Partial differential equations related to the Gauss-Bonnet-Chern integrand on 4-manifolds

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0. Introduction

In this article we discuss some recent developments in conformal geometry of four dimension. The main theme is the study of curvature invariants which form part of the integrand of Gauss-Bonnet-Chern on compact 4-manifold by means of fourth order non-linear differential equations and a related fully nonlinear second order equation. We will report on two aspects of this development. The first deals with conformal metrics on locally conformally flat 4-manifolds, the primary examples arising from the quotients of Kleinian groups. We are interested in developing finiteness criteria for such conformal structures. We will discuss some joint work with Jie Qing [CQY-1, CQY-2, CQY-3], as well as some results of Hao Fang [F] extending some well known finiteness criteria of Cohn-Vossen [CV] and Huber [H] for surfaces to dimension four. The second aspect is concerned with the Ricci tensor. The natural fourth order conformal curvature invariant is a quadratic expression in the Ricci tensor. Under natural positivity assumptions on the conformal structure, it is possible to prescribe the quadratic invariant. This result has implication for the existence of conformal metrics of strongly positive Ricci tensor. We will report on joint work with Matt Gursky [CGY-2, CGY-3] on this problem.

A basic tool in our study is the partial differential equations associated with conformally covariant operators. On a Riemannian manifold (M^n, g) of dimension n, the Laplace operator is the natural geometric operator. Under conformal change of metric $g_w = e^{2w}g$, when the dimension is two, Δ_{g_w} is related to Δ_g by the simple formula:

$$\Delta_{g_w}(\varphi) = e^{-2w}\Delta_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^2) \tag{0.1}$$

In dimension greater than two, similar transformation property continues to hold for a modification of the Laplacian operator called the conformal Laplacian operator $L \equiv -\Delta + \frac{n-2}{4(n-1)}R$ where R is the scalar curvature of the metric. We have

$$L_{g_w}(\varphi) = e^{-\frac{n+2}{2}\omega} L_g\left(e^{\frac{n-2}{2}\omega}\varphi\right) \tag{0.2}$$

for all $\varphi \in C^\infty(M)$. In general, we call a metrically defined operator A conformally covariant of bidegree (a, b) if, under the conformal change of

*Research supported in part by NSF Grant DMS-0070542 and a Guggenheim Foundation Fellowship.

†Research supported in part by NSF Grant DMS-0070526 and the Ellentuck Fund.

metric $g_\omega = e^{2\omega} g$, the pair of corresponding operators A_ω and A are related by

$$A_\omega(\varphi) = e^{-b\omega} A(e^{a\omega}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n). \tag{0.3}$$

A particularly interesting such operator is a fourth order operator on 4-manifolds discovered by Paneitz [P] in 1983:

$$P\varphi \equiv \Delta^2\varphi + \delta\left(\frac{2}{3}Rg - 2\mathrm{Ric}\right) d\varphi \tag{0.4}$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor of the metric. The *Paneitz* operator P is conformal covariant of bidegree $(0,4)$ on 4-manifolds, i.e.

$$P_{g_w}(\varphi) = e^{-4w} P_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^4). \tag{0.5}$$

A fourth order curvature invariant $Q = \frac{1}{12}\{-\Delta R + R^2 - 3|Ric|^2\}$ is associated to the Paneitz operator:

$$Pw + 2Q = 2Q_w e^{4w}. \tag{0.6}$$

In dimension four, the Paneitz equation has close connection with the Gauss-Bonnet-Chern formula. For a compact oriented 4-manifold,

$$\chi(M) = \frac{1}{4\pi^2}\int_M \left(\frac{|W|^2}{8} + Q\right) dv \tag{0.7}$$

where $\chi(M)$ denotes the Euler characteristic of the manifold M, and W the Weyl tensor, and $|W|^2$ the norm squared of the Weyl tensor. Since $|W|^2 dv$ is a pointwise invariant under conformal change of metric, Qdv is the term which measures the conformal change in formula (0.7).

The Paneitz operator plays the same role on manifold of dimension four as the Laplacian on surfaces; with the Q curvature replacing the role played by the Gaussian curvature in conformal geometry.

In dimension four, the Q curvature is also related to the second elementary symmetric function $\sigma_2(A)$ where $A = Ric - \frac{1}{6}Rg$ is sometimes called the conformal Ricci tensor:

$$Q = \frac{1}{2}\sigma_2 + \frac{1}{12}(-\Delta R).$$

The decomposition of the curvature tensor may be expressed as

$$Rm = W + \frac{1}{2} A \owedge g$$

where $\owedge$ indicates the Kulkarni-Nomizu product of bilinear forms. Elementary symmetric functions of the tensor A has been studied by Viaclosvky [V-1] in his thesis. Thus the Yamabe problem is to prescribe the first symmetric function of the tensor A. It is fruitful to ask if one can prescribe the other symmetric functions of the tensor A as well. In particular, in low dimensions, the second symmetric functions of A has strong implications for the Ricci tensor. In dimension four, we can use the fourth order equation as a regularization of the second order equation to prescribe the second elementary symmetric functions $\sigma_2(A)$ where A is the conformal Ricci tensor $A = Ric - \frac{1}{6}Rg$. In a joint work with Gursky, we take up this idea to give a simple criteria for existence, in a given four dimensional conformal class, of a metric with strongly positive Ricci tensor. The differential structures that support conformal classes satisfying the conformally invariant conditions $\int \sigma_2(A)dv > 0$ and having positive Yamabe invariant has the same known obstructions as those differential structures supporting positive Einstein metrics: the Hitchin-Thorp inequality

$$2\chi + 3\tau > 0, \quad \text{and} \quad 2\chi - 3\tau > 0. \tag{0.8}$$

It is an interesting question to ask if these two classes coincide and if (0.8) gives the only obstructions.

In section one below, we will describe our joint work with Jie Qing ([CQY-1], [CQY-2], [CQY-3]) on some finiteness criteria of locally conformally flat four manifolds. In section two we will give some background material for the work in [CGY-2] and [CGY-3]. In section 3, we will sketch the proof of the main result in [CGY-2], in section 4, an outline of the proof in [CGY-3].

The authors would like to thank the organizers for the invitation to take part in the conference and to write this survey for the proceedings. The authors would also like to thank their collaborators, in particular Matt Gursky and Jie Qing, for sharing with us their inspiration and labor in taking part in these development.

1. Finiteness of conformally flat structures

It is well known that two dimensional metrics g are locally conformally flat: there exists coordinate system x in which g has the form $g = e^{2w}|dx|^2$. We first review the basic finiteness results for conformal structures in dimension two: the Cohn-Vossen Huber inequality, Ahlfors' finiteness theorem and the Bishop-Jones characterization of geometrically finite Kleinian groups. We discuss the extension of some of these results to dimension four.

To recall the Cohn-Vossen [C-V] inequality for complete surfaces, suppose (M, g) is a complete surface with integrable Gauss curvature K, then

$$\int_M K dA \leq 2\pi\chi. \tag{1.1}$$

In addition, Huber [Hu] has shown that such a surface has a conformal compactification $M = \tilde{M}\backslash\{P_1, \dots, P_n\}$ where $\tilde{M}$ is a compact Riemann surface. At each puncture P_i by inverting a conformal disc $D_i\backslash\{P_i\}$, Finn [Fi] studied the isoperimetric ratio $\nu_i = \lim_{r\to\infty} \frac{(Length(\partial D_r))^2}{4\pi Area(D_r)}$, to account for the deficit in the inequality above:

$$\chi(M) - \frac{1}{2\pi}\int_M K dA = \sum_{i=1}^{n} \nu_i. \tag{1.2}$$

A completely analogous situation holds in dimension four provided we restrict ourselves to conformally flat 4-manifolds of positive scalar curvature. Let us recall that Schoen-Yau [ScY] showed that for such manifolds, the holonomy cover embed conformally as domain $\tilde{M}$ in S^4 with a boundary of Hausdorff dimension less than or equals to one. Thus by going to a covering of such manifolds we may assume that we are dealing with domains in $\mathbb{R}^4$.

Theorem 1.1 (CQY-1). *Let $e^{2w}|dx|^2$ be a complete metric on $\Omega = \mathbb{R}^4\backslash\{P_1, \dots, P_n\}$ with nonnegative scalar curvature near the punctures. Suppose in addition that Q is integrable. Then we have*

$$\chi(\Omega) - \frac{1}{4\pi^2}\int_\Omega Q dv = \sum_{i=1}^{n} \nu_i \tag{1.3}$$

where at each puncture P_i a conformal disk $D_i\backslash\{P_i\}$ is inverted and

$$\nu_i = \lim_{r\to\infty} \frac{(vol(\partial B_r))^{4/3}}{4(2\pi^2)^{1/3} vol(Br)}. \tag{1.4}$$

To give some idea of the proof of Theorem 1.1, we explain the situation on $\mathbb{R}^4$. The proof is based on an idea of Finn, to compare the conformal factor with the biharmonic potential derived from the measure Qdv. The positivity of the scalar curvature at infinity implies that the conformal factor agrees with the potential up to a constant. Working then with the expression of the potential as a logarithmic integral, a delicate analysis shows that the isoperimetric ratio ν can be compared with that of the symmetrized potential. In the latter case the required identity follows from an analysis of the resultant ODE.

The finiteness of the Q integral together with the embedding result of Schoen-Yau has strong implication for the underlying topology:

Theorem 1.2 (CQY-2). *Let (M^4, g) be a simply connected complete conformally flat manifold satisfying scalar curvature $R \geq c > 0$, $Ric \geq -c$, and $\int |Q| dv < \infty$; then M is conformally equivalent to $S^4\backslash\{P_1, \dots, P_k\}$. In case M^4 is not assumed simply connected, under the additional assumption that M^4 is geometrically finite as a Kleinian manifold, then M is conformally equivalent to $\tilde{M}\backslash\{P_1, \dots, P_k\}$, where $\tilde{M}$ is a compact conformally flat manifold. In addition, we have*

$$\chi(M) = \frac{1}{4\pi^2}\int_M Q dv + k \tag{1.5}$$

A consequence of this finiteness criteria is a determination of complete conformal metrics defined on domains in S^4, which satisfy the curvature conditions in Theorem 1.2, and in addition has constant Q curvature which are integrable. There are only three such metrics: the standard metric on S^4, the flat metric on $\mathbb{R}^4$ and the cylindrical metric on $\mathbb{R}^4\backslash\{0\}$.

To give an idea of the proof of Theorem 1.2 let us suppose Ω is a domain in $\mathbb{R}^4$ on which there is a conformal metric $g = u^2|dx|^2 = e^{2w}|dx|^2$ satisfying the assumptions of Theorem 1.2. A key ingredient in the proof of Theorem 1.2 is the following size estimate of the conformal factor $u(x)$ for $x \in \Omega$ in terms of the Euclidean distance $d(x) = distance(x, \partial\Omega)$.

$$\frac{1}{C}d(x)^{-1} \leq u(x) \leq Cd(x)^{-1} \quad \text{for all} \quad x \in \Omega \tag{1.6}$$

We remark that the lower bound follows from the Harnack estimate of Yau [ScY]. The upper estimate is derived using a blow up argument for the Paneitz equation, together with the following uniqueness result.

Lemma 1.3. *On* $(\mathbb{R}^4, u^2|dx|^2)$*, the only metric with* $Q \equiv 0$ *and* $R \geq 0$ *at infinity is isometric to* $(\mathbb{R}^4, |dx|^2)$*.*

The main idea of the argument is that the finiteness of the Q integral should imply a rate of growth of volume of the sublevel sets of the conformal factor u. More precisely consider the sets

$$U_\lambda = \{x : u(x) \leq \lambda\} \text{ and} \tag{1.7}$$

$$S_\lambda = \{x : u(x) = \lambda\}, \tag{1.8}$$

for large values of λ. Apply the Gauss-Bonnet-Chern formula (0.7) for the domain U_λ, we obtain

$$C \geq \lambda \frac{d}{d\lambda} V(\lambda) \tag{1.9}$$

where

$$V(\lambda) = \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J(\partial_n w) e^{2w} d\sigma + 2\int_{U_\lambda} J|\nabla u|^2 dx. \tag{1.10}$$

The positivity of the scalar curvature then implies that

$$V(\lambda) \geq C \int_{U_\lambda} u^4 dx. \tag{1.11}$$

Since the level sets of u are comparable to the complements of the balls covering the limit set Λ, a measure theoretic argument then shows that Λ consists of a finite number of points.

This result has been extended in two directions in the thesis of Fang. The first extension replaces integrability of Q by the integrability of $\sigma_2(A)$.

Theorem 1.4 (F)**.** *Let* $\Omega \subset S^4$ *be a domain on which there is a conformal metric* g *satisfying the curvature bounds:* $0 < c \leq R \leq C$; $|\nabla R| \leq C$, $-C \leq$ *Ric and*

$$\int_\Omega |\sigma_2(A)| dv < \infty; \tag{1.12}$$

then $\Omega = S^4 \backslash \{P_1, ..., P_k\}$, *and*

$$4\pi^2\chi(\Omega) = \int Qdv + k. \tag{1.13}$$

The second extension is to the higher dimensional setting. In general even dimensions $n = 2k$, the existence of a conformally covariant operator of degree n is shown by the work of Fefferman and Graham ([FG], see also [GJMS]), the general expression is necessarily complicated. However in the case of locally conformally flat metric, the relation between the operator and the Q curvature invariant can be computed due to its conformal covariance property. Also to obtain the analogue of Theorem 1.2, it is necessary to impose additional higher curvature bounds. Hao Fang [F] has worked out the precise condition for dimension six:

Theorem 1.5 (F). *Let* $\Omega \subset S^6$ *be a domain on which there is a complete conformal metric* g *satisfying the curvature bounds:* $0 < c \leq R \leq C$; $|\nabla R| \leq C$, $-C \leq Ric$ *and*

$$-\Delta J - 11J^2 + 56\sigma_2 - 64\frac{\sigma_2^2}{J^2} - 3|\nabla J|^2 \geq c > 0, \tag{1.14}$$

where $J = \frac{1}{10}R$. *Then* $\Omega = S^6 - \{P_1, ..., P_k\}$ *and*

$$c_6\chi(\Omega) = \int Qdv + k. \tag{1.15}$$

We remark that condition (1.14) is a "pinching condition" on the curvatures, which is satisfied, for example, for metrics which is in a small neighborhood of the standard metrics on S^4 on the region Ω.

We turn our attention to the general conformally flat manifolds which arise from the quotient of Kleinian groups. In the analytic study of Kleinian groups, a desirable condition that provides good control of the geometry and analysis of the Kleinian manifold is that of geometric finiteness. We recall that the conformal group $Conf(S^n)$ of the n-sphere also acts as hyperbolic isometrics of the hyperbolic $(n+1)$-ball. A discrete subgroup $\Gamma \subset Conf(S^n)$ is called a Kleinian group. The set of limit points of Γ is called the limit set Λ and its complement Ω is called the domain of discontinuity. In dimension two, a celebrated result of Ahlfors and Bers says that if Γ is finitely generated discrete subgroup, then the Kleinian quotient Ω/Γ is a finite union of Riemann surfaces of finite type, i.e. compact Riemann surfaces with at most a finite number of punctures. Such a Kleinian group is said to be analytically finite. The analogous result in higher dimensions is not available. In fact Kapovich [K] has given examples of finitely generated Kleinian group with quotient having infinitely many cusp ends. The geometric finiteness condition says that there is a fundamental domain consisting of a finite polyhedra for the group action of Γ on the hyperbolic ball. There has been extensive study (see [B]) of the notion of geometric finiteness resulting in several equivalent formulation of this condition in terms of the geometry of the hyperbolic quotient. Recently Bishop and Jones gave a characterization of geometric finiteness in two dimension in terms of the size of the limit set.

Theorem 1.6 (BJ). *Suppose Γ is an analytically finite Kleinian group on S^2, then it is geometrically finite if and only if its limit set Λ has Hausdorff dimension strictly less than two.*

A large part of the Bishop-Jones argument carries over to the higher dimensional situation. In fact their argument proves that geometrically finite groups in general dimension n have limit sets of Hausdorff dimension strictly smaller than n. What is missing for the converse direction is a suitable formulation of analogue of the analytical finiteness assumption. In [CQY-3] we provide a natural analogue which we call conformal finiteness: the ends of the Kleinian quotient should be at most finite in number and each end is a standard cusp end which is conformally equivalent to $(R^m - B^m) \times K^{n-m}$ where K^{n-m} is a compact flat manifold. Under this conformal finiteness condition we extend the criteria of Bishop-Jones to general dimensions:

Theorem 1.7 (CQY-3). *Suppose Γ is a conformally finite Kleinian group on S^n, then it is geometrically finite if its limit set Λ has Hausdorff dimension strictly less than n.*

The main ingredient of the argument as in the two dimension case is the construction of an equivariant hypersurface $G(\Gamma)$ in the hyperbolic ball which divide the geometrically finite ends from the convex core of the hyperbolic manifold. The hypersurface is the boundary of the domain in B^{n+1} which is formed by the union of the balls with center at points $x \in \Omega$ with a radius given by a small multiple of the distance of x to the limit set Λ and their images under the group Γ. The key analytic fact we need is the following distortion lemma that can be proved by elementary considerations:

Lemma 1.8. *Suppose that Γ is a non-elementary Kleinian group, $\Omega(\Gamma)$ is its domain of discontinuity and Λ is its set of limit points. Then there exists a positive number C such that*

$$\frac{1}{C}\frac{dist(\gamma(x), L(\Gamma))}{dist(x, L(\Gamma))} \leq |\gamma'(x)|_s \leq C\frac{dist(\gamma(x), L(\Gamma))}{dist(x, L(\Gamma))}$$

for all $x \in \Omega(\Gamma)$ and all $\gamma \in \Gamma$.

The hypersurface serves to transfer geometric bounds on the boundary quotient Ω/Γ to the interior of the hyperbolic manifold. The dividing hypersurface upon radial projection to the boundary sphere can be viewed as a graph over the domain of discontinuity Ω:

$$F(x) = f(x)x : \Omega \to G(\Gamma),$$

where $f(x) = 1$ for $x \in \Lambda$, and f is a Lipschitz function. An important element in this argument is the existence of an invariant metric on Ω satisfying the bounds of equation (1.6). The existence of such metric when Γ is a conformally finite group is easy to arrange, since on each standard end the natural conformal metric satisfies the requirement. In this situation the radial projection is a Lipschitz equivalence with respect to the metric on G induced from the hyperbolic metric and the invariant metric on Ω. In fact using this dividing hypersurface we have conversely the following criteria for conformal finiteness.

Theorem 1.9. *Suppose Γ is a non-elementary Kleinian group. Then Γ is conformally finite if and only if the Kleinian quotient Ω/Γ possess a complete Riemannian metric of finite volume and satisfying the curvature bounds*

$$|R| + |\nabla R| \leq C, \quad \textit{and} \quad Ric \geq -C.$$

We briefly outline the Bishop Jones argument to use the dividing hypersurface to prove Theorem 1.7. Assume to the contrary that a conformally finite Kleinian group with limit set of Hausdroff dimension smaller than n is not geometrically finite. Then according to a characterization of Thurston of geometrically infinite group, there is sequence of points $\{x_k\}$ in the thick part of the hyperbolic quotient that tends to infinity in the convex core. To make use of this fact consider hyperbolic harmonic measure of the set Ω i.e. the hyperbolic harmonic function $\omega(x)$ with boundary value given by χ_Ω. If one can find a point x where $\omega(x) < 1$, this means that Λ has positive measure hence a contradiction to the assumption. Since ω is invariant under the group Γ, it can be considered a harmonic function on the hyperbolic quotient B/Γ. An application of the Green's identity shows that

$$\omega(x) = \frac{1}{2^{n-1}|S^n|} \int_{G/\Gamma} (-\frac{\partial G}{\partial n}(x,y)) d\sigma(y) \tag{1.16}$$

where $\frac{\partial}{\partial n}$ is the hyperbolic normal derivative of the hypersurface G/Γ in B^{n+1}/Γ, and $d\sigma$ is the induced one from B^{n+1}/Γ. The Harnack inequality then reduces the estimate for this integral to an estimate for the Green's function on the manifold B^{n+1}/Γ:

$$\omega(x) \leq C \int_{G/\Gamma} G(x,y) d\sigma(y) \tag{1.17}$$

The latter integral can be estimated using the bounds for the heat kernel sy, so that we find:

$$\omega(x) \leq C \int_{G\Gamma} e^{-\sqrt{\frac{4(\lambda_0 - 2\delta)}{4+\delta}} d_H(x,y)} |B_1(x)|^{-\frac{1}{2}} |B_1(y)|^{-\frac{1}{2}} d\sigma(y) \tag{1.18}$$

where λ_0 is the bottom of the spectrum of B/Γ which is known to be positive ([BJ]). Using the bilipschitz equivalence it is a simple matter to control the integral of the negative fractional power of the volume of the y ball and then choosing the point x among the sequence of points $\{x_k\}$ far into the thick part of the convex core, so that the exponential decay in the distance can be exploited to show that at such points $\omega(x) < 1$.

The basic question to look for intrinsic group theoretic condition that characterizes finiteness of the geometric quotient remains open.

2. Background on σ_2

In the thesis of J. Viaclovsky [V-1], a family of fully nonlinear differential equations are introduced as generalizations of the Yamabe equation that pertain to the conformal structure of a Riemannian manifold. Consider the conformal Ricci tensor: $A = Ric - \frac{1}{2(n-1)}Rg$. The k-th elementary symmetric function of the eigenvalues of the matrix A is denoted by $\sigma_k(A)$. In particular σ_1 is a multiple of the scalar curvature. In dimension four,

$$\sigma_2 = -\frac{1}{2}|E|^2 + \frac{1}{24}R^2. \tag{2.1}$$

is part of the Gauss-Bonnet integrand and is also related to the fourth order curvature invariant

$$Q = -\frac{1}{12}\Delta R + \frac{1}{2}\sigma_2. \tag{2.2}$$

$$8\pi^2\chi(M^4) \;=\; \int \frac{1}{4}|W|^2\, dv \;+\; \int \sigma_2(A)dv, \tag{2.3}$$

where dv denotes the volume form. Thus $\int \sigma_2(A)dv$ is conformally invariant because the integrand $|W|^2 dv$ is invariant under conformal change of metric.

This is actually a special case of a more general phenomenon. Let (M^{2k}, g) be a compact, LCF Riemannian manifold of dimension $n = 2k$. If we define $A = Ric - \frac{1}{2(n-1)}Rg = Ric - \frac{1}{2(2k-1)}Rg$, then the integral

$$\int \sigma_k(A)\, dv$$

is conformally invariant (see [V-1]). Moreover,

$$\chi(M^{2k}) \;=\; c_k \int \sigma_k(A)\, dv.$$

Before we state the main result in [CGY-2], we recall the definition of another conformally invariant quantity, namely the *Yamabe* invariant $Y(g_0)$:

$$Y(g_0) = inf \frac{\int R_g dv_g}{(\int dv_g)^{\frac{n-2}{n}}}$$

where $g = e^{2w}g_0$ varies over all metrics g conformal to g_0.

Theorem 2.1 (CGY-2)**.** *Let (M^4, g_0) be a compact four-manifold satisfying*

(i) $Y(g_0) > 0$,
(ii) $\int \sigma_2(A_0)dv_0 > 0$.

Then there is a conformal metric $g = e^{2w}g_0$ with $\sigma_2(A_g) > 0$.

Corollary 2.2. *Under the assumptions of Theorem 2.1, there is a conformal metric $g = e^{2w}g_0$ with*

(i) $Ric > 0$,

(ii) $S = -Ric + \frac{1}{2}Rg > 0$.

In terms of examples, since such manifolds must have finite fundamental group, it suffices to consider simply connected examples. According to the homeomorphism classification of simply connected smooth 4-manifolds and the Hitchin-Thorp inequality, the simply connected 4-manifolds that admit a positive conformal structure with $\int \sigma_2(A) > 0$ must be homeomorphic to $k(\mathbb{CP}^2) \# \ell\overline{(\mathbb{CP}^2)}$ or $k(S^2 \times S^2)$ where k, l are positive integers satisfying $0 < k < 4 + 5l$. We remark that Lebrun-Nayatani-Nitta ([LNN]) has constructed metrics of with positive σ_2 on $k(\mathbb{CP}^2)$ for $0 < k \leq 3$. Previously Sha-Yang [ShYa] have constructed metrics of positive Ricci curvature on $k(\mathbb{CP}^2) \# \ell\overline{(\mathbb{CP}^2)}$ and $k(S^2 \times S^2)$ without constraints on k, ℓ. Thus, the class of 4-manifolds admitting metrics with positive σ_2 are necessarily a proper subset of those admitting positive Ricci curvature metrics. It is not clear if the list of such manifolds is finite.

The problem to conformally deform a metric with $\sigma_2(A) > 0$ to one with $\sigma_2(A) \equiv$ constant is addressed - but not resolved - in [V-2], where degree-theoretic arguments are used. What is lacking are L^∞-estimates for solutions. In section 4 below we present an alternative approach in [CGY-3], including a priori L^∞-bounds for solutions w of the equation $\sigma_2(A_{g_w}) = f$ for some positive function f on manifolds that are not conformally equivalent to the round four-sphere. More precisely, we have proved the following result:

Theorem 2.3 (CGY-3). *Let (M^4, g_0) be a compact four-manifold. Suppose the conformal metric $g = e^{2w}g_0$ satisfies*

(i) $R(g) > 0$,

(ii) $\sigma_2(A_g) = f > 0$.

If (M^4, g_0) is not conformally equivalently to the round sphere, then there is a constant $C = C(||f||_{C^1}, (min f)^{-1}, g_0)$ such that

$$||w||_{L^\infty} \leq C.$$

We then use the degree theory developed in [L] for fully nonlinear equations to prove the following result.

Corollary 2.4. *If (M^4, g_0) is a compact four-manifold satisfies the assumptions as in the statement of Theorem 2.1, then there exists a solution $g = e^{2w}g_0$ with $\sigma_2(A_g) \equiv 1$.*

In the following subsections we provide some background and preliminary results.

a. The Curvature of Four-Manifolds.

To begin, let (M^4, g) be a compact four-manifold. The curvature tensor will be denoted Rm, and usually viewed as a $(0,4)$-tensor. We let $E = Ric - \frac{1}{4}Rg$ denote the trace-free Ricci tensor, then

$$Rm = W + \frac{1}{2}E \owedge g + \frac{1}{24}R\, g \owedge g \tag{2.4}$$

where $\owedge$ is the Kulkarni-Nomizu product (see [Be, 1.110]). Alternatively, if $A = Ric - \frac{1}{6}Rg$, then we have the simpler decomposition

$$Rm = W + \frac{1}{2} A \owedge g. \tag{2.5}$$

The Gauss-Bonnet-Chern formula can be expressed as

$$8\pi^2 \chi(M^4) = \int \left(\frac{1}{4}|W|^2 + \sigma_2(A) \right) dv. \tag{2.6}$$

If M^4 is oriented, let $* : \Omega^p(M^4) \to \Omega^{4-p}(M^4)$ denote the Hodge operator. Then we have the splitting $\Omega^2(M^4) = \Omega^2_+(M^4) \oplus \Omega^2_-(M^4)$ into the sub-bundles of self-dual and anti-self-dual two-forms. This splitting induces a decomposition of the Weyl curvature into $W^\pm : \Omega^2_\pm(M^4) \to \Omega^2_\pm(M^4)$, viewed as as bundle endomorphism. Combining the signature formula

$$12\pi^2 \tau(M^4) = \int \frac{1}{4}(|W^+|^2 - |W^-|^2)$$

with (2.6) we obtain

$$4\pi^2(2\chi(M^4) + 3\tau(M^4)) = \int (\frac{1}{2}|W^+|^2 + \sigma_2(A)). \tag{2.7}$$

It is clear from (2.6) and (2.7) that the positivity of $\sigma_2(A)$ implies global topological information. But it also implies local geometric information, as the following lemmas show.

Lemma 2.5. *$R^2 \geq 24\sigma_2(A)$ with equality if and only if $E = 0$. In particular, if $\sigma_2(A) > 0$ on M^4 then either $R > 0$ or $R < 0$ on M^4.*

Proof. This is immediate, since

$$\sigma_2(A) = -\frac{1}{2}|E|^2 + \frac{1}{24}R^2 \leq \frac{1}{24}R^2.$$

□

Lemma 2.6. *Let $P \in M^4$ and $X \in T_P M^4$ be a tangent vector at P. If the scalar curvature R of g is positive at P, then*

$$\begin{aligned} S(X,X) &= -Ric(X,X) + \frac{1}{2} g(X,X) \\ &\geq \frac{3}{R}\sigma_2(A)\, g(X,X), \end{aligned} \tag{2.8}$$

$$Ric(X,X) \geq \frac{3}{R}\sigma_2(A)\, g(X,X). \tag{2.9}$$

Proof. To simplify notation we often denote $g(X,X) = |X|^2 = \langle X, X\rangle$. In terms of the trace-free Ricci tensor,

$$S = -E + \frac{1}{4} Rg \tag{2.10}$$

so that

$$S(X,X) \;=\; -E(X,X) \;+\; \frac{1}{4}\,Rg(X,X).$$

Since E is trace-free, we have the sharp inequality $|E(X,X)| \leq \frac{\sqrt{3}}{2}\,|E|\,|X|^2$ (see [SW, p.234]). Thus

$$\begin{aligned}
S(X,X) \;\;&\geq\;\; -\frac{\sqrt{3}}{2}\,|E|\,|X|^2 \;+\; \frac{1}{4}\,R|X|^2 \\
&\geq\;\; -\left(|E|\sqrt{\frac{3}{2R}}\right)^2 |X|^2 \;-\; \left(\sqrt{\frac{R}{8}}\right)^2 |X|^2 \;+\; \frac{1}{4}\,R|X|^2 \\
&=\;\; \frac{3}{R}\,\sigma_2\,(A)\,|X|^2.
\end{aligned}$$

The proof of (2.9) is essentially the same. We begin with

$$Ric \;=\; E \;+\; \frac{1}{4}\,Rg. \tag{2.11}$$

Then

$$Ric(X,X) \;\geq\; -\frac{\sqrt{3}}{2}\,|E|\,|X|^2 \;+\; \frac{1}{4}\,R|X|^2,$$

and we can argue as before. $\square$

Arguing exactly as in the proof of Lemma 2.6 we have

Lemma 2.7. *Let $P \in M^4$ and $X \in T_P M^4$. If $R < 0$ at P then*

$$\begin{aligned}
S(X,X) \;\;&\leq\;\; \frac{3}{R}\,\sigma_2(A)\,g(X,X), \\
Ric(X,X) \;\;&\leq\;\; \frac{3}{R}\,\sigma_2(A)\,g(X,X).
\end{aligned}$$

Combining the preceding lemmas we conclude

Corollary 2.8. *If $\sigma_2(A) > 0$ on M^4 then either $S > 0$ and $Ric > 0$ on M^4, or $S < 0$ and $Ric < 0$ on M^4, depending on the sign of the scalar curvature (which is necessarily a constant by Lemma 2.5).*

b. CONFORMAL CHANGES OF METRIC.

Now denote our four-manifold by (M^4, g_0). We will usually write conformal metrics in the form $g = e^{2w} g_0$. Also, metric-dependent quantities which have 0 as a subscript or superscript are understood to be with respect to g_0, while those without are with respect to g. For example, $\nabla_0^2\varphi$ denotes the Hessian of φ with respect to g_0 and $\Delta_0\varphi = tr_{g_0}\nabla_0^2\varphi$ the Laplacian; while $\nabla^2\varphi$ and $\Delta\varphi = tr_g\nabla^2\varphi$ denote the Hessian and Laplacian with respect to g.

Of basic importance are the transformation laws for the various components of the curvature tensor under a conformal change of metric:

$$R \;=\; e^{-2w}(R_0 - 6\Delta_0 w - 6|\nabla_0 w|^2), \tag{2.12}$$

$$Ric \;=\; Ric_0 \,-\, 2\nabla_0^2 w - \Delta_0 w g_0 \,+\, 2dw\otimes dw \,-\, 2|\nabla_0 w|^2 g_0, \tag{2.13}$$

$$A \;=\; A_0 \,-\, 2\nabla_0^2 w \,+\, 2dw\otimes dw \,-\, |\nabla_0 w|^2 g_0, \tag{2.14}$$

$$S \;=\; S_0 + 2\nabla_0^2 w \,-\, 2\Delta_0 w g_0 \,-\, 2dw\otimes dw \,-\, |\nabla_0 w|^2 g_0. \tag{2.15}$$

The Bach tensor plays a prominent role in our analysis. It is defined by (see [De])

$$B_{ij} \;=\; \nabla^k\nabla^\ell W_{kij\ell} \,+\, \frac{1}{2}R^{k\ell}W_{kij\ell}.$$

Using the Bianchi identities, this can be rewritten as

$$\begin{aligned} B_{ij} \;=\; &-\frac{1}{2}\Delta E_{ij} \,+\, \frac{1}{6}\nabla_i\nabla_j R \,-\, \frac{1}{24}\Delta R g_{ij} - E^{k\ell}W_{ikj\ell} \\ &+E_i^k E_{jk} \,-\, \frac{1}{4}|E|^2 g_{ij} \,+\, \frac{1}{6}RE_{ij} \end{aligned} \tag{2.16}$$

where $\Delta E_{ij} = g^{k\ell}\nabla_k\nabla_\ell E_{ij}$. Although it has several interesting properties, for our purposes the most important feature of the Bach tensor is its conformal invariance: if $g = e^{2w}g_0$, then

$$B \;=\; e^{-2w}B_0. \tag{2.17}$$

c. EQUATIONS OF MONGE-AMPERE TYPE.

If we fix a background metric g_0, then by (2.14) we are attempting to solve the equation

$$\sigma_2(A_0 \,-\, 2\nabla_0^2 w \,+\, 2dw\otimes dw \,-\, |\nabla_0 w|^2 g_0) \;=\; f \tag{2.18}$$

for some $f > 0$. This is an example of a fully non-linear equation of Monge-Ampere type (see [CNS-1, CNS-2, CKNS]). Many of the relevant properties of (2.18) are summarized by the following result:

Proposition 2.9. *The equation (2.18) is elliptic at a solution w if $f > 0$. The linearized operator is given by*

$$L[\varphi] \;=\; -2S^{ij}\,\nabla_i^0\,\nabla_j^0\,\varphi, \tag{2.19}$$

where $S^{ij} \;=\; (g_0)^{ik}\,(g_0)^{j\ell}\,S_{k\ell}$, and

$$\begin{aligned} S_{k\ell} \;=\; &S_{k\ell}^0 \,+\, 2\,\nabla_k^0\,\nabla_\ell^0 w \,-\, 2(\Delta_0 w)(g_0)_{k\ell} \\ &-\, 2\nabla_k^0 w\,\nabla_\ell^0 w \,-\, |\nabla_0 w|^2(g_0)_{k\ell} \end{aligned}$$

is given by (2.15). If the scalar curvature R of $g = e^{2w} g_0$ is positive, then the ellipticity constants of L satisfy

$$\frac{1}{2} R f |\xi|^2 \ \geq \ S_{ij}\xi^i\xi^j \ \geq \ \frac{3}{R} f\,|\xi|^2 \tag{2.20}$$

Proposition 2.9 follows from a straight forward computation. We only remark that the estimates (2.20) follow from Lemma 2.6.

For the record, we display the equation of $\sigma_2(A)$ for $A = A_g$ and $g = e^{2w} g_0$ in terms of the metric g_0

$$\begin{aligned} \sigma_2(A)e^{4w} &= \sigma_2(A_0) - 2|\nabla_0^2 w|^2 + 2(\Delta_0 w)^2 \\ &\quad + 2(\nabla_0 w, \nabla_0 |\nabla_0 w|^2) + 2\Delta_0 w |\nabla_0 w|^2 \\ &\quad - 2Ric^0(\nabla_0 w, \nabla_0 w) - \ 2 < S_0, \nabla_0^2 w > . \end{aligned} \tag{2.21}$$

Also, in terms of the conformal metric g, we have

$$\sigma_2(A) = \frac{1}{2} S^{ij} A_{ij}. \tag{2.22}$$

d. The Functional Determinant

Let (M^4, g_0) be a compact four-manifold. A metrically defined differential operator A is said to be conformally covariant of bidegree (a, b) if under the conformal change of metric $g = e^{2w} g_0$,

$$A_g(\varphi) \ = \ e^{-bw} A_0(e^{aw}\varphi). \tag{2.23}$$

In [BO] an explicit formula for $F[w] = \log(\det A_g / \det A_0)$ is computed, which may be expressed as

$$F[w] \ = \ \gamma_1 \mathrm{I}[w] \ + \ \gamma_2 \,\mathrm{II}\,[w] \ + \ \gamma_3 \,\mathrm{III}\,[w] \tag{2.24}$$

where $\gamma_i = \gamma_i(A)$ are constants and

$$\mathrm{I}\,[w] \ = \ \int 4|W_0|^2 dv_0 \ - \left(\int |W_0|^2 dv_0\right) \log ⨍ e^{4w} dv_0, \tag{2.25}$$

$$\mathrm{II}\,[w] \ = \ \int w P_0 w \, dv_0 \ + \int 4 Q_0 w \, dv_0 \ - \left(\int Q_0 \, dv_0\right) \log ⨍ e^{4w} \, dv_0, \tag{2.26}$$

$$\mathrm{III}\,[w] \ = \ \frac{1}{3}\left[\int R^2 \, dv \ - \int R_0^2 \, dv_0\right]. \tag{2.27}$$

where P denotes the *Paneitz* operator [P] as defined in (0.4); $P_0 = P_{g_0}$.

Before we discuss the existence theory some remarks are in order, explaining the significance of these formulas. First, if we consider the functional II alone, then critical points satisfy

$$P_0 w \ + \ 2Q_0 \ = 2\left(\int Q_0 \, dv_0\right) e^{4w}.$$

In general, if $g = e^{2w} g_0$ is a conformal change of metric, then the quantity Q transforms according to the formula

$$P_0 w \,+\, 2Q_0 \;=\; 2Qe^{4w}$$

where $Q = Q(e^{2w} g_0)$. We therefore conclude that critical points of II are precisely those metrics which satisfy $Q \equiv$ constant.

It is easy to see that critical points of III satisfy $\Delta R \equiv$ constant. Since M^4 is compact, this implies that R is constant. Thus III is the quadratic version of the Yamabe functional.

In order to state the relevant existence result of [CY-1], we need to further define the conformal invariant

$$\kappa_d \;=\; \gamma_1 \int |W_0|^2 \, dv_0 \,+\, \gamma_2 \int Q_0 \, dv_0. \tag{2.28}$$

Theorem 2.10. ([CY-1, Theorem 1.1]). *Let (M, g_0) be a compact four-manifold. If $\gamma_2, \gamma_3 > 0$ and $\kappa_d < 8\gamma_2\pi^2$, then inf $F(w)$ is attained by some function $w \in W^{2,2}$ and the metric $g = e^{2w} g_0$ satisfies*

$$\gamma_1 |W|^2 \,+\, \gamma_2 Q \,-\, \gamma_3 \Delta R \;=\; \kappa_d \, \mathrm{vol}(g)^{-1}. \tag{2.29}$$

Furthermore, g is smooth ([CGY1]).

A recent work [UV] of J. Viacovsky and K. Uhlenbeck studies the regularity of weak $W^{2,2}$ solution of some 4-th order equation of divergence type defined on 4-manifold, which in particular includes equation of type (2.29), and thus in particular they have obtained the regularity for all critical points of the functional F.

Theorem 2.11 (UV). *Let (M, g_0) be a compact four-manifold. Suppose $g = e^{2w} g_0$ is a critical metric of the functional $F(w)$ as in Theorem 2.10 above, then g is smooth.*

To apply Theorem 2.10, we now make a specific choice of $(\gamma_1, \gamma_2, \gamma_3)$

$$\begin{aligned}
\gamma_1 &= -\int Q_0 dv_0 \Big/ \int |W|_0^2 \, dv_0 \\
&= -\frac{1}{2} \int \sigma_2(A_0) \, dv_0 \Big/ \int |W|_0^2 \, dv_0, \\
\gamma_2 &= 1, \\
\gamma_3 &= \frac{1}{24}\,(3\delta - 2).
\end{aligned} \tag{2.30}$$

Notice that if $\int \sigma_2(A_0) dv_0 > 0$ then $\gamma_1 < 0$. With this choice of $(\gamma_1, \gamma_2, \gamma_3)$ we have

$$\kappa_d \;=\; \gamma_1 \int |W|_0^2 \, dv_0 \,+\, \int Q_0 dv_0 \;=\; 0. \tag{2.31}$$

To write down the corresponding functional, let us introduce the quantity

$$\begin{aligned} U_0^\delta &= U^\delta(g_0) \\ &= \gamma_1 |W|_0^2 + Q_0 - \frac{1}{24}(3\delta - 2)\,\Delta_0 R_0. \end{aligned} \tag{2.32}$$

Then according to (2.24), (2.30) and (2.32),

$$\begin{aligned} F[w] = F_\delta[w] &= \gamma_1 \mathrm{I}[w] + \mathrm{II}[w] + \frac{1}{24}(3\delta - 2)\,\mathrm{III}[w] \\ &= \int 4U_0^\delta\, w\, dv_0 + \int w P_0 w\, dv_0 \\ &\quad + \frac{1}{2}(3\delta - 2)\, Y[w], \end{aligned} \tag{2.33}$$

where

$$Y[w] = \int (\nabla_0 w + |\delta_0 w|^2)^2 dv_0 - \frac{1}{3}\int R_0 |\delta_0 w|^2 dv_0.$$

Note that F_δ is scale-invariant; i.e., $F_\delta[w + c] = F_\delta[w]$ for any constant c. By (2.29) and (2.30) the corresponding Euler equation for F_δ is

$$\gamma_1 |W|^2 + Q - \frac{1}{24}(3\delta - 2)\,\Delta R = 0 \qquad (*)_\delta$$

which can be rewritten as either

$$\delta \Delta R = 8\gamma_1 |W|^2 - 2|E|^2 + \frac{1}{6}R^2 \qquad (*)_\delta$$

or

$$\sigma_2(A) = \frac{\delta}{4}\Delta R - 2\gamma_1 |W|^2. \qquad (*)_\delta$$

The latter is relevant for our purpose. If $\delta = 0$, then $(*)_0$ becomes $\sigma_2(A) = -2\gamma_1 |W|^2$. Now recall that $\int \sigma_2(A_0) dv_0 > 0$ implies that $\gamma_1 < 0$, so in this case we conclude that $\sigma_2(A) \geq 0$ and $\sigma_2(A) > 0$ at the points where ${|W|_0}^2$ is not zero. In order to ensure that $\sigma_2(A) > 0$ everywhere on the limiting metric, we may replace the term $|W|^2$ in the expression for $I[w]$ above by the norm of a section of the bundle of symmetric $(0, 2)$ -tensors on M^4. Ths suggests the following strategy to construct a conformal metric with $\sigma_2(A) > 0$, it suffices to show that F_δ admits a critical point when $\delta = 0$. This approach, however, presents some serious technical difficulties. In some sense F_δ actually degenerates as $\delta \to 0$. One can see that this is the case by writing down just the highest order terms in F_δ:

$$\begin{aligned} F_\delta[w] = \int 3\delta(\Delta_0 w)^2 &+ 3(3\delta - 2)\,\Delta_0 w |\nabla_0 w|^2 + 2(3\delta - 2)|\nabla_0 w|^4 \\ &+ \text{(lower order terms)}. \end{aligned}$$

When $\delta = 0$ the leading term is absent. This behavior is reflected in the Euler equation for F_δ: when $\delta \neq 0$ then $(*)_\delta$ is fourth order in the metric, but only second order when $\delta = 0$.

Instead of studying F_0 directly we rely on a limiting argument. That is, we begin by showing that for any sufficiently small $\delta > 0$, $(*)_\delta$ admits a smooth solution with positive scalar curvature. Even when $\delta > 0$, though, things are hardly routine: notice that when $\delta > \frac{2}{3}$ then $\gamma_3 > 0$ while $\gamma_2 \equiv 1$, so the existence result Theorem 2.10 applies. At the point $\delta = \frac{2}{3}$, $\gamma_3 = 0$, the functional F_δ becomes a combination of functional I and II. In this case, we have the following existence result for functional II.

Theorem 2.12. ([CY-1, Theorem 1.2]). *Let (M, g_0) be a compact four-manifold. Assume that $\int Q < 8\pi^2$, assume further that the Paneitz operator P is positive, with $Ker\, P = \{constants\}$, then inf $II[w]$ exists with the extremal metric satisfying $Q \equiv constant$.*

It is worthwhile to point out some examples of compact 4-manifolds on which the Paneitz operator has negative eigenvalues. Let Σ be a hyperbolic surface of genus two with a small eigenvalue λ for the Laplacian. Then on the product 4-manifold $\Sigma \times \Sigma$ the Paneitz operator is given by $P = \Delta^2 + \Delta$ so that if λ is small enough, the operator P will have negative eigenvalue.

A key step in our proof of Theorem 2.1 above is the following result of Gursky–which indicates that the conformally invariant conditions (i) and (ii) in the statement of Theorem A is sufficient to ensure the positivity of the Paneitz operator.

Theorem 2.13. ([G-2, Theorem A]). *Let (M, g_0) be a compact four-manifold which satisfies the conditions (i) and (ii) as in the statement of Theorem 2.1. Then the Paneitz operator $P_0 = P_{g_0}$ is a positive operator, with $Ker\, P_0 = \{constants\}$. In addition, we have*

$$\int \sigma_2 dV \leq 16\pi^2 \tag{2.34}$$

where equality holds if and only if (M, g) is conformally the standard 4-sphere.

In the following we outline the proof of Theorem 2.1. We will use method of continuity. Fix a $\delta_0 \in (0,1)$ and define

$$S = \{\delta \in [\delta_0, 1] | (*)_\delta \text{ admits a smooth solution with positive scalar curvature}\}.$$

In Section 3 we will use the continuity method to show that $S = [\delta_0, 1]$. Since δ_0 is arbitrary, we will conclude that $(*)_\delta$ always admits a smooth solution of positive scalar curvature for any $\delta \in (0,1]$. The next (and most difficult involved) step is to obtain a priori estimates for solutions of $(*)_\delta$. this will be achieved using the divergence structure of the equation σ_2. For technical reasons, the optimal estimates we can derive $W^{2,s}$-bounds on solutions with $s < 5$. This is sufficient to apply heat equation techniques and obtain a smooth conformal metric with $\sigma_2(A) > 0$.

We end this section with a preliminary result which uses the existence theory of [CY-1] for the functional determinant in order to show that S is non-empty.

Proposition 2.14. *If $\int \sigma_2(A_0)dv_0 > 0$ and $Y(g_0) > 0$, then $1 \in S$.*

Proof. When $\delta = 1$, $\gamma_3 = \frac{1}{24}$. It follows from Theorem 2.10 that there is a smooth extremal metric $g = e^{2w}g_0$ satisfying $(*)_1$. In particular,

$$\Delta R \,=\, 8\gamma_1|W|^2 \,-\, 2|E|^2 \,+\, \frac{1}{6}R^2.$$

Also, $\int \sigma_2(A_0)dv_0 > 0$ implies that $\gamma_1 < 0$. Thus

$$\Delta R \,\leq\, \frac{1}{6}R^2$$

on M^4. It follows from [G-1, Lemma 1.2] that $R > 0$ on M^4. □

3. Deforming σ_2 to a positive function

In this section, we outline the arguments to prove Theorem 2.1.

Let F_δ denote the functional as defined in (2.33). That is, F_δ is the functional (2.24) with coefficients $\gamma_1, \gamma_2, \gamma_3$ chosen as in (2.30). The a-priori estimates proceed in two steps, in the first step, we obtain enough a priori control of the δ equation to prove the existence and regularity of the δ equation.

Proposition 3.1. *Suppose $g = e^{2w}g_0$ is a smooth solution of $(*)_\delta$ with positive scalar curvature, normalized so that $\int w dv_0 = 0$. Then there exist constants C_0, C_1 , depending only on g_0, so that*

$$w \,\geq\, C_0, \tag{3.1}$$

$$\int [\delta(\Delta_0 w)^2 \,+\, |\nabla_0 w|^4]\, dv_0 \,\leq\, C_1, \tag{3.2}$$

Lemma 3.2. *Suppose $g = e^{2w}g_0$ is a solution of $(*)_\delta$. Then for any $\varphi \in W^{2,2}(M^4)$,*

$$\begin{aligned}\int \frac{3}{2}\,\delta\Delta_0 w\,\Delta_0\varphi \;\;+\;\; &\frac{1}{2}(3\delta-2)\;\big[\Delta_0\varphi|\nabla_0 w|^2 \,+\, 2\Delta_0 w\langle\nabla_0\varphi, \nabla_0 w\rangle_0 \\ &+ 2|\nabla_0 w|^2\langle\nabla_0\varphi, \nabla_0 w\rangle_0\big] \\ = \;\; &\int \; -2U_0^\delta\varphi \,+\, 2Ric_0(\nabla_0\varphi, \nabla_0 w) \,+\, \frac{1}{2}(\delta-2)R_0\langle\nabla_0\varphi, \nabla_0 w\rangle.\end{aligned} \tag{3.3}$$

Remark Although we implicitly assume in the proof that w is smooth, it follows from a standard limiting argument that (3.3) is valid if $w \in W^{2,2}(M^4)$. Indeed, we shall take (3.3) as our definition of a (weak) $W^{2,2}$-solution of $(*)_\delta$.

The above Lemma follows from a straight forward computation (cf [CY-1, (1.8)] or [BO]) for critical point of the functional F_δ.

The proof of the a priori estimate (3.2) in Proposition 3.1 then follows from Lemma 3.2 by taking $\varphi = w$ and using the consequence of the fact that $R \geq 0$

$$\Delta_0 w \, + \, |\nabla_0 w|^2 \, \leq \, \frac{1}{6} \, R_0, \tag{3.4}$$

We will show that for all sufficiently small $\delta > 0$, $(*)_\delta$ admits a smooth solution with positive scalar curvature. To accomplish this, we will apply the continuity method and show that for each $\delta_0 > 0$ the set S defined in section 2 is both open and closed. Since we already saw that $1 \, \in S$ by Proposition 2.14 the desired result will follow.

Proposition 3.3. *If $\int \sigma_2(A_0) dv_0 > 0$ then S is open.*

Proof. The proof of this fact relies (as usual) on a perturbation result. Consequently, we will need to study the following linearized problem.

Lemma 3.4. *Let $\mathcal{L}_\delta$ denote the linearization of $(*)_\delta$ at a solution g of positive scalar curvature. Then for any $\varphi \in W^{2,2}$,*

$$\langle \varphi, \mathcal{L}_\delta \varphi \rangle_{L^2} \, \geq \, \int \frac{3}{4} \, \delta^2 (\Delta \varphi)^2 \, + \, \frac{\delta}{16} \, R |\nabla \varphi|^2. \tag{3.5}$$

In particular, Ker $\mathcal{L}_\delta = \mathbb{R}$.

Remark The kernel of $\mathcal{L}_\delta$ is due to the scale-invariance of F_δ.

Remark Lemma 3.4 is a generalization of [G-2, Theorem A], which considered the case where $\delta = \frac{2}{3}$. This corresponds to an eigenvalue estimate for the Paneitz operator. It is remarkable that, despite the coefficient δ in the leading term of F_δ, one can still show that $\mathcal{L}_\delta$ is invertible (modulo constants) for *all* $\delta > 0$.

Now suppose that $\delta_1 \in S$, and that $g_1 = e^{2w_1} g_0$ is a smooth solution of $(*)_{\delta_1}$ with positive scalar curvature. By the preceding lemma, Ker $\mathcal{L}_{\delta_1} = \mathbb{R}$, and it follows from [ADN, Theorem 13.1] that there is a unique (up to scaling) smooth solution of $(*)_\delta$ for all δ sufficiently close to δ_1. Moreover, since the scalar curvature of g_1 is positive, by taking solutions in a small enough $C^{2,\alpha}$-neighborhood of g_1 we may conclude that the solutions of $(*)_\delta$ will also have positive scalar curvature, for δ close enough to δ_1. It follows that S is open, and the proof of Proposition 3.3 is complete. $\square$

Proposition 3.5. *S is closed.*

Proof. . The proof of Proposition 3.5 consists of two parts. First, an a priori estimate for solutions of $(*)_\delta$ with positive scalar curvature. A consequence of this estimate will be the following: if $\{\delta_k\}$ is a sequence in S, and $\delta_k \to \bar{\delta}$, then $(*)_{\bar{\delta}}$ admits a weak $W^{2,2}(M^4)$-solution. The second part of the proof follows from a local estimate which, when combined with the regularity theory for extremals of the functional determinant developed in [CGY], will allow us to conclude that this weak solution of $(*)_{\bar{\delta}}$ is actually smooth with positive scalar curvature. Or one can apply Theorem 2.11, the result of [UV], to conclude directly that the weak $W^{2,2}$ solutions which are critical points of F_δ are in fact smooth. It then follows that S is closed. $\square$

Our next step is to prove that solution of $(*)_\delta$ have a-priori $W^{2,3}$ bounds independent of δ.

Proposition 3.6. *Let $g = e^{2w} g_0$ be a solution of $(*)_\delta$ with positive scalar curvature, normalized so that $\int w dv_0 = 0$, and assume*

$$\int \sigma_2(A_0) dv_0 \; = \; \int \sigma_2(A) dv \; > \; 0. \tag{3.6}$$

Then there are constants $C = C(g_0)$ and $0 < \delta_0 < 1$ such that

$$\int |\nabla_0^2 w|^3 dv_0 \; + \; \int |\nabla_0 w|^{12} dv_0 \; \leq \; C \tag{3.7}$$

for $\delta < \delta_0$. In particular, for any $\alpha \in (0, \frac{2}{3})$ there is a constant $C_\alpha = C(\alpha, g_0)$ such that

$$||w||_{C^\alpha} \leq C_\alpha. \tag{3.8}$$

The proof of (3.8) is quite complicated, it involves a series of lemmas and propositions. The basic estimate we will need is

Proposition 3.7. *Under the same hypotheses of Theorem 2.1, there is a constant $C = C(g_0)$ such that*

$$\int \left(\frac{R}{6}\right)^3 dv \; \leq \; (1 + C\delta) \int |\nabla w|^6 dv \; + \; C \int R^2 dv \; + \; C \tag{3.9}$$

for δ sufficiently small.

Our arguments are guided by the a priori C^2-estimates for Monge-Ampere equations as described in [CNS-1], [CNS-2], [Ev], and [K]. However, the estimates in these references are pointwise in nature and involve the maximum principle. Since our regularized equation is fourth order, such techniques cannot work for us. Instead, we rely on integral estimates which present their own difficulties. Since our calculations are quite involved, we will give here an overview of the argument.

We begin with a simple identity. Let $f \in C^\infty(M^4)$. Then by the divergence theorem,

$$\begin{aligned} 0 \; &= \; \int \nabla_i (S_{ij} \, \nabla_j f) \\ &= \; \int \nabla_i S_{ij} \, \nabla_j f \; + \; S_{ij} \, \nabla_i \nabla_j f. \end{aligned}$$

Also, the contracted second Bianchi identity implies that S is divergence-free: $\nabla_i S_{ij} = \nabla_i(-R_{ij} + \frac{1}{2} R g_{ij}) = -\nabla_i R_{ij} + \frac{1}{2} \nabla_j R = 0$. Therefore,

$$0 = \int S_{ij} \nabla_i \nabla_j f, \text{ any } f \in C^\infty(M^4). \tag{3.10}$$

We will apply (3.10) to two different choices of f, resulting in two different inequalities. First, we let $f = R$. Then differentiating $(*)_\delta$ twice and using (3.10) we obtain the inequality

$$0 = \int S_{ij} \nabla_i \nabla_j R \geq \int 6\, trE^3 + \frac{1}{12} R^3 + (\text{lower order terms}),$$

where $trE^3 = E_{ij}E_{ik}E_{jk}$. Next, we let $f = 12|\nabla w|^2$, resulting in the inequality

$$\begin{aligned} 0 &= \int S_{ij} \nabla_i \nabla_j (12|\nabla w|^2) \\ &\geq \int -6trE^3 + \frac{1}{12} R^3 - 6R|\nabla w|^4 + (l.o.t.). \end{aligned} \tag{3.11}$$

Adding (3.11) to (3.12), the term trE^3 cancels to give

$$\int \left(\frac{R}{6}\right)^3 \leq \int \left(\frac{R}{6}\right) |\nabla w|^4 + (l.o.t.),$$

and (3.9) can be shown to follow from this inequality.

A key identity in the derivation of the inequality (3.11) is the following:

Lemma 3.8. *Let (M^4, g) be any Riemannian 4-manifold. Then*

$$\begin{aligned} S_{ij}\nabla_i\nabla_j R &= 3\Delta\sigma_2(A) + 3\left(|\nabla E|^2 - \frac{1}{12}|\nabla R|^2\right) \\ &+ 6\, trE^3 + R|E|^2 - 6W(E,E) - 6 < E, B > \end{aligned} \tag{3.12}$$

where B_{ij} denotes the Bach tensor.

(3.13) can be derived directly from the Bach tensor identity (2.16) which we have mentioned in section 2.

We then take $f = R^{p+1}$ in the scheme above for all $0 \leq p < 2$, we the obtain a prior R^s estimate for $s = p+3$ or equivalently the $W^{2,s}$ estimate for the solutions w_δ for all $s < 5$.

Theorem 3.2. *Let $g = e^{2w} g_0$ be a solution of $(*)_\delta$ with positive scalar curvature, normalized so that $\int w dv_0 = 0$. Assume*

$$\int \sigma_2(A_0)\, dv_0 = \int \sigma_2(A)\, dv > 0. \tag{3.13}$$

Then there are constants $C_s = C(g_0, s)$ and $\delta_0 < 1$ such that

$$\int |\nabla_0^2 w|^s\, dv_0 \leq C_s \tag{3.14}$$

for any $s < 5$ and $\delta \leq \delta_0$.

We remark that as a direct corollary of the Sobolev embedding theorem, we have $\mathcal{C}^{1,\alpha}$ a priori bound for the solution w of $(*)_\delta$.

However, for technical reasons it appears difficult to improve on this. Instead we show that once $s > 4$, we can use the Yamabe flow to smooth solutions of $(*)_\delta$ and obtain metrics with $\sigma_2(A) > 0$.

Proposition 3.9. *Let $g = e^{2w} g_0$ be a solution of $(*)_\delta$ with positive scalar curvature, normalized so that $\int w dv_0 = 0$. Assume $\int \sigma_2(A) dv > 0$. If δ is sufficiently small, then there is a smooth conformal metric $h = e^{2v} g$ such that $\sigma_2(A_h) > 0$.*

The proof of Proposition 3.10 is based on estimates for solutions of the Yamabe flow (e.g. work of [Ye]) using parabolic Moser iteration. Let g be any of the solutions of the equation $(*)_\delta$, we consider the Yamabe flow:

$$\begin{cases} \frac{\partial h}{\partial t} \;=\; -\frac{1}{3} Rh, \\[2ex] h(0,\cdot) \;=\; g \;=\; e^{2w} g_0. \end{cases}$$

The key facts concerning the Yamabe flow have been worked out by Hamilton [Ha] and Ye [Ye]. Namely there is a time interval in which the equation (3.16) has a unique smooth solution. We record some basic equations governing the evolution of important quantities:

$$\frac{\partial}{\partial t} dv \;=\; -\frac{2}{3} R dv, \tag{3.15}$$

$$\frac{\partial R}{\partial t} \;=\; \Delta R + \frac{1}{3} R^2. \tag{3.16}$$

These equations give control of the decay of volume, as well as assuring the positivity of scalar curvature as the metrics evolve in time. To control the Ricci tensor, we compute the evolution of the Ricci tensor (making use of the Bach tensor) and derive: for $t \le T_1$

$$\frac{\partial}{\partial t} |Ric| \;\le\; \Delta |Ric| \;+\; C\,|Ric|^2. \tag{3.17}$$

Then making use of the fact that for $0 < t < T$, there is a uniform Sobolev constant for the metrics up to that time, we can prove: for each fixed $s \in (4,5)$ there is a time T_1 so that the solution metrics up to that time satisfies:

(i) $\| Ric_h \|_{L^s} \le 2 \| Ric_g \|_{L^s}$;
(ii) $\| Ric_h \|_\infty \le C_2 t^{-\frac{2}{s}}$, where $C_2 = C_2(g_0)$;
(iii) $\| v \|_\infty \le C(g_0)$.

Now let $f = \sigma_2 + 2\gamma_1 |\eta|^2$ and consider the evolution of the quantity $\frac{f}{R}$:

$$\frac{\partial}{\partial t}\left(\frac{f}{R}\right) \;\geq\; \Delta\left(\frac{f}{R}\right) + \frac{2}{R}trE^3 + \frac{1}{3}|E|^2 - \frac{1}{3}f$$
$$- \;\; 2R^{-1}W(E,E) - 2R^{-1} < B, E > -C. \qquad (3.18)$$

Consequently, for $\varphi = max\{-\frac{f}{R}, 0\}$ and $t \leq T_1$ we have

$$\frac{\partial\varphi}{\partial t} \leq \Delta\varphi + C_1|Ric|\varphi + C_1|Ric| \qquad (3.19)$$

where $C_1 = C_1(g_0)$.

¿From (3.20) and (3.21), we may obtain by comparing with the function

$$\varphi_1(t) = exp\{\frac{s}{s-2}C_1C_2t^{\frac{s-2}{s}}\} - 1$$

that

$$||\varphi - \varphi_1||_\infty \leq C\delta^{1/2}t^{-2/s}.$$

Hence, unraveling the definition, we find

$$\sigma_2(A) \geq C_4 - C_3t^{1-(4/s)} - C_3\delta^{1/2}t^{-4/s}$$

for $t \leq T_1$. By taking t_0 fixed but sufficiently small making sure that

$$C_3t_0^{1-(4/s)} \leq C_4.$$

then for δ sufficiently small, we find that $\sigma_2 \geq (1/2)C_4 > 0$. This finishes the smoothing argument.

A curious feature of the analysis is the necessity of a priori L^s bounds for the curvature of the initial data with $s > 4$. Typically, the smoothing effects of semi-linear heat flows like the Yamabe or Ricci flows only require $s > \frac{n}{2} = 2$. But to obtain in addition a positive lower bound for $\sigma_2(A)$ we actually need $s > 4$.

4. Deforming σ_2 to constant

In this section we will prove Therorem 2.3 in section 2. First we will describe the deformation argument to obtain from a conformal metric of positive σ_2 a conformal metric of constant positive σ_2. We make use of the degree theory developed for fully nonlinear elliptic equations in [L]. The key fact of the degree theory of fully nonlinear equations is that the degree remains invariant under continuous deformations of the equation as long as there is a uniform a-priori estimates for all solutions of the equation and a uniform bound for the ellipticity of the corresponding equation.

On (M^4, g_0), we begin with a solution of the equation

$$\sigma_2(A_g) = f > 0, \quad R_g > 0, \qquad (4.1)$$

where $g = e^{2w}g_0$ is a metric conformal to g_0. Consider a one parameter family of equations:

$$\sigma_2(A_{g_t}) = tf + (1-t), \;\; R_{g_t} > 0 \tag{4.2}$$

We label the above equation as Σ_t. We need to provide a priori estimates for the family of metrics $g_t = e^{2w_t}g_0$; i.e. denote $w = w_t$, we need to prove that w satisfy:

$$||w||_{4,\alpha} \leq C; \; and \; S_{e^{2w}g_0}(\xi,\xi) \geq C|\xi|^2 \tag{4.3}$$

where C is a constant independent of t. We let

$$O_c = \{w \in C^{4,\alpha} | \; ||w||_{4,\alpha} \leq C, \quad \text{and } \; S_{e^{2w}g}(\xi,\xi) \geq C|\xi|^2\}.$$

The degree theory of [L] assures that the topological degree $deg(\Sigma_t, 0, O_C)$ is independent of t. We need to do a calculation verifying that for $t = 1$ the degree of the equation is non-zero. In order to do this we deform the equation to one whose degree is easy to determine. It is useful to write the σ_2 equation in a suggestive form. Suppose $g = e^{2w}g_0$, denote

$$M_{ij} = 2S^0_{ij} + 2\nabla^0_i\nabla^0_j w - 2\Delta_0 w g^0_{ij} - 2\nabla^0_i w \nabla^0_j w.$$

Then, after some computation, the equation (4.1) may be written in the form

$$-\nabla^0_i\{M_{ij}(w)\nabla^0_j w\} + \sigma_2(A_{g_0}) = \sigma_2(A_g)e^{4w} = \; fe^{4w}. \tag{4.4}$$

It is important to note the identity

$$M_{ij}(w) = S_{ij} + S^0_{ij} + |\nabla_0 w|^2 g^0_{ij}$$

so that it is clear that when both S_{ij} and S^0_{ij} are positive definite, then M_{ij} is positive definite. It is also convenient to rewrite the equation, on account of the conformal covariance property, using the solution metric g of the equation as the background metric:

$$-\nabla_i\{M_{ij}(w)\nabla_j w\} + f = fe^{4w} \tag{4.5}$$

so that $w = 0$ is a solution to this equation satisfying $R > 0$.

Without loss of generality, we may assume the normalizing condition $\int dv = 1$. We begin with the following deformation:

$$-\nabla_i\{M_{ij}(w)\nabla_j w\} + f = (1-t)f\int e^{4w} + tfe^{4w} \tag{4.6}$$

Let's label the equation by Γ_t. When $t = 1$ we recover the equation (4.5), and when $t = 0$ we have the "linear" equation

$$-\nabla_i\{M_{ij}(w)\nabla_j w\} + f = f\int e^{4w}. \tag{4.7}$$

We then continue with the following deformation: for a positive λ,

$$-\nabla_i\{M_{ij}(w)\nabla_j w\} + sf + (1-s)\lambda = (sf + (1-s)\lambda)\int e^{4w}. \tag{4.8}$$

We will label this equation by Λ_s. This family of equations agrees for $s = 1$ with the equation (4.7) and reduces to the simpler equation for $s = 0$:

$$-\nabla_i\{M_{ij}(w)\nabla_j w\} + \lambda = \lambda \int e^{4w}. \tag{4.9}$$

To analyze equation (4.9), we integrate it over the manifold to find that $\int e^{4w} = 1$. Hence the equation reduces to

$$-\nabla_i\{M_{ij}(w)\nabla_j w\} = 0.$$

By the maximum principle, $w = 0$ is the unique solution. To calculate the degree of this equation, we assume for the moment that we have established the apriori estimates for all solutions of the equations Γ_t and Λ_t for all relevant values of the parameters s and t; so that we find $deg(\Gamma_1, O_C, 0) = deg(\Lambda_0, O_C, 0)$. We need to find the linearization of the equation (4.9) at the unique solution $w = 0$:

$$\mathcal{L}(\delta w) = -\nabla_i\{M_{ij}\nabla_j \delta w\} - 4\lambda \int \delta w.$$

The degree is given by $(-1)^k$, where k is the number of negative eigenvalues of the linearized operator $\mathcal{L}$. It is easy to see that the constant functions are the only eigenfunctions with negative eigenvalue. Hence $deg(\Lambda_0, O_C, 0) = -1$.

In the remaining part of this section, we will first provide an outline to prove a priori estimates for solutions of the equation Σ_t for all $0 \leq t \leq 1$, and then the required estimates for solutions of the equation Γ_t and Λ_s. Suppose the conformal metric $g = e^{2w}g_0$ satisfies

$$\sigma_2(A_g) = f > 0,\; R_g > 0. \tag{4.10}$$

The first step is to prove that there is a sup-norm bound of w in terms of f unless the manifold is the standard 4-sphere. In the case of 4-sphere with the standard metric g_c, all metrics $g = e^{2w}g_c = \phi^*(g_c)$ for a conformal transformation ϕ of the 4-sphere satisfy the equation $\sigma_2(A_g) \equiv 6$. Since the group of conformal transformation of the 4-sphere is not compact, the collection of such w does not have an apriori sup-norm bound. A basic philosophy in treating these conformally invariant equations is to prove the sphere is the only exceptional case; that is, unless the manifold is the 4-sphere with the standard metric, we can establish the sup-norm bound through some blow-up analysis. Namely, given a sequence of solutions to (4.1), for which the sup-norm estimate fails we dilate the solutions to construct a new sequence which converges to a smooth solution of (4.1) on $(\mathbb{R}^4, ds^2)$ with $f \equiv$ constant. We then classify the solutions of such equations on $(\mathbb{R}^4, |dx|^2)$ according to whether the constant is zero or positive. We can rule out the first case easily because our blowup procedure is chosen to avoid such a possibility. The latter possibility would imply that the manifold (M^4, g_0) is conformally equivalent to the standard 4-sphere, the case being ruled out by our assumptions.

This argument is fairly standard in outline. The main technical difficulty in our case is the absence of a Harnack inequality for solutions of (4.1). Consequently, if we simply dilate our solutions in order to obtain a sequence which is uniformly bounded above, we are unable to conclude that the sequence has a uniform lower bound (even locally). This makes it difficult to construct a non–trivial limiting solution to (4.1) on $\mathbb{R}^4$ from our original sequence. To overcome this difficulty we dilate in an unusual way, which we now describe. Let $g_k = e^{2w_k} g_0$ be a sequence of solutions to

$$\sigma_2(A_{g_k}) = f_k \ , \tag{4.11}$$

and we assume that $\{f_k\}$ satisfies

$$0 < c_0 \leq f_k \ \leq c_0^{-1}, \|f_k\|_{C^1(M)} \leq c_1 \ . \tag{4.12}$$

We claim that there exists some constant C, $C = C(c_0, c_1)$ such that

$$max_M[|\nabla_0 w_k| + e^{w_k}] \leq C \tag{4.13}$$

for all k. Assume not, then

$$max_M[|\nabla_0 w_k| + e^{w_k}] \to \infty \quad as \quad k \to \infty \ . \tag{4.14}$$

Let us assume that $P_k \in M$ is a point at which $(|\nabla_0 w_k| + e^{w_k})$ attains its maximum. By choosing normal coordinates $\{\Phi_k\}$ centered at P_k, we may identify the coordinate neighborhood of P_k in M with the unit ball $B(1) \subset \mathbb{R}^4$ such that $\Phi_k(P_k) = 0$. Given $\varepsilon > 0$, we define the dilations $T_\varepsilon \colon \mathbb{R}^4 \to \mathbb{R}^4$ by $x \mapsto \varepsilon x$, and consider the sequence $w_{k,\varepsilon} = T_\varepsilon^\star w_k + \log \varepsilon$. Note that

$$|\nabla_0 w_{k,\varepsilon}| + e^{w_{k,\varepsilon}} = \varepsilon(|\nabla_0 w_k| + e^{w_k}) \circ T_\varepsilon \ .$$

Thus, for each k we can choose ε_k so that

$$|(\nabla_0 w_{k,\varepsilon_k})| + e^{w_{k,\varepsilon_k}}\big|_{x=0} = 1 \ . \tag{4.15}$$

Note that $\nabla_0 w_{k,\varepsilon_k}$ is defined in $B_{\frac{1}{\varepsilon_k}}(0)$, and

$$|(\nabla_0 w_{k,\varepsilon_k})| + e^{w_{k,\varepsilon_k}} \leq 1 \quad on \quad B_{\frac{1}{\varepsilon_k}}(0) \ .$$

To simplify notation, let us denote w_{k,ε_k} by w_k and call the new sequence $\{w_k\}$ the dilated sequence. Note that $g_k^\star \equiv e^{2w_k} T_{\varepsilon_k}^\star g_0 \equiv e^{2w_k} g_0^k$ satisfies

$$\sigma_2(A_{g_k^\star}) = f_k \circ T_{\varepsilon_k} \ .$$

Furthermore, $g_0^k = T_{\varepsilon_k}^\star g_0 \to ds^2$ in $C^{2,\beta}$ on compact sets.

There are now two possibilities to consider, depending on the behavior of the exponential term in (4.14). First, suppose that

$$\lim_k e^{w_k(0)} = 0 \ .$$

After choosing a subsequence (also denoted $\{w_k\}$) with $e^{w_k(0)} \to 0$, we let $\tilde{w}_k(x) = w_k(x) - w_k(0)$. We call the sequence $\{\tilde{g}_k\}$ the dilated and rescaled sequence. We observe

$$\sigma_2(A_{\tilde{g}_k}) = e^{4w_k(0)}(f_k \circ T_{\varepsilon_k}) \ . \tag{4.16}$$

It is easy to verify that $\tilde{w}_k$ is bounded in the C^1–topology on compact sets in $\mathbb{R}^4$.

Alternatively, suppose $\limsup e^{w_k(0)} = \delta_0 > 0$. Then it is easy to verify that there is a subsequence (also denoted $\{w_k\}$) which is bounded in the C^1–topology on compact sets in $\mathbb{R}^4$.

Both $\tilde{w}_k$ and $\{w_k\}$ converge; but the type of convergence differs: for the sequence $\tilde{w}_k$ the equation (4.11) will not be uniformly elliptic, because the RHS tends to zero as $k \to \infty$ (see [CGY2, Prop. 1.5]). On the other hand, the sequence w_k does satisfy a (uniformly) elliptic equation; hence we have better control of the convergence. In addition, the convexity of the equation ${\sigma_2}^{\frac{1}{2}}$ allows us to bring in the work of Evans [Ev] and Krylov [Kr]. We summarize our discussion above in the following.

Proposition 4.1. *(i) For the dilated and rescaled sequence $\tilde{g}_k = e^{2\tilde{w}_k} g_0$, there exists a $C^{1,1}$ conformal metric $\tilde{g} = e^{2\tilde{w}} ds^2$ with (a subsequence of) $w_k \to \tilde{w}$ in $C^{1,\beta}$ on compact subsets of $\mathbb{R}^4$, where $0 < \beta < 1$. Furthermore, $\tilde{g}$ satisfies*

$$R_{\tilde{g}} \geq 0, \;\; \sigma_2(A_{\tilde{g}}) = 0, \tag{4.17}$$

$$|\nabla_0 \tilde{w}|(0) = 1\,. \tag{4.18}$$

(ii) For the dilated sequence $g_k^\star = e^{2w_k} g_0^k$, there exists a C^∞ conformal metric $g = e^{2w} ds^2$ such that (a subsequence of) $w_k \to w$ in $C^{2,\beta}$ on compact subsets of $\mathbb{R}^4$. Furthermore, after possibly rescaling, g satisfies

$$R_g \geq 0, \;\; \sigma_2(A_g) = 6. \tag{4.19}$$

We discuss the classification of solutions of the equation $\sigma_2(A_g) \equiv$ constant on $\mathbb{R}^4$. When the constant is zero, we have the following conclusion.

Proposition 4.2. *Suppose $g = e^{2w} ds^2$ is a conformal metric on $\mathbb{R}^4$ with $w \in C^{1,1}$, satisfying*

$$\sigma_2(A_g) = 0 \;, R_g \geq 0\,.$$

Then $w \equiv$ constant.

As a consequence, we see that the limiting metric $\bar{g} = e^{2w} ds^2$, defined as the $C^{1,1}$–limit of the blow–up sequence defined in (4.16), cannot occur.

Proposition 4.2 can be proved by using an estimate similar to that of (3.2) in Proposition 3.1 in section 3. Namely, fixing a ball B_ρ of radius ρ in $\mathbb{R}^4$, we multiply the equation (3.3) in Lemma 3.2 by $\phi = (w - \bar{w})\eta$, where $\bar{w}$ denotes the mean value of w over B_ρ, and η a cut off function supported on B_ρ with $\eta \equiv 1$ on $B_{\rho/2}$. We then let ρ tend to infinity and prove that $\int_{R^4} |\nabla w|^4 dv \equiv 0$.

Proposition 4.3. *Suppose $g = u^2 |dx|^2$ is a solution of (4.19). Then $u(x) = (a|x|^2 + b_i x^i + c)^{-1}$ for constants a, b_i, c. In particular, g is obtained by pulling back to $\mathbb{R}^4$ the round metric on S^4 (or its image under a conformal map).*

Our method is inspired by the corresponding uniqueness result of Obata [O] for the scalar curvature equation. Recall the formula:

$$E = -2u^{-1}\nabla_g^2 u + \frac{1}{2}u^{-1}\Delta_g u\, g \;;$$

and introduce, for conformally flat metrics,

$$L = \frac{1}{2}\sigma_2(A)g + \frac{1}{3}RA - A^2 \;.$$

The relevant properties of the tensor L are

$$tr_g L = 0, \quad \delta L = d(\sigma_2(A)) \tag{4.20}$$

and for metrics satisfying $\sigma_2(A) > 0$:

$$< L, E >= 0 \;\text{ implies }\; E = 0. \tag{4.21}$$

Then following Obata, we consider the integral

$$\int_M < L, E > u dV = -2\int_M < L, \nabla^2 u > dV = -2\int_{\partial M} L(\nabla u, \nu) d\sigma \tag{4.22}$$

In our situation, we introduce a cut off function, and instead of the boundary integral, we need to control the extra term coming from the cut off. Without going into the detail, the inequality in Proposition 4.4 below provides the key estimate which allows us to handle (i.e. let it vanish) the remaining term.

Proposition 4.4. *There is a constant c_2 such that for any $\rho > 0$,*

$$\int_{B(2\rho)\setminus B(\rho)} R|\nabla u|^2 u^{-1} dx \le c_2\rho^2 \;. \tag{4.23}$$

We refer the readers to [CGY-3] for details.

We are ready to prove the main a priori estimates.

We begin by assuming that for some sequence of conformal metrics $\{g_k = e^{2w_k}g_0\}$ the bound (4.13) fails. We then dilate as described before to obtain a sequence $\{g_k^\star = T_{\varepsilon_k}^\star(e^{2w_k}g_0) \equiv e^{2w_k}g_0^k\}$ on $\mathbb{R}^4$ satisfying

$$\max\{e^{w_k} + |\nabla_0 w_k|\} = e^{w_k(0)} + |\nabla_0 w_k(0)| = 1 \;.$$

Recall that if $k \to \infty$, $\lim e^{w_k(0)} = 0$, then by Proposition 4.1 (a subsequence of) the rescaled sequence $\tilde{w}_k = w_k - w_k(0)$ converges to $w \in C^{1,1}(\mathbb{R}^4)$, with $\tilde{g} = e^{2\tilde{w}}ds^2$ satisfying

$$\sigma_2(A_{\tilde{g}}) \equiv 0 \,, R_{\tilde{g}} \ge 0 \;, |\nabla_0 \tilde{w}(0)| = 1 \,.$$

However, Proposition 4.2 implies that $\tilde{w} \equiv$ constant, contradicting (4.18). Consequently, we may assume that $\limsup e^{w_k(0)} > 0$. In this case, by proposition 4.1 (a subsequence of) $w_k \to w \in C^\infty(\mathbb{R}^4)$. After rescaling if necessary, the limiting metric $g = e^{2w}ds^2$ satisfies

$$\sigma_2(A_g) \equiv 6 \,, R_g > 0 \,.$$

According to Proposition 4.3, g is obtained by pulling back the round metric on S^4 via steriographic projection. In particular, it follows that

$$\int_{\mathbb{R}^4} \sigma_2(A_g) dvol_g = 16\pi^2 \,, \int_{\mathbb{R}^4} dvol_g = \frac{8}{3}\pi^2 \,.$$

Now, given any fixed ball $B(\rho) \subset \mathbb{R}^4$,

$$\int_{B(\rho)} \sigma_2(A_{g_k^\star}) dvol_{g_k^\star} \to \int_{B(\rho)} \sigma_2(A_g) dvol_g$$

as $k \to \infty$.

On the other hand, since $g_k^\star = T_{\varepsilon_k}^\star g_k$, we have

$$\int_{B(\rho)} \sigma_2(A_{g_k^\star}) dvol_{g_k^\star} = \int_{B(\rho)} T_{\varepsilon_k}^\star (\sigma_2(A_{g_k}) dvol_{g_k}) \quad \le \int_{M^4} \sigma_2(A_{g_k}) dvol_{g_k} \,. \tag{4.24}$$

If we let $k \to \infty$ in (4.24), then let $\rho \to \infty$, it follows that

$$16\pi^2 \le \int_{M^4} \sigma_2(A_{g_0}) dvol_{g_0} \,. \tag{4.25}$$

However , by [G-2, Theorem B], $\int \sigma_2(A_{g_0}) dvol_{g_0} \le 16\pi^2$ with equality if, and only if, (M^4, g_0) is conformally equivalent to the round sphere. Thus, the bound (4.13) holds unless (M^4, g_0) is conformally the round sphere, as claimed.

This finishes the apriori estimates for solutions of the equations Σ_t. To obtain corresponding estimates for solutions of the equations Γ_t and Λ_t, we need to proceed a bit differently in the situation where the new σ_2 is not clearly a priori bounded between two positive constants. The idea is to obtain enough control of the conformal factors w so that the new σ_2 is a priori bounded between two positive constants. A combination of blow up argument with the classification results yield the required bounds for solutions of the equations Λ_s and Γ_t for $0 < t_0 \le t \le 1$ for any $t_0 > 0$. We shall discuss the remaining case $0 < t \le t_0$ for t_0 small. Thus for the equation Γ_t we begin by integrating the equation over the manifold to obtain an upper bound for

$$\int_M e^{4w} \le C.$$

As a consequence, we also find a lower bound

$$\int_M e^{4w} \ge C.$$

Hence, we find an upper bound for the mean value

$$\bar{w} \le C. \tag{4.26}$$

Multiply the equation (4.4) by $w - \bar{w}$ and integrate over the manifold, we find on account of (4.4),

$$\begin{aligned}\int_M \{S_{ij}\nabla_i w\nabla_j w &+& \tilde{S}_{ij}\nabla_i w\nabla_j w + |\nabla w|^4\} \\ &=& \int_M \nabla_i w M_{ij}\nabla_j w + f(w-\bar{w}) \\ &=& \int_M \{(1-t)f(w-\bar{w})(\int e^{4w}) + tf(w-\bar{w})e^{4w}\} \qquad (4.27)\end{aligned}$$

It follows, using the concavity of the logarithm, that

$$\int |\nabla w|^4 \leq C + Ct_0 \log \int e^{5(w-\bar{w})}.$$

Hence the Moser-Trudinger inequality ([M] and [T]) can be used to show a bound for $\int |\nabla w|^4$, and that in turn gives a bound for $\int e^{4(w-\bar{w})}$ as well as for $\int e^{-4(w-\bar{w})}$. Consequently, we find a lower bound for $\bar{w}$. This together with (4.26) show that $\bar{w}$ is apriori bounded, hence the positivity of the scalar curvature shows that $\inf w$ is bounded from below.

We then apply the argument of Theorem 3.9 to show that there is a bound of the form

$$||w||_{2,4+\epsilon} \leq C$$

and hence via the Sobolev inequality pointwise bound for w. Therefore for solutions w of the equation (4.6) for the parameters t satisfying $0 < t_0 \leq t \leq 1$, there exists constants c and C so that

$$0 < c \leq \sigma_2(A_{e^{2w}g}) = (1-t)f(\int e^{4w})e^{-4w} + tf \leq C.$$

The previous argument for a priori estimates then holds. We refer the reader to the original article [CGY-3] for further detail.

References

[ADN] S. Agmon, A. Douglis and L. Nirenberg; "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II", Comm. Pure Appl. Math. 17 (1964) pp 39-92.

[B] L. Bers; "On Ahlfors' finiteness theorem", Amer. Jour. Math. 89 (1967), pp 1078-1082.

[Be] A. Besse; "Einstein Manifolds", Springer-Verlag, Berlin (1987).

[BJ] C. Bishop and P. Jones; "Housdorff dimension and Kleinian groups' ', Acta Math., 179 (1997) pp1-39.

[BO] T. Branson and B. Ørsted; "Explicit functional determinants in four dimensions", Proc. Amer. Math. Soc., vol 113, (1991), pp 669-682.

[CNS-1] L. Caffarelli, L. Nirenberg and J. Spruck; "The Dirichlet problem for nonlinear second order elliptic equations, I: Monge-Ampere equations", CPAM, vol 37, (1984), pp 369-402.

[CNS-2] L. Caffarelli, L. Nirenberg and J. Spruck; "The Dirichlet problem for nonlinear second order elliptic equations, III: Functions of the eigenvalues of the Hessian", Acta Math., vol 155 (1985), no. 3-4, pp 261-301.

[CKNS] L. Caffarelli, J. Kohn, L. Nirenberg and J. Spruck; "The Dirichlet problem for nonlinear second order elliptic equations, II: Complex Monge-Ampere, and uniformly elliptic equations", CPAM 38(1985), pp 209-252.

[CGY-1] S.-Y. A. Chang, M. Gursky and P.Yang; "On the regularity of a fourth order PDE with critical exponent", Amer. J. of Math., 121 (1999), pp 215-257.

[CGY-2] S.-Y. A. Chang, M. Gursky and P.Yang; "An equation of Monge- Ampere type in conformal geometry, and four-manifolds of positive Ricci curvature", preprint 1999.

[CGY-3] S.-Y. A. Chang, M. Gursky and P. Yang; "A prior estimtes for a class of fully non-linear equations on 4-manifolds", preprint, 2001.

[CQY-1] S.Y.A. Chang, J. Qing and P. Yang; "On the Chern-Gauss-Bonnet integral for conformal metrics on R^4", Duke Math J, vol 103, No. 3 (2000), pp 523-544.

[CQY-2] S.Y.A. Chang, J. Qing and P. Yang; "Compactification for a class of conformally flat 4-manifold", Inventiones Mathemticae, 142 (2000), pp 65-93.

[CQY-3] S.Y.A. Chang, J. Qing and P. Yang; " On finiteness of Kleinian group in space", preprint, 2001.

[CY-1] S.-Y. A. Chang and P. Yang; "Extremal metrics of zeta functional determinants on 4-Manifolds", Annals of Math. 142(1995), pp 171-212.

[CV] S. Cohn-Vossen; "Kürzest Wege und Totalkrümmung auf Flächen", Compositio Math. 2 (1935), pp 69-133.

[De] A. Derdzinski; "Self-dual Khler manifolds and Einstein manifolds of dimension four", Compositio Math. 49 (1983), no. 3, pp 405-433.

[Ev] C. Evans; "Classical solutions of fully non-linear, convex, second order elliptic equations", CPAM, XXV (1982), pp 333-363.

[F] H. Fang; Thesis, Princeton Univ., 2001.

[FG] C. Fefferman and C.R. Graham; "Conformal invariants", In: *Élie Cartan et les Mathématiques d'aujourd'hui. Asterisque* (1985), pp 95-116.

[Fi] R. Finn; "On a class of conformal metrics, with application to differential geometry in the large", Comm. Math. Helv. 40 (1965), pp 1-30.

[GJMS] C. R. Graham, R. Jenne, L. Mason, and G. Sparling; "Conformally invariant powers of the Laplacian, I: existence", J. London. Math. Soc. (2) 46, (1992), pp 557-565.

[G-1] M. Gursky; "The Weyl functional, deRham cohomology and Kahler-Einstein metrics", Ann. of Math. 148 (1998), pp 315-337.

[G-2] M. Gursky; "The principal eigenvalue of a conformally invariant differential operator , with an application to semilinear elliptic PDE", preprint 1998, to appear in Comm. Math. Phys.

[GV] M. Gursky and J. Viaclovsky; "Quadratic Riemannian functionals on 3-manifolds", preprint 1999, to appear in Inventiones Math.

[Ha] R. Hamilton; "The Ricci flow on surfaces", *Comp. Math.* **71** Mathematics and General Relativity, AMS, 1988.

[Hu] A. Huber; "On subharmonic functions and differential geometry in the large", Comm. Math. Helv. 32 (1957), pp 13-72.

[K] M. Kapovich; "On the absence of Sullivan's cusp finiteness theorem in higher dimensions. Algebra and analysis ", (Irkutsk, 1989), 77–89, Amer. Math. Soc. Transl. Ser. 2, 163,
Amer. Math. Soc., Providence, RI, 1995

[Kr] N.V. Krylov; "Boundedly nonhomogeneous elliptic and parabolic equations", Izv. Akad. Nak. SSSR Ser. Mat. 46, (1982), pp 487-523; English transl. in Math. USSR Izv. 20, (1983), pp 459-492.

[LNN] C. Lebrun, S. Nayatani, and T. Nitta; "Self-dual manifolds with positive Ricci curvature", Math. Z. 224 (1997), no. 1, pp 49-63.

[L] Y. Li; "Degree theroy for second order non-linear ellptic equations", Comm. in PDE, 14 (1989), pp 1541-78.

[M] J. Moser; "A Sharp form of an inequality by N. Trudinger", Indiana Math. J., 20, (1971), pp 1077-1091.

[P] S. Paneitz; "A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds", Preprint, 1983.

[ScY] R. Schoen and S.T. Yau; "Lectures on Differential Geometry", International Press, 1994.

[ShYa] J. Sha and D. Yang; "Positive Ricci curvature on compact simply connected 4-manifolds", Differential geometry: Riemannian geometry (Los Angeles, CA, 1990), pp 529-538, Proc. Sympos. Pure Math., 54, Part 3, Amer. Math. Soc., Providence, RI, 1993.

[SW] E. Stein and G. Weiss; "Introduction to Fourier Analysis on Euclidean spaces", Princeton University Press, 1971.

[T] N. Trudinger; "On embedding into Orlicz spaces and some applications", J. Math. Mech, 17(1967), pp 473-483.

[UV] K. Uhlenbeck and J. Viaclovsky: " Regularity of weak solutions to critical exponent variational equations", Math. Res. Lett., 7 (2000), pp 651-656.

[V-1] J. Viaclovsky; "Conformal geometry, contact geometry and the calculus of variations", Duke Math. J., to appear

[V-2] J. Viaclovsky; "Estimates and existence results for some fully nonlinear elliptic equations on Riemannian manifolds", preprint 1999.

[Ye] R. Ye; "Global existence and convergence of the Yamabe flow", J. Differential Geom. 39 (1994), no. 1, pp 35-50.

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