The Inequality of Moser and Trudinger and applications to conformal geometry

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Dedicated to the memory of Jürgen Moser

0 Introduction

In the study of conformal structures on manifolds, the application of method of nonlinear partial differential equation may be said to begin with the work of Poincare. In the paper [43], Poincare solved the uniformization problem for Riemann surfaces of genus greater than one by solving for a conformal metric with constant negative curvature -1. The analogous question for surfaces of positive curvature was first studied successfully by Moser in two fundamental papers ([37], [38]) in which he obtained with precise constants, a sharp version of a limiting case of the Sobolev inequality that are now commonly referred to as the Moser-Trudinger inequalities. The inequality asserts (for precise statement, see section one) that for two dimensional domains, compactly supported functions with Dirichlet integrals bounded by a constant automatically belong to exponentially integrable class. This inequality makes it possible to obtain bounds for the Dirichlet integral of a minimizing family of functions in a variational problem associated with the problem to prescribe the Gauss curvature of a conformal metric on the real projective plane.

Subsequent development depends in part on understanding the role of the Mobius group in the critical exponent inequalities. Since the variational functional in Moser's approach has a natural meaning in spectral geometry, there is beautiful development concerning the compactness of isospectral family of metrics in low dimension. The theory is enriched by the introduction of higher order invariants by the work of Paneitz ([44]) and Fefferman-Graham ([27]) , and more recently our own work on the σ_2 equation ([11]). In this brief article, we will survey some work in conformal geometry in which the Moser-Trudinger inequality plays a role, as well as extensions and generalizations of this inequality to higher dimensions. In the last section we will present a new inequality on the 4-sphere which is the natural generalization of the original Moser's inequality for the 2-sphere.

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1 The inequality of Moser and Trudinger

For a smooth domain $\Omega \subset R^n$, let $W_0^{\alpha,q}(\Omega)$ denote the closure of functions of compact support in Ω with derivatives of order α in L^q under the norm $||u||_{\alpha,q,\Omega} = (\int_\Omega \sum_{|\beta|\leq\alpha} |D^\beta u|^q dx)^{1/q}$. The classical Sobolev embedding theorem states that $W_0^{\alpha,q}(\Omega) \subset L^p(\Omega)$ where $\frac{1}{p} = \frac{\alpha}{n} - \frac{1}{q}$ for $\alpha q < n$ and $p > 1$. In the limiting case when $\alpha q = n$, one can easily see that the corresponding inclusion cannot hold. For example, when $\alpha = 1, q = n = 2$, one may take Ω to be the unit ball B in R^2, and let $u(x) = \log(1 + \log\frac{1}{|x|})$, then one checks easily that u belongs to $W_0^{1,2}(B)$ but it does not belong to $L^\infty(B)$. Trudinger pointed out that functions in $W_0^{\alpha,\frac{n}{\alpha}}$ are in the exponential class (i.e. in the Orlicz space).

Theorem 1. *(Trudinger [53], 1967) There exist constants β, C depending only on the dimension n, so that for functions $u \in W_0^{1,n}(\Omega)$ satisfying the normalization $\int_\Omega |\nabla u|^n dx \leq 1$, we have*

$$\int_\Omega exp(\beta|u|^{\frac{n}{n-1}})dx \leq C|\Omega|. \tag{1.1}$$

For application to the prescribed Gauss curvature equation, one requires a particular value for the best constant β_0. In connection with his work on the Gauss curvature equation, Moser sharpened the above result of Trudinger:

Theorem 2. *(Moser [37], 1971) There exists sharp constant $\beta_0 = \beta_0(n) = n\omega_{n-1}^{\frac{1}{n-1}}$ and $C = C(n)$ so that for $u \in W_0^{1,n}(\Omega)$ satisfying $\int_\Omega |\nabla u|^n\, dx \leq 1$, the inequality (1.1) holds for all $\beta \leq \beta_0(n)$. The constant β_0 is sharp in the sense that for all $\beta > \beta_0$ there is a sequence of functions $u_k \in W_0^{1,n}(\Omega)$ satisfying $\int_\Omega |\nabla u_k|^n dx \leq 1$, but the integrals $\int_\Omega exp(\beta|u_k|^{\frac{n}{n-1}})dx$ grow without bound.*

Subsequently, Carleson and Chang [8] found that, contrary to the situation for Sobolev inequality, there is an extremal function realizing the equality when $\beta = \beta_0(n)$ and Ω is the unit ball in Euclidean space. This fact remains true for simply connected domains in the plane was shown by Flücher ([28]), and for some domains in the n-sphere by Soong ([48]).

Since the argument of Moser was based on the symmetrization procedure, the nature of the constant β_0 is expected to be related to the isoperimetric constant. This is made explicit in the articles ([19], [21]) in the former it was shown that the constant β_0 is determined by certain isoperimetric constant associated to two dimensional piecewise smooth domains and in the latter for two dimensional orbitfolds.

2 Prescribing Gaussian curvature for surfaces

The problem to characterize the Gauss curvature function on the 2-sphere is commonly attributed to Nirenberg. Since the 2-sphere has a unique conformal structure, this problem can be interpreted as to find a conformal metric with a prescribed curvature function. Let S^2 denote the unit sphere in R^3 with the standard metric g_0 of constant Gauss curvature one. Consider conformal metric $g_w = e^{2w} g_0$ whose Gaussian curvature K_w is given by the following equation:

$$-\Delta w + 1 = K_w e^{2w}. \tag{2.1}$$

Here and later on Δw and ∇w, etc are taken with respect to the background metric g_0. In addition to the obvious sign requirement imposed by the Gauss-Bonnet theorem, there is an obstruction discovered by Kazdan and Warner ([34])

$$\int_{S^2} \nabla K_w \cdot \nabla x \; e^{2w} dA_0 = 0 \tag{2.2}$$

where x is any of the ambient coordinate function. Moser realized that the implicit integrability condition is satisfied if the conformal factor has antipodal symmetry and that in fact there is no further integrability condition in that case:

Theorem 3. *(Moser [38], 1971) Let K be a function with antipodal symmetry and positive somewhere on the 2-sphere. Then there is a smooth function w also with antipodal symmetry for which the equation (2.1) holds for $K = K_w$.*

Moser studied the variational functional

$$J[w] = \frac{1}{4\pi} \int_{S^2} (|\nabla w|^2 + 2w) dA_0 - \log\{\frac{1}{4\pi} \int_{S^2} K e^{2w} dA_0\} \tag{2.3}$$

and proved, in the same paper, a version of the inequality (1.1) for functions on the 2-sphere:

Theorem 4. *(Moser [38]) Let w be a smooth function on the 2-sphere satisfying the normalizing conditions: $\int_{S^2} |\nabla w|^2 dA_0 \leq 1$ and $\bar{w} = 0$ where $\bar{w}$ denotes the mean value of w, then*

$$\int_{S^2} e^{\beta w^2} dA_0 \leq C \tag{2.4}$$

where $\beta \leq 4\pi$ and C is a fixed constant. If w has antipodal symmetry then the inequality holds for $\beta \leq 8\pi$.

The general inequality (2.4) with $\beta = 4\pi$ shows the functional $J[w]$ is bounded from below. However due to the action of the Mobius group, a minimizing sequence in general will not satisfy the Palais-Smale property. But within the class of functions satisfying antipodal symmetry, the inequality (2.4) hold with the better value $\beta = 8\pi$, hence there is compactness in a minimizing sequence. Thus the functional J achieves a minimum within the class of functions with antipodal symmetry.

The inequality of Moser shows that there is a lower bound for the functional $J[w]$, Onofri ([40]) determined the best lower bound, in his study of the volume element in string theory integrals, using an estimate of Aubin ([2]):

$$J[w] = \frac{1}{4\pi}\int_{S^2}(|\nabla w|^2 + 2w)dA_0 - \log\{\frac{1}{4\pi}\int_{S^2} e^{2w}dA_0\} \geq 0 \tag{2.5}$$

and equality holds precisely for conformal factors w of the form $e^{2w}g_0 = T^* g_0$ where T is a Mobius transformation of the 2-sphere. This inequality was also obtained independently by Hong [33].

A fascinating implication of Moser's inequality is associated with the fact that $J[w]$ for $K = 1$ computes the logarithm of the regularized determinant of the Laplacian as defined by Ray-Singer ([46]) see also ([45]):

$$J[w] = -\frac{1}{6}log(\frac{det\Delta_{g_w}}{det\Delta_{g_0}}). \tag{2.6}$$

Independently of Onofri, Osgood-Philips-Sarnak ([41], [42]) arrived at the same sharp inequality in their study of the log-determinant of the Laplacian. This inequlity also plays an important role in their proof of the C^∞ compactness of isospectral metrics on compact surfaces. The reader is also referred to the lecture notes ([9]) for connections between Moser-Onofri inequality and other isospectral problems in conformal geometry.

Returning to the solvability question of the Nirenberg problem, we devised a degree count ([18], [19], [10]) associated to the function K and the Mobius group on the 2-sphere, that is motivated by the Kazdan-Warner condition. This degree actually computes the Leray-Schauder degree of the equation as a nonlinear Fredholm equation. In the special case that K is a Morse function satisfying the condition $\Delta K(x) \neq 0$ at the critical points x of K, this degree can be expressed as:

$$\sum_{\nabla K(q)=0, \Delta K(q)<0} (-1)^{ind(q)} - 1. \tag{2.7}$$

The latter degree count is also obtained later by Chang-Liu [17] and Han [32].

More recently, there is an extensive study of a generalization of the equation (2.1) to compact Riemann surfaces. Since Moser's argument is readily applicable to a compact surface (M, g_0), a lower bound for similiarly defined functional J on (M, g_0) continues to hold in that situation. The Chern-Simons-Higgs equation in the Abelian case is the study of the equation on M:

$$\Delta w = \rho e^{2w}(e^{2w} - 1) + 2\pi \sum_{i=1}^{N} \delta_{p_i}. \tag{2.8}$$

The mean field equation is the study of the equation :

$$\Delta w + \rho(\frac{he^{2w}}{\int he^{2w}} - 1) = 0, \tag{2.9}$$

where ρ is a real parameter that is allowed to vary.

There is active development on these equations by several group of researchers including Caffarelli-Y.Yang ([7]), Ding-Jost-Li-Wang ([23]), Tarantello ([51]), Struwe and Tarantello ([49]), C.-C. Chen and C.S. Lin ([22]); and most recently by Y. Yang ([56]) on systems of such equations.

In higher dimensional Kähler geometry, the Moser-Trudinger inequality also plays a role in the study of Kähler Einstein metrics. The reader is referred to the articles of Siu ([47]), Ding-Tian ([24]) and Tian ([52]).

3 Fully nonlinear equations in conformal geometry in dimension four

In dimensions greater than two, the natural curvature invariants in conformal geometry are the Weyl tensor W, and the Weyl-Schouten tensor $A = Rc - \frac{R}{2(n-1)}g$ that occur in the decomposition of the curvature tensor.

$$Rm = W \oplus \frac{1}{n-2} A \owedge g \tag{3.1}$$

Since the Weyl tensor W transform by scaling under conformal change $g_w = e^{2w}g$, only the Weyl-Schouten tensor depends on the derivatives of the conformal factor. It is thus natural to consider $\sigma_k(A_g)$ the k-th symmetric function of the eigenvalues of the Weyl-Schouten tensor A_g as curvature invariants of the conformal metrics. As a differential invariant of the conformal factor w, $\sigma_k(A_{g_w})$ is a fully nonlinear expression involving the Hessian and the gradient of the conformal factor w. We have abbreviating A_w for A_{g_w}:

$$A_w = \{-2\nabla^2 w + 2dw \otimes dw - \frac{|\nabla w|^2}{2}\} + A_g. \tag{3.2}$$

The equation

$$\sigma_k(A_w) = 1 \tag{3.3}$$

is a fully nonlinear version of the Yamabe equation. When $k \neq \frac{n}{2}$ and the manifold (M, g) is locally conformally flat, Viaclovsky ([54]) showed that the equation (3.3) is the Euler equation of the variational functional $\int \sigma_k(A_{g_w})dV_{g_w}$. In the exceptional case $k = n/2$, the integral $\int \sigma_k(A_g)dV_g$ is a conformal invariant. For a symmetric $n \times n$ matrix A, we say $A \in \Gamma_k^+$ if $\sigma_k(A) > 0$ and A may be joined to the identity matrix by a path consisting entirely of matrices A_t such that $\sigma_k(A_t) > 0$. We say $g \in \Gamma_k^+$ if the corresponding Weyl-Schouten tensor $A_g(x) \in \Gamma_k^+$ for every point $x \in M$. For $k = 1$ the Yamabe equation for prescribing scalar curvature

$$-\frac{4(n-1)}{n-2}\Delta u + R_0 u = R_{g'} u^{\frac{n+2}{n-2}} \tag{3.4}$$

is a semilinear one in the conformal factor u where $g = u^{\frac{4}{n-2}} g_0$; hence the condition for $g \in \Gamma_1^+$ is the same as requiring the operator $L = -\frac{4(n-1)}{n-2}\Delta + R_0$ be a positive operator. The criteria for existence of a conformal metric $g \in \Gamma_k^+$ is not as easy for $k > 1$ since the equation is a fully nonlinear one. However when $n = 4, k = 2$ the invariance of the integral $\int \sigma_2(A_g) dV_g$ is a reflection of the Chern-Gauss-Bonnet formula

$$8\pi^2 \chi(M) = \int_M (\sigma_2 + \frac{1}{4}|W|^2) dV. \tag{3.5}$$

In dimension 4, we also recall that

$$\sigma_2(A_g) = \frac{1}{2}(|Trace A_g|^2 - |A_g|^2) = \frac{R^2}{24} - \frac{|E|^2}{2}, \tag{3.6}$$

where E denotes the traceless Ricci tensor.

In this case it is possible to find a criteria:

Theorem 5. *([11]) For a closed 4-manifold (M, g) satisfying the following conformally invariant conditions:*
(i) L is a positive operator, and
(ii) $\int \sigma_2(A_g) dV_g > 0$;
then there exists a conformal metric $g_w \in \Gamma_2^+$.

Remark: In dimension four, the condition $g \in \Gamma_2^+$ implies that $R > 0$ and an easy computation shows that Ricci is positive everywhere. Thus such manifolds have finite fundamental group. In addition, the Chern-Gauss-Bonnet formula and the signature formula shows that this class of 4-manifolds satisfy the same conditions as that of an Einstein manifold with positive scalar curvatures. Thus it is the natural class of 4-manifolds in which to seek an Einstein metric.

The existence result depends on the solution of a family of fourth order equations involving the Paneitz operator ([43]). In the following we briefly outline this connection. In dimension four, the Paneitz operator

$$P = \Delta^2 + div(\frac{2R}{3} g - 2Rc)\nabla \tag{3.7}$$

enjoys conformal covariance:under conformal change of metric $g_w = e^{2w} g_0$

$$P_{g_w} = e^{-4w} P_{g_0}. \tag{3.8}$$

The Paneitz operator computes a fourth order curvature called the Q-curvature:

$$P_0 w + 2Q_0 = 2Q_{g_w} e^{4w} \tag{3.9}$$

where

$$Q = \frac{-1}{12}\Delta R + \frac{1}{2}\sigma_2. \tag{3.10}$$

In an elegant paper [31], Gursky showed that the positivity of the operator is a consequence of the assumptions (i) and (ii) of Theorem 5, and of equal significance, such manifolds satisfy the condition

$$\int_M \sigma_2(A_g)dV_g \leq 16\pi^2, \tag{3.11}$$

and equality holds if and only if M is conformally diffeomorphic to S^4. In an earlier article [20], we showed that for such a 4-manifold M, the Q-curvature may be prescribed to be a constant by a conformal metric. The main ingredient in that existence theory is the generalized Moser-Trudinger inequality of D. Adams ([1]; on manifolds [26]): For any bounded domain Ω in R^4, there is a constant $C = C(n)$ so that for a function $w \in C_0^2(\Omega)$ satisfying the normalization $\int |\Delta w|^2 \leq 1$, we have

$$\int_\Omega e^{32\pi^2 w^2} dx \leq C|\Omega|. \tag{3.12}$$

A corresponding inequality can be shown to hold for a function w on a closed 4-manifold whose Paneitz operator is positive, $\int_M w dV = 0$ and the normalization $\int_M Pw \cdot w dV \leq 1$. This then is the starting point of a continuity argument in which we solve the family of equations

$$\sigma_2(A_g) = \frac{\delta}{4}\Delta R - \gamma|W|^2 \tag{3.13}$$

where γ is chosen so that $\int \sigma_2(A_g)dV_g = -\gamma \int |W|_g^2 dV_g$. The bulk of the analysis consist in estimating the solution as δ tends to zero, showing essentially that in the equation (3.13) the term $\frac{\delta}{4}\Delta R$ is small in the weak sense. The proof ends by applying the Yamabe flow to the metrics g_δ which satisfies (3.13) to show that for sufficiently small δ the smoothing provided by the Yamabe flow yields a metric $g \in \Gamma_2^+$.

The equation (3.3) becomes meaningful for 4-manifolds which admits a metric $g \in \Gamma_2^+$. In the article [12], we provide apriori estimates for solutions of the equation

$$\sigma_2(A_g) = f \tag{3.14}$$

where f is a given positive smooth function. Then we use the following 1-parameter family of equations

$$\sigma_2(A_{g_t}) = tf + (1-t) \tag{3.15}$$

to deform the original metric to one with constant $\sigma_2(A_g)$.

In terms of geometric application, this circle of ideas may be applied to characterize a number of interesting conformal classes in terms of the the relative size of the conformal invariant $\int \sigma_2(A_g)dV_g$ compared with the Euler number.

Theorem 6. *([14]) Suppose (M^4, g) is a closed 4-manifold whose conformal Laplacian is positive. If*

$$\int_M \sigma_2(A_g) dV_g > \frac{1}{4}\int_M |W|_g^2 dV_g, \tag{3.16}$$

then M is diffeomorphic to a quotient of the standard 4-sphere.
If M is not diffeomorphic to the standard 4-sphere and

$$\int_M \sigma_2(A_g) dV_g = \frac{1}{4}\int_M |W|_g^2 dV_g, \tag{3.17}$$

then M is conformally equivalent to a quotient of CP^2 or $S^1 \times S^3$.

This first part of Theorem 6 applies the existence argument to find a conformal metric g' which satisfies the pointwise inequality

$$\sigma_2(A_g') > \frac{1}{4}|W'|^2. \tag{3.18}$$

The diffeomorphism assertion follows from Margerin's ([36]) precise convergence result for the Ricci flow: such a metric will evolve under the Ricci flow to one with constant curvature. Therefore such a manifold is diffeomorphic to a quotient of the standard 4-sphere.

For the second part of the assertion, we argue that if such a manifold is not diffeomorphic to the 4-sphere, then the conformal structure realizes the minimum of the quantity $\int |W|^2 dV$, and hence its Bach tensor vanishes. There are two possibilities depending on whether the Euler number is zero or not. In the first case, an earlier result of Gursky ([30]) shows the metric is conformal to that of the space $S^1 \times S^3$. In the second case, we solve the equation

$$\sigma_2(A_g') = \frac{1}{4}|W'|^2 + \epsilon \tag{3.19}$$

and let ϵ tends to zero. We obtain in the limit a $C^{1,1}$ metric which satisfies the equation on the open set $\Omega = \{x | W(x) \neq 0\}$:

$$\sigma_2(A_g') = \frac{1}{4}|W'|^2. \tag{3.20}$$

Then a long Lagrange multiplier computation, inspired in part by the corresponding computation of Margerin, shows that the curvature tensor of the limit metric agrees with that of the Fubini-Study metric on the open set where $W \neq 0$. Therefore $|W'|$ is a constant on Ω thus W cannot vanish at all. It follows that the curvature tensor of the limit metric agrees with that of Fubini-Study metric everywhere.

4 A Moser-Onofri inequality for the 4-sphere

In [3], Beckner generalized the sharp inequality (2.5) of Moser-Onofri to n-spheres. Denote by (S^n, g_0) the n-sphere in R^{n+1} with the standard metric g_0; Beckner's inequality bounds the volume of the metric $g_w = e^{2w} g_0$ by an energy term with leading order term of the form $\int (\Delta w)^{\frac{n}{2}} dV_0$. In our work ([20]), we gave an alternative argument for this inequality based on the conformal covariance of the general n-th order Paneitz operator. For example in case of the 4-sphere, the inequality takes the form:

$$\frac{1}{|S^4|}\int_{S^4} \{(\Delta w)^2 + 2|\nabla w|^2 + 12w\} dV_0 - 3log\{\frac{1}{|S^4|}\int_{S^4} e^{4w} dV_0\} \geq 0, \tag{4.1}$$

where $|S^4|$ ($= \frac{8\pi^2}{3}$) denotes the volume of the 4-sphere. The equality hold if and only if the metric g_w is isometric to the standard metric g_0.

In this section we discuss another extension of the sharp Moser-Onofri inequality to S^n when $n = 2k$ is even, and for a class of functions whose associated conformal metrics belong to the class Γ_k^+.

For a compact surface (M^2, g_0), consider the functional

$$J[w] = \int |\nabla w|^2 + 2K_0 w dV_0 \tag{4.2}$$

under the volume constraint that $\int e^{2w} dV_0 = Vol(g_0)$, where K_0 denotes the Gaussian curvature of the metric g_0. Then

$$J'[w](\phi) = \frac{d}{d\epsilon}|_{\epsilon=0} J[w + \epsilon\phi] = 2\int (-\Delta w + K_0)\phi dV_0 \tag{4.3}$$

for all $\phi \in C^\infty(M)$ with $\int e^{2w}\phi dV_0 = 0$. It follows from the Gaussian curvature equation

$$-\Delta w + K_0 = K_w e^{2w} \tag{4.4}$$

that at a critical point w of the functional J:

$$0 = J'[w](\phi) = 2\int K_w e^{2w} \phi dV_0 \tag{4.5}$$

for all $\phi \in C^\infty(M)$ with $\int e^{2w}\phi dV_0 = 0$. We say $\frac{1}{2}J$ is a conformal primitive of the Gaussian curvature K.

On a compact n-manifold ($n \geq 3$), a similar computation shows that the functional

$$F[w] = \frac{1}{n-2}\int_M R_{g_w} dV_{g_w} \tag{4.6}$$

is the conformal primitive of the scalar curvature R. Using this terminology, the result of Viaclovsky which we have mentioned in the previous section can be restated as:

Theorem 7. *([54]) On a compact (M^n, g),*
(a) in the case $n \neq 2k$, the functional $F_k[w] = \frac{1}{n-2k}\int_M \sigma_k(A_{g_w})dV_{g_w}$ is the conformal primitive of $\sigma_k(A_g)$;
(b) in the remaining case $n = 2k$, and assume also that (M^n, g) is locally conformally flat, then $\int_M \sigma_k(A_{g_w})dV_{g_w}$ is conformally invariant.

In view of the statement (a), it is natural to ask for the existence of a functional which is the conformal primitive of $\sigma_k(A_g)$ when $n = 2k$. In our previous work on the log determinant functional ([5] for the 4-sphere; [20] for general 4-manifolds), it was observed that such a functional exists in the case $n = 4 = 2k$. To describe the functional, let us define for a compact 4-manifold M, $g_w = e^{2w}g$:

$$\begin{aligned} II[w] &= \int Pw \cdot w + 4\; Q_g w dV_g \\ III[w] &= \frac{1}{3}(\int R_{g_w}^2 dV_{g_w} - \int R_g^2 dV_g), \end{aligned}$$

where P is the Paneitz operator, and $Q = -\frac{1}{12}\Delta R + \frac{1}{2}\sigma_2$ as defined in section 3. In fact, using equation (3.9) one can easily check that II is the conformal primitive of the fourth order curvature $4Q$. By another straightforward calculation, one can also check that III is the conformal primitive of $-4\Delta R$. Therefore, the conformal primitive of σ_2 is given by

$$F_2[w] = \frac{1}{2}(II[w] - \frac{1}{12}III[w]). \tag{4.7}$$

It is thus natural to ask if one can study the problem to prescribe the curvature invariant $\sigma_2(A_g)$ by a variational method using the conformal primitive. We remark that, in general this cannot be an easy task since the functional F_2 is the difference of functionals II and III which are both coercive (in the cases we consider) and of higher order, although there is total cancellation of the fourth order terms. In particular on the 4-sphere, both II and III are extremized by the standard metric ([5], see also [3]). It is not clear how to study $\inf F_2[w]$. It is our purpose in this section to study this problem for the restrictive class of metrics $g_w \in \Gamma_2^+$. We will use a parabolic equation introduced by Guan and Wang ([29]):

$$\frac{d}{dt}g = -(log(\sigma_k(g)) - log(r_k(g))) \cdot g \tag{4.8}$$

where $\log(r_k(g)) = \int log(\sigma_k(g))dV_g$ and the initial metric $g(0) = g_0$. When the manifold M^4 is conformally flat, the argument of Ye ([57]), shows that there is apriori C^1 estimates for solutions of equation (4.8). In the article [29], Guan and Wang showed the longtime existence as well as the uniform C^2 estimates for solutions of the equation. We now follow the arguments in [50] and modify them to the functional F_2. First it is easy to see that under the flow (4.8) we have

$$\frac{d}{dt}F_2[g(t)] = -\frac{1}{2}\int (\sigma_2(g) - r_2(g))(log\sigma_2(g) - log(r_2(g))dV_g. \tag{4.9}$$

Therefore the functional F_2 decreases under the flow (4.8). In addition, we have

$$\int_0^T \int_M (\sigma_2(g) - r_2(g))(\log(\sigma_2(g)) - \log(r_2(g))dV_g dt \leq 2|F_2[g(t)] - F_2[g(0)]|. \tag{4.10}$$

It follows that under the flow $\sigma_2(g)$ and $F_2[g]$ remain bounded and

$$\int_0^\infty \int_M (\sigma_2(g) - r_2(g))^2 dV_g dt < \infty. \tag{4.11}$$

Then there exists a sequence of times $\{t_l\}$ for which

$$\int_M (\sigma_2(g(t_l)) - r_2(g(t_l)))^2 dV_g \to 0. \tag{4.12}$$

On account of the uniform C^2 bounds for the metrics $g(t_l)$, a subsequence will converge in $C^{1,\alpha}$ to a $C^{1,1}$ metric g_∞ which is a viscosity solution of the equation $\sigma_2(g) = constant$; this constant is positive due to the conformal invariance of the integral $\int \sigma_2(A_g)dV_g$. Since such solutions are in fact smooth according to Evans-Krylov ([25], [35]), the classification provided in ([55], see also [13]) shows that g_∞ must be standard, hence the constant curvature metric on S^4 realizes the infimum for F_2. We summarize this conclusion in the following:

Theorem 8. *On the 4-sphere (S^4, g_0), if $g_w = e^{2w}g_0$ is a conformal metric lying in the set Γ_2^+, then we have*

$$\frac{1}{|S^4|}\int_{S^4}\{-2\Delta w|\nabla w|^2 - |\nabla w|^4 + 6|\nabla w|^2 + 12w\}dV_0 - 3\log\{\frac{1}{|S^4|}\int_{S^4} e^{4w}dV_0\} \geq 0. \tag{4.13}$$

Remark: The conditon $g_w \in \Gamma_2^+$ cannot be removed as we see easily that by taking w to be a large multiple of any first eigenfunction on the 4-sphere makes the quantity in (4.13) an arbitrarily large negative number. We thank the referee for pointing out this example. However, it is reasonable to ask if the inequality continues to hold for metrics in the set Γ_1^+, that is, metrics with positive scalar curvature.

More generally, we now describe a possible procedure to find a functional $F_{k,n}$, which is a conformal primitive of σ_k when $n = 2k$ for a conformally flat structure. We illustrate the method by deriving the functional $F_{2,4} = \frac{1}{4}F_2$ for the functional F_2 in (4.7) in dimension 4. Thus the conformal flow (4.8) can be applied to derive, in principle, an extension of the Moser-Onofri inequality to all even dimensional spheres.

We first set up the notations. To be consistent with the notations of Viaclovsky, let us denote $C_{ij} = \frac{1}{n-2}A_{ij}$ and $\sigma_{k,n}$ the k-th symmetric function of the eigenvalues of C_{ij}. Thus for $n = 4$, we have $\sigma_{2,4} = \frac{1}{4}\sigma_2$ for σ_2 defined in (3.6).

We define the conformal primitive for $\sigma_{k,n}$, using the "analytic continuation in dimension" method that we learned from Tom Branson. In [4], Branson used a similar method to calculate the conformal primitive of the Q-curvatures. Let us denote for $n \neq 2k$, $\sigma_{k,n}(w) = \sigma_{k,n}(C_{g_w})$ and define for $g_w = e^{2w}g_0$

$$F_{k,n}[w] = \frac{1}{n-2k}\int_M (\sigma_{k,n}(w)e^{nw} - \sigma_{k,n}(0))dV_0. \tag{4.14}$$

Then according to Viaclovsky,

$$\frac{d}{d\epsilon}|_{\epsilon=0}F_{k,n}[w+\epsilon\phi] = \int \sigma_{k,n}(w)\phi dV_w. \tag{4.15}$$

We write for $n \neq 2k$,

$$\begin{aligned}
\sigma_{k,n}(w)e^{nw} - \sigma_{k,n}(0) &= e^{(n-2k)w}\sigma_{k,n}(w)e^{2kw} - \sigma_{k,n}(0) \\
&= (e^{(n-2k)w}-1)\sigma_{k,n}(w)e^{2kw} + (\sigma_{k,n}(w) - \sigma_{k,2k}(w))e^{2kw} \\
&+ (\sigma_{k,2k}(w)e^{2kw} - \sigma_{k,2k}(0)) + (\sigma_{k,2k}(0) - \sigma_{k,n}(0)).
\end{aligned}$$

Notice that the second to the last term in the above expression is zero after integration over the manifold M. Therefore we divide the equation above by $n-2k$ and take limit as n tends to $2k$:

$$F_{k,2k}[w] = \lim_{n\to 2k} F_{k,n}[w] = \int (w\sigma_{k,2k}(w)e^{2kw} + \frac{d}{dn}|_{n=2k}\{\sigma_{k,n}(w)e^{2kw} - \sigma_{k,n}(0)\})dV_0. \tag{4.16}$$

Remarks:
1. The quantity $F_{k,n}[w]$ has the following scaling property: if in the definition of C_{ij} we put $C_{ij} = c_n A_{ij}$ for some choice of c_n, and denote the resulting quantity by $\tilde{\sigma}_{k,n}$ and repeat the same steps to define the corresponding functional $\tilde{F}_{k,n}$. Then $\tilde{F}_{k,n}[w] = c_n F_{k,n}[w]$. This is clear when $n \neq 2k$. When $n = 2k$, we observe that in the formula (4.16) we have

$$\frac{d}{dn}|_{n=2k}\tilde{\sigma}_{k,n}[w] = (\frac{d}{dn}|_{n=2k}c_n)\sigma_{k,2k}[w] + c_n\frac{d}{dn}|_{n=2k}\sigma_{k,n}[w]. \tag{4.17}$$

Thus, due to the conformal invariance of the integral $\int \sigma_{k,2k}(g)dV_g$, we have $\tilde{F}_{k,2k}[w] = c_n F_{k,2k}[w]$.
2. We need to explain the justification in taking the derivative $\frac{d}{dn}$ in the above formula. When viewed as formal algebraic expressions in the various derivatives of w in an appropriate tensor space, the quantity $\sigma_{k,n}(w)$ may be expanded using the formula of A_{ij} as in (3.2) into a polynomial expression in various derivatives of w with coefficients that are rational expressions in n. Viewed as function of n such an expression is rational in n with no pole at $n = 2k$. Since the formula (4.15) for the conformal primitive may be viewed as an identity in the corresponding rational expression in n, it may be differentiated at $n = 2k$ to derive the equation of conformal primitive for $\sigma_{k,2k}$.

To illustrate this procedure, we carry out the computation of $F_{1,2}$ in (4.16). Recall

$$\sigma_{1,n} = \frac{1}{n-2} tr(R_{ij} - \frac{1}{2(n-1)} R g_{ij}) = \frac{1}{2(n-1)} R. \tag{4.18}$$

Let us denote the conformal metric by $g_w = e^{2w} g = u^{\frac{4}{n-2}} g$. Recall the scalar curvature equation (3.4):

$$\sigma_{1,n}(w) = (-\frac{2}{n-2}\Delta u + \sigma_{1,n}(0)u) u^{-\frac{n+2}{n-2}}, \tag{4.19}$$

which may also be written as

$$\sigma_{1,n}(w) = -\{\frac{n-2}{2}|\nabla w|^2 + \Delta w - \sigma_{1,n}(0)\} e^{-2w}. \tag{4.20}$$

Thus

$$\frac{d}{dn}\sigma_{1,n}|_{n=2} e^{2w} - \frac{d}{dn}|_{n=2}\sigma_{1,n}(0) = -\frac{1}{2}|\nabla w|^2. \tag{4.21}$$

It follows from (4.20) that

$$\int w\sigma_{1,2} e^{2w} dV_0 = \int \{-\Delta w + K\} w dV_0 = \int \{|\nabla w|^2 + Kw\} dV_0. \tag{4.22}$$

Combining equations (4.21) and (4.22) into (4.16), and compare to the formula (4.2), we find:

$$F_{1,2} = \frac{1}{2}\int \{|\nabla w|^2 + 2Kw\} dV_0 = \frac{1}{2} J[w]. \tag{4.23}$$

We remark that a similar, but more tedious computation also shows that $F_{2,4} = \frac{1}{4} F_2$

In view of the validity of Beckner's inequality for spheres of all dimension, one has to ponder what should be an appropriate analogue of the inequality (4.13) for odd dimensional spheres. In the articles [15], [16] sharp versions of the Moser-Onofri inequality for a third order operator on the 3-sphere as a boundary operator was obtained. Such considerations may be relevant to this question. We hope to return to this question on a later occasion.

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