**Theorem.** *If $k_p < 8\pi^2$, and assume that $P$ is an non-negative operator with kernel of $P$ consisting of constant functions. Then $\inf_{w \in W^{2,2}} II(w)$ is achieved by some function $w_p$ and the metric $g_p = e^{2w_p} g$ satisfies the equation*

$$Q_p = k_p \; Vol(g_p)^{-1}; \tag{16}$$

*and all functions $\varphi \in W^{2,2}$ satisfy the inequality*

$$k_p \log \fint e^{4(\varphi - \tilde{\varphi})} dV_p \leq \langle P\varphi, \varphi \rangle \tag{17}$$

*where $\tilde{\varphi}$ is the mean value of $\varphi$ with respect to the metric $g_p$. Further the equation (16) may be expressed in terms of the conformal factor $w_p$:*

$$P w_p + 2Q = 2Q_p \, e^{4w_p}. \tag{18}$$

In particular for the operator $L$ and $\not\nabla^2$ , we obtain existence results for extremal metrics of the corresponding log-determinant functional. We also obtain existence of extremals for the functional $II$ in many cases. Thus for a large class of conformal 4-manifolds, we have the existence of several extremal metrics in addition to the Yamabe metric. It is an interesting problem to study the relation among these metrics. For example, we found in [BCY] that on $S^4$ all these extremal metrics coincide. In order to identify these extremal metrics in special circumstances, we provide some uniqueness results ([CY-5]):

**Theorem.** *If $k_d \leq 0$, the extremal metric $g_d$ for the functional $F$ corresponding to the conformal Laplacian operator $L$ is unique.*

**Theorem.** *If $k_p \leq 0$, $P$ is non-negative and $KerP = \{constants\}$, then the extremal metric $g_p$ is unique.*

These uniqueness assertions are obtained as consequences of the convexity of the corresponding functionals. We were able to identify some of the extremal metrics with known metrics in special circumstances. We remark that:

**Remark.**
*(1) Theorem 1.1 applies to all functionals of the form $\gamma_1 I + \gamma_2 II + \gamma_3 III$. What is required in the statement of Theorem 1.1 is that $\gamma_2\gamma_3 > 0$. In case $\gamma_2 > 0, \, , \gamma_3 > 0$ we substitute infimum for supremum of $F[w]$ in the statement of the theorem.*

*(2) The main examples of functionals $F[w]$, which appear in the form of Theorem 1.1 above, are quotients of two zeta function determinants*

$$F[w] = \log(\det\, A_w / \det\, A),$$

*where $A_w$ is the operator $A$ evaluated w.r.t. to the conformal metric $g_w = e^{2w}g$. In [BO], for operators $A$ or powers of $A$ which satisfies the conformal covariance property:*

$$A_w(\varphi) = e^{-bw} A\,(e^{aw}\varphi),$$

*explicit formulas for quotient of functional determinant are computed and are all of the form $F[w]$. For up to a multiplicative constant, the coefficients $\gamma_1$,$\gamma_2$,$\gamma_3$ in $F[w]$ are computed for $A = -\triangle + \frac{R}{6}$ the conformal Laplacian operator, for which the conformal covariance property holds with $b = 3$ and $a = 1$, and for $A = \nabla\!\!\!/^2$, the square of the Dirac operator for which the conformal covariance property holds with $b = \frac{3}{2}$ and $a = \frac{5}{2}$.*

We now examine the situations in which we can identify some of the extremal metrics. Let $g_d$ denote the extremal metric for the determinant of the conformal Laplacian, $g_p$ denote the extremal metric for the functional II, and $g_y$ denote the Yamabe metric. Here are some examples: We denote $k_p' = 4k_p$.

**Examples.**
*(1) For $S^4$ we have $|C|^2 = 0$, $\chi = 2$ hence $k_d(S^4) = k_p'(S^4) = 32\pi^2$. In this case all three extremal metrics exist and we have the coincidence of all three with the standard metric: $g_d = g_p = g_y = g_0$.*

*(2) For $\mathbb{H}^2 \times \mathbb{H}^2$, we have $\chi = (4\pi^2)^{-1} \cdot volume$, $|C|^2 = \frac{16}{3}$. For any lattice $\Gamma$, $k_d(\mathbb{H}^2 \times \mathbb{H}^2/\Gamma) = -4 \cdot volume < 0$, $k_p' = \frac{4}{3}\, volume$. So we have $g_0 = g_d = g_y$. When $\Sigma$ is a genus 2 hyperbolic surface, we have $k_p'(\Sigma \times \Sigma) = \frac{4}{3}\, 16\pi^2 < 32\pi^2$. In the standard metric for $\Sigma \times \Sigma$ we have*

$$P = \triangle^2 + \triangle.$$

*If we take $\Sigma$ to have a pinching neck so that $\lambda_1(\Sigma) \ll 1$, then we see that $\lambda_1(P)$ is small negative. For such conformal structure on $\Sigma \times \Sigma$, the functional $II$ is not bounded from below, and the standard metric is a saddle point for $II$.*
*(3) $S^1 \times S^3$ is known to be conformally flat and by varying the radius $t$ of $S^1$ we have a 1-parameter family of conformally flat structures $S^1(t) \times S^3$. We have $\chi = 0$ and $|C|^2 = 0$, so that $k_d(S^1 \times S^3) = k'_p(S^1 \times S^3) = 0$. It follows that $g_0 = g_d = g_p$. The last equality holds since the Euler equation is a linear equation. We recall that Schoen [Sc-2] has shown that there exists a $t_0$ such that for $t \le t_0$ , $g_y = g_0$; while $g_y \neq g_0$ when $t > t_0$. This is an instance where the extremal metric $g_d$ is different from the Yamabe metric.*

## C. Sharp inequalities

The main analytic tool in the study of the zeta functional determinants on 4-manifolds is the following sharp Sobolov inequality , which is a generalized form of Moser's inequality, due to Adam's [Ad]: On a compact 4-manifold M without boundary we have:

$$\int_M e^{32\pi^2|w-\bar{w}|^2} \le C_M, \text{ if} \int_M |\Delta w|^2 \le 1. \tag{19}$$

Applying the elementary inequality

$$4(w-\bar{w}) \le \frac{32\pi^2|w-\bar{w}|^2}{\int_M |\Delta w|^2} + \frac{1}{8\pi^2}\int_M |\Delta w|^2$$

one obtains

$$\log \fint_M e^{4(w-\bar{w})} \le \frac{1}{8\pi^2}\int_M |\Delta w|^2 + \log C_M. \tag{20}$$

Inequality (20) is the key inequality which used in the derivation of the compactness results in Section B above.

In the special cases when $k_d = (-\gamma_2)8\pi^2$ and $k_p = 8\pi^2$, as is the case on $(S^4, g_c)$ and when $A$ is the conformal Laplacian operator $L$. Instead of using (20), we may employ the following sharp inequality due to Beckner [Be-2]: On $(S^4, g_c)$,

$$\log \fint e^{4(w-\bar{w})} \le \frac{1}{3}\fint_{S^4} wP_4 w \tag{21}$$

with equality holds if and only if $e^{2w}g_c$ is isometric to the standard metric $g_c$ of $S^4$.

**Theorem 5.** *On $(S^4, g_c)$ if $\gamma_2 < 0$, and $\gamma_3 < 0$, then $F[w] \le 0$ and $F[w] = 0$ if and ony if $e^{2w}g_c$ is isometric to $g_c$.*

*Proof.* On $S^4$

$$\begin{aligned} F[w] &= \gamma_1 \left\{ 2\int_{S^4} wP_4w + 24\int_{S^4} w - 3\pi^2 \log \fint_{S^4} e^{4w} \right\} \\ &= \gamma_2 \left\{ \int_{S^4} R_w^2 dV_w - \int_{S^4} R_0^2 dV_0 \right\}. \end{aligned}$$

Thus the result follows from inequality (21) above and the sharp Yamabe inequality

$$\int_{S^4} R_w dV_w \geq \int_{S^4} R_0 dV_0,$$

where equality holds if and only if $e^{2w}g_c$ is isometric to $g_c$ on $S^4$.

To end this chapter, we would like to mention that inequality (21) is a special case of general set of inequalities established by Beckner on $(S^n, g_c)$ for all $n \geq 2$. To state the inequalities, we set

$$\mathbb{P}_n = \prod_{0}^{\frac{n-2}{2}} (-\Delta + k(n-k-1)), \text{ when } n \text{ is even}$$

$$\mathbb{P}_n = \left(-\Delta + \left(\frac{n-1}{2}\right)^2\right)^{\frac{1}{2}} \prod_{0}^{\frac{n-3}{2}} (-\Delta + k(n-k-1)), \text{ when } n \text{ is odd.}$$

Notice that $\mathbb{P}_n$ is the conformal pulling via the sterographic projection from $R^n$ with Euclidean metric to $S^n$ of the operator $(\Delta)^{\frac{n}{2}}$.

**Theorem.** *([Be-2])*

$$\log \fint_{S^n} e^{n(w-\bar{w})} \leq \frac{n}{2(n-1)!} \fint_{S^n} w\mathbb{P}_n w \tag{22}$$

*where equality holds if and only if* $e^{2w}g_c$ *is isometric to* $g_c$.

Beckner's original proof of inequality (22) uses methods in Fourier analysis and is quite ingenious. It turns out one can also give an alternative proof (22) using the geometric method employed by Onofri in his proof of the inequality for the special case $n = 2$. The proof exploits the conformal covariant property of the *Paneitz* type operators. We now briefly outline the proof here:

*Step 1.* Let $S_n[w] = \fint_{S^n} w\mathbb{P}_n w + 2(n-1)! \fint_{S^n} w$. Then $S_n[w]$ is a conformally invariant quantity. That is, if $e^{2v}g_0 = \phi^*(e^{2u}g_0)$ for some conformal transformation $\phi$ of $S^n$, then $S_n[v] = S_n[u]$.

*Step 2.* $\inf_{\fint e^{uw}=1} S_n[w] = \inf_{w\in S} S_n[w]$ and is attained. Here $S$ denotes the set of "balanced" metrics which satisfy

$$\fint_{S^n} e^{uw} = 1 \text{ and } \int_{S^n} e^{nw} x_j = 0, \quad j = 1, \ldots, n+1.$$

*Step 3.* If $S_n[w_0] = \inf_{w\in S} S_n[w]$, let $\lambda_1(\mathbb{P}_n^{w_0})$ be the first eigenvalue of the operator $\mathbb{P}_n^{w_0} = e^{-nw_0}\mathbb{P}_n$, then $\lambda_1(\mathbb{P}_n^{w_0}) \geq n!$.

*Step 4.* For any metric $g_w = e^{2w} g_c,\ w \in S,\ \lambda_1(\mathbb{P}_n^w) \le n!$ with equality holds if and only if $w \equiv 0$.

¿From above four steps we may conclude easily that

$$S_n[w] \ge 0 \text{ if } \fint_{S^n} e^{nw} = 1,$$

which implies inequality (22).

**Chapter 3. The formula on compact manifolds with boundary**

In the last two chapters we have discussed the generalized Ray-Singer-Polyakov formulas of the zeta functional determinants for conformally covariant geometric differential operators on manifolds without boundary. As applications we have also discussed extremal metrics of the zeta functional determinants and some related isospectral problems. In this chapter we will discuss similar problems on compact manifolds with boundary. Here we are reporting some recent joint work of Jie Qing and myself. [CQ-1], [CQ-2].

Let $(A, B)$ be a pair of conformally covariant geometric operators on $(M, \partial M)$, where $B$ is a boundary operator and we require that the space of functions $w$ with $Bw = 0$ on the boundary $\partial M$ does not depend on the metrics within a given conformal class. We know, under some suitable ellipticity assumptions of the operators $(A, B)$, we have

$$Tr(fe^{-tA_B}) \sim \sum_0^\infty a_k(f, A, B) t^{\frac{k-n}{2}} \quad ast \to 0^+ \tag{1}$$

where

$$a_k(f, A, B) = \int_M fU_k(A) + \sum_{\nu=0}^{n-1} \oint_{\partial M} (N^\nu f) U_{k,\nu}(A; B) \tag{2}$$

$U_k(A)$ is a local invariant of order $k$, $U_{k,\nu}(A, B)$ is a local invariant of order $k-\nu-1$, and $N^\nu$ is the $\nu$th covariant derivative in the inner normal direction. The typical example of such pairs of operators is $(L, \mathcal{R})$, where $L$ is the conformal Laplacian and $\mathcal{R} = \frac{\partial}{\partial n} - \frac{n-2}{2(n-1)} H$, is the Robin boundary operator, where $H$ denotes the mean curvature on $\partial M$ and $n$ is the dimension of the manifold $M$.

**Theorem.** *[BG-1], [BG-2] On $(M, \partial M)$, suppose $A$ is an elliptic operator of order $2l$, and $(A, B)$ satisfies some suitable ellipticity and conformal invariant properties, then*

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} a_k(A, B)[u + \varepsilon f] = (k-n) a_k(f, A, B)[u]$$

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \zeta'_{(A,B)[u+\varepsilon f]} = 2l a_n(f, A, B)[u]. \tag{3}$$

For the purpose of this talk, we may assume $(A, B)$ is the pair of operators $(L, R)$.

**A. 2-dimensional case.**

We will start the discussion with the case of compact surfaces with boundary. For simplicity, we will consider the the pair of operators $\left(-\Delta, \frac{\partial}{\partial n}\right)$. Recall

$$a_2\left(f,-\Delta,\frac{\partial}{\partial n}\right)=\frac{1}{12\pi}\left(\int_M fK+\oint_{\partial M} fk\right)+\frac{1}{8\pi}\oint_{\partial M}\frac{\partial}{\partial n}f \tag{4}$$

where $K$ is the Gaussian curvature of $M$, $k$ is the geodesic curvature of the boundary $\partial M$. As a consequence of (3) above, we have

$$F[w]=\log\frac{\det\left(-\Delta,\frac{\partial}{\partial n}\right)[w]}{\det\left(-\Delta,\frac{\partial}{\partial n}\right)[0]}=-2\int_0^1 a_2\left(w,-\Delta,\frac{\partial}{\partial n}\right)[tw]dt. \tag{5}$$

Applying the relations

$$\begin{cases} -\Delta w+K[0]=K[w]e^{2w} \\ \frac{\partial}{\partial n}w+k(0)=k[w]e^{w} \end{cases}$$

we get

$$\begin{aligned} F[w]= & -\frac{1}{12\pi}\left\{\int_M|\nabla w|^2+2\int_M K[0]w+\oint_{\partial M}k[0]w\right\} \\ & -\frac{1}{4\pi}\left\{\oint_{\partial M}k[w]-\oint_{\partial M}k[0]\right\}. \end{aligned} \tag{6}$$

To consider the extremum of $F[w]$, one needs to assume that

$$\oint_{\partial M}k[w]e^w-\oint_{\partial M}k[0]=0, \tag{7}$$

which appears due to the term $\oint_{\partial M}\frac{\partial}{\partial n}f$ in the expression of $a_2\left(f,-\Delta,\frac{\partial}{\partial n}\right)$. A more geometric reason for imposing condition (7) is given in [OPS-2]. Notice that without imposing the condition (7), the functional $F[w]$ is not be bounded from either above or below.

There are two types of variational problem for manifolds with boundary. We will call the conformal changes of metrics which preserve vol$(M)$ by the conformal variations of type I, and call the conformal changes of metrics which preserve vol$(\partial M)$ by the conformal variations of type II. Then, when restricted to metrics conformal to a fixed one, the extremal metrics for the conformal variations of type I have constant Gaussian curvature on $M$ and vanishing geodesic curvature on $\partial M$, while the extremal metrics for the conformal variations of type II have vanishing Gaussian curvature on $M$ and constant geodesic curvature on $\partial M$. For simply-connected surface, the extremal metrics are metrics of the canonical metrics of hemi-sphere and disk for type I and type II problems respectively. In [OPS-2], Osgood, Phillips and Sarnak have studied these variational problems also for the log-determinant functional $F[w]$ in (6) and in particular they have established the existence of the extremal metrics for the functional. The role played by the Onofri's inequality is replaced by the following Milin-Lebedev inequality in studying the extremals of $F[w]$ on compact surface with boundary. was the following Lebedev-Milin inequality

$$\log \oint_{S^1} e^{(w-\bar{w})} \frac{d\theta}{2\pi} \leq \frac{1}{4} \left\{ \int_D w(-\Delta) w \frac{dxdy}{\pi} + 2 \oint_{S^1} w \frac{\partial w}{\partial n} \frac{d\theta}{2\pi} \right\} \tag{8}$$

where $\bar{w}$ denotes the average of $w$ over $S^1$, and $W$ ia any extension of $w$ inside the unit disc $D$, and with equality holds if and only if $(D, e^w g_0)$ is isometric to the standard metric $(D, g_0)$.

**B. 3-dimensional case.**

On compact 3-manifolds with boundary, we again consider general conformally covariant, geometric, elliptic pair of operators $(A, B)$. Via Weyl's invariant theory, we can write

$$\begin{aligned} a_3(f, A, B) = & \oint_{\partial M} f(\alpha_1 \tilde{K} + \alpha_2 H^2 + \alpha_3 (|L|^2 + \alpha_4 F) \\ & + \beta \oint_{\partial M} f_N H + \gamma \oint_{\partial M} f_{;NN} \end{aligned} \tag{9}$$

where $\tilde{K}$ is the Guassian curvature of $\partial M$, $N$ denotes the inward normal covariant differentiation $a$ denotes the index $\{1, 2\}$ in the tangential direction of the boundary $\partial M$, $f_{;NN}$ denotes (double) covariant differentiation in the $N$ direction and $F = R_{aNaN}$.

Applying the conformal covariance property of $(A, B)$, one may reduce the expression of $a_3$ in (9) above as:

$$a_3(f, A, B) = \oint_{\partial M} f(\alpha_1 \tilde{K} + \alpha_2 (H^2 - |L|^2)) + \beta \oint_{\partial M} f_N H + \gamma \oint_{\partial M} f_{;NN}. \tag{10}$$

We may then apply the relation (2) to compute the log-determinant of zeta functional for quotient of two conformal metrics as:

$$\begin{aligned} \log \frac{\det(A, B)[w]}{\det(A, B)[0]} = & -2\alpha_1 \left\{ \frac{1}{2} \oint_{\partial M} |\tilde{\nabla} w|^2 + \oint_{\partial M} \tilde{k}_0 w \right\} \\ -2\alpha_2 \oint_{\partial M} w(H^2 - 2|L^2|) + & \frac{\beta}{2} \left\{ \oint_{\partial M} (H^2 ds)[w] - \oint_{\partial M} (H^2 ds)[0] \right\} \\ + \gamma \left\{ \oint_{\partial M} \left( \left( F + \frac{1}{4} H^2 \right) ds \right) [w] - \oint_{\partial M} \left( \left( F + \frac{1}{4} H^2 \right) ds \right) [v] \right\}. & \end{aligned} \tag{11}$$

In the following we will briefly indicate how to derive the formula (11) from the expression of $a_3$ in (10). For demonstration purpose, we will indicate the computation for the term with $\gamma$ as its coefficient. We first write

$$f_{;NN}[w] e^{2w} = f_{;NN}[0] - f_N w_N + f_a w_a. \tag{12}$$

Therefore

$$\int_0^1 \left( \oint_{\partial M} w_{;NN}[tw] e^{2tw} \right) dt = \oint_{\partial M} w_{;NN} - \frac{1}{2} \oint_{\partial M} (w_N)^2 + \frac{1}{2} \oint_{\partial M} |\tilde{\nabla} w|^2. \quad (13)$$

To further express the terms in (13) into geometric forms, we first observe that

$$\begin{aligned} \frac{1}{2}\left(F + \frac{1}{4}H^2\right)[w]e^{2w} &= \frac{1}{2}\left(F + \frac{1}{4}H^2\right)[0] - w_{;NN} \\ &\quad + \frac{1}{2}\tilde{\Delta} w - \frac{1}{2}|\tilde{\nabla} w|^2 + \frac{1}{2}(w_N)^2. \end{aligned} \quad (14)$$

Thus

$$\begin{aligned} \int_0^1 \left( \oint_{\partial M} w_{;NN}[tw] e^{2tw} \right) dt &= \frac{1}{2} \left\{ \oint_{\partial M} \left( \left( F + \frac{1}{4}H^2 \right) ds \right) [0] - \right. \\ &\quad \left. \oint_{\partial M} \left( \left( F + \frac{1}{4}H^2 \right) ds \right) [w] \right\}. \end{aligned} \quad (15)$$

Since the formula (10) only involves boundary terms, one can handle the study of the extremal metrics in this case using the Moser-Trudinger and Onofri inequalities on 2-manifolds; in particular the study of the formula does not provide new insight of how to control the behavior of a conformal metric on the interior of the manifold from its boundary behavior. We will soon see that this is not the case on 4-manifolds with boundary.

**C. 4-dimensional case.**

It is natural that one seeks results about functional determinants on 4-manifolds with boundary analogous to the above mentioned ones on compact surfaces with boundary. For the convenience of this chapter we shall assume that $(A, B) = (L, \mathcal{R})$ from now on. We first modify the definitions of conformal variations. For conformal variations of type I, we require the mean curvature $H$ stays vanishing, and for conformal variations of type II we require the mean curvature $H$ stays constant. In addition, we always require that $\oint_{\partial M}(T\ ds)[w]$ fixed for all conformal variations (this corresponds to the requirement that $\oint_{\partial M} kds$ stays constant for all conformal variations in the cases considered in [OPS] in dimension 2). It turned out that, to establish the existence of extremum of the functional determinant under conformal variations of type I, the following sharp inequality of Adams type is crucial.

**Theorem.** *[CQ-2] For any function $\phi$ on a given 4-manifolds $(M, \partial M)$ with Neumann boundary condition $N\phi = 0$,*

$$\int_M e^{\alpha|\phi - \bar{\phi}|^2} dx < C, \quad (16)$$

*if $\int_M |\Delta\phi|^2 dx \leq 1$, for any $\alpha < 16\pi^2$.*

**Remark.** *Although for the purpose here it is sufficient that (16) holds for any* $\alpha < 16\pi^2$*, in view of the Moser inequalities etc before, one would expect that the inequality (16) holds for* $\alpha = 16\pi^2$*, which remains to be verified.*

Apply the inequality (16), one can establish the existence of extremum for $F[w]$ for conformal variations of type I relatively easily. In this case, one can also establish the uniqueness in some special classes of metrics.

**Theorem.** *The maximum of* $F[w]$ *is achieved by some* $w_d \in W^{2,2}$ *if* $-a_4(L, \mathcal{R}) < 16\pi^2$*. Moreover, if* $a_4(L, \mathcal{R}) \geq 0$*, then* $w_d$ *is unique.*

**Remark.** *One borderline case is* $H^4$ *with standard metric* $g_0$*, where* $-a_4(L, \mathcal{R}) = 16\pi^2$*. We observed that, under conformal variations of type I,* $F[w] \leq 0$ *and the* $\sup_{\omega \in \mathcal{E}} F[w] = 0$ *is attained by and only by* $e^{2w} g_0 \in \phi^* g_0$ *for some conformal transformation* $\phi$ *of the standard sphere* $S^4$ *which maps the upper hemisphere to itself.*

The problem of studying the functional $F[w]$ for metrics conformal variations of type II is a much more involved one. It is clear that one would need some sharp inequality of the Lebedev-Milin type which relates the behavior of a metric inside a manifold to its boundary behavior. In an effort to search a correct formulation for such an inequality for compact 4-manifolds $(M, \partial M)$, we (in [CQ-1] and [CQ-2]) have taken a geometric point of view. First we view the original Lebedev-Milin inequality (8) as an inequality which relates the the pair of local invariants $(K, k)$ to the pair of operator$(\Delta, \frac{\partial}{\partial n})$. As we have mentioned before on compact surfaces, the relationship between $(K, k)$ and $(\Delta, \frac{\partial}{\partial n})$ under conformal changes of metrics can be expressed via the partial differential equations:

$$\Delta w + K[0] = K[w] e^{2w}$$

and

$$\frac{\partial w}{\partial n} + k[0] = k[w] e^{w}.$$

Therefore, on compact 4-manifolds, we start with the search for the right pairs of curvature functions and their corresponding differential operators. As the study in the case of compact closed manifolds suggest, one natural interior local invariant of order 4 which should appear is the *Paneitz* quantity

$$12Q = -\Delta R + R^2 - 3|\mathrm{Ric}|^2,$$

which has been discussed in Chapter 2. The differential operator corresponds to $Q$ is the *Paneitz* operator

$$P_4 = (-\Delta)^2 + \mathrm{div}\left(\frac{2}{3} R\ Id - 2R_{ij}\right).$$

The relation between $Q$ and $P_4$ can be seen via the following differential equation.

$$P_4 w + 2Q[0] = 2Q[w] e^{4w} \text{ in } M.$$

It turns out that on 4-manifolds there also exists a boundary local invariant of order 3 and a conformal covariant operator $P_3$ of bidegree $(0,3)$, the relation of $(Q,T)$ to $(P_4, P_3)$ on 4-manifolds is parallel to that of $(K,k)$ to $(\Delta, \frac{\partial}{\partial n})$ on compact surfaces. More specifically we have a local invariant of order 3 defined as

$$T = \frac{1}{12}\frac{\partial}{\partial n}R + \frac{1}{6}RH - FH + \frac{1}{9}H^3 - \frac{1}{3}TrL^3 + \frac{1}{3}\tilde{\Delta}H \tag{17}$$

and a corresponding operator

$$P_3 = \frac{1}{2}\frac{\partial}{\partial n}\Delta + \tilde{\Delta}\frac{\partial}{\partial n} + \frac{2}{3}H\tilde{\Delta} + L_{ab}\tilde{\nabla}_a\tilde{\nabla}_b + (\frac{1}{3}R - F)\frac{\partial}{\partial n} + \frac{1}{3}\tilde{\nabla}H\cdot\tilde{\nabla}$$

such that

$$P_3 w + T[0] = T[w]e^{3w} \text{ on } \partial M,$$

and

$$P_3[w] = e^{-3w}P_3[0].$$

The best way to understand how $T$ and $P_3$ were discovered in [CQ-2] is via the Chern-Gauss-Bonnet formula for 4-manifolds with boundary:

$$\chi(M) = (32\pi^2)^{-1}\int_M (|C|^2 + 4Q)dx + (4\pi^2)^{-1}\oint_{\partial M}(T - \mathcal{L}_4 - \mathcal{L}_5)dy. \tag{18}$$

where $\mathcal{L}_4$ and $\mathcal{L}_5$ are boundary invariant of order 3 which are invariant under conformal change of metrics. Hence with a fixed conformal class of metrics,

$$\frac{1}{2}\int_M Qdv + \oint_{\partial M} Tds$$

is a fixed constant. We would like to remark that in the original Chern-Gauss-Bonnet formula $T$ is not exactly the term as we have defined in (17) above, actually it differs from $T$ by $\frac{1}{3}\tilde{\Delta}H$, which does not affect the integration formula.

Thus on 4-manifolds with boundary it is natural to study the energy functional

$$E[w] = \frac{1}{4}\int wP_4w + \frac{1}{2}\int wQ + \frac{1}{2}\oint_{\partial M} wP_3w + \oint_{\partial M} wT. \tag{19}$$

In view of the complicated expressions of the operators $P_4$, $P_3$, $Q$ and $T$, in general it is difficulty to study the functional $E[w]$ defined as above at this moment. But in the special case of $(B^4, S^3)$ with the standard metrics, we have

$$P_4 = (\Delta)^2, P_3 = \frac{1}{2}N\Delta + \tilde{\Delta}N + \tilde{\Delta}, \text{ and } Q[0] = 0, \text{ and } T[0] = 3. \tag{20}$$

Thus the expression in $E[w]$ becomes relatively simple. Thus in this special case, we are able to study the conformal variation problem of type II of the functional $F[w]$. The main analytic tool is the following sharp inequality of Lebedev-Milin type on $(B^4, S^3)$.

**Theorem.** *Suppose $w \in C^\infty(\bar{B}^4)$ and that $e^{2w}g_0$ is a conformal variation of type II with respect to the standard metric $g_0$ of $B^4$. Then*

$$\log\left\{\frac{1}{2\pi^2}\oint_{S^3} e^{3(w-\bar{w})}dy\right\} \leq \frac{3}{4\pi^2}\left\{\frac{1}{4}\int_{B^4} w\Delta^2 w + \frac{1}{2}\oint_{S^3} wP_3 w + \frac{1}{4}\oint w_N + \frac{1}{4}\oint w_{;NN}\right\}, \tag{21}$$

*under the boundary assumptions*

$$\int_{S^3}\tau[w]ds[w] = 0$$

*where $\tau$ is the scalar curvature of $M$. Moreover the equality holds if and only if $e^{2w}g_0$ on $B^4$ is isometric to the canonical metric $g_0$.*

**Remark.** *The additional boundary condition assumed in the above is necessary in the sense that the left side of the inequality (21) would not have any lower bound otherwise. The choices of the right geometric conditions, like we imposed here, depend on the analytic expressions of transformation formulas for curvatures under conformal changes of metrics.*

The key step in the proof of theorem is the following analytic lemma.

**Lemma.** *Suppose $w$ solves*

$$\begin{cases} \Delta^2 w = 0 & \text{in } R^4 \\ w\,|_{S^3} = u & \\ w_N\,|_{S^3} = \phi. & \end{cases}$$

*Then*

$$\Delta w\Big|_{\partial B^4} = 2\tilde{\Delta}u - 2\left\{(-\tilde{\Delta}+1)^{\frac{1}{2}} + 1\right\}\phi \tag{22}$$

$$N\Delta w\Big|_{\partial B^4} = 2\mathbb{P}_3 u + 2\tilde{\Delta}u - 2\tilde{\Delta}\phi \tag{23}$$

*where $\mathbb{P}_3 = (-\tilde{\Delta}+1)^{\frac{1}{2}}(-\tilde{\Delta})$ is the same as the $\mathbb{P}_3$ operator defined on $S^3$ by Beckner in Chapter 1.*

We would also like to remark that the second term in (22) above, i.e. the term $\left\{(-\tilde{\Delta}+1)^{\frac{1}{2}}+1\right\}$ is the Dirichlet to Neumann operator on $(B^4, S^3)$.

**Corollary.** *On $B^4$,*

$$P_3[0](w) = \mathbb{P}_3(w) \text{ on } \partial B^4,$$

*provided that $\Delta^2 w = 0$ in $B^4$.*

We can then state a result about extremal metrics of conformal covariant type II on $(B^4, S^3, g_0)$.

**Theorem.** *On $B^4$, suppose $g_w$ is a metric conformal variation of type II with respect to the standard metric $g_0$. And assume further that $Q[w]=0$. Then*

$$
\begin{aligned}
F[w] = \; & -8\left\{\frac{1}{4}\int_{B^4} wP_4w + \frac{1}{2}\oint_{S^3} wP_3w + 2\oint_{S^3} w\right\} \\
& - 4\int_{B^4} R^2[w]e^{4w} \\
& - 4\left\{\oint_{S^3}(\tilde{\tau}dy)[w] - \oint_{S^3}(\tilde{\tau}dy)[0]\right\} \\
& - \frac{50}{3}\left\{\left\{\int_{B^4}(Rdx)[w_h] + 2\oint_{S^3}(Hdy)[w]\right\}\right. \\
& \quad \left. - \left\{\int_{B^4}(Rdx)[0] + 2\oint_{S^3}(Hdy)[0]\right\}\right\}. \qquad (24)
\end{aligned}
$$

*Hence $F[w]$ is non-positive and attains its maximum at and only at metrics $g_w = e^{2w}g_0$ which is isometric to the standard metric $g_0$ within the class of conformal variations of type II with $Q[w]=0$.*

Here $F[w]$ denotes a constant multiple of the functional determinant. Above result is a consequence of three sharp inequalities. The non-positivity of the first line in (24) above is a consequence of the sharp inequality of Lebedev-Milin type which we have established in (21), the non-positivity of the third line in (24) follows from the sharp Sobolev inequality on $S^3$ for the Yamabe problem, and the non-positivity of the fourth term in (24) follows from the sharp Trace Sobolev inequality established by Escobar ([E-1], [E-2]) and Beckner [Be-1].

The problem of studying the extremal metrics for conformal variations of type II problem for the log-determinant functional on general compact manifolds with boundary remains open.

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