Q-curvature and Conformal Covariant operators

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Question: What are the general conformal invariants? What are the conformal covariant operators and their related curvature invariants?

• Second order invariants:

1. (Δ_g, K_g) on (M^2, g) satisfying

$$\Delta_{g_w} = e^{-2w}\Delta_g$$

and

$$-\Delta_g w + K_g = K_{g_w} e^{2w}.$$

2. (L_g, R_g) on (M^n, g), $n \geq 3$, where

$$L_g = -c_n \Delta_g + R_g$$

where R_g the scalar curvature, satisfying

$$L_{g_w} = e^{-\frac{n+2}{2}w} L_g(e^{\frac{n-2}{2}w}\cdot),$$

Yamabe equation

$$L_g(e^{\frac{n-2}{2}w}) = R_{g_w} e^{\frac{n+2}{2}w}.$$

• 4th order invariants:

When $n = 4$, Paneitz operator: (1983)

$$P\varphi \equiv \Delta^2\varphi + \delta[\left(\frac{2}{3}Rg - 2Ric\right) d\varphi]$$

Satisfying:

$$P_{g_w} = e^{-4w} P_g$$

$$P_g w + 2Q_g \ = \ 2Q_{g_w} e^{4w}$$

$$Q \ = \ \frac{1}{12}(-\Delta R + R^2 - 3|Ric|^2)$$

3. On (M^n, g), $n \neq 4$.

Existence of 4-th order
conformal Paneitz operator P_4^n,

$$P_4^n = \Delta^2 + \delta \left(a_n R g + b_n \mathrm{Ric}\right) d + \frac{n-4}{2} Q_4^n.$$

For $\bar{g} = u^{\frac{4}{n-4}} g$: $P_4^n u = \bar{Q}_4^n u^{\frac{n+4}{n-4}}$.

• P_4^n is conformal covariant of bidegree $(\frac{n-4}{2}, \frac{n+4}{2})$.

• Q_4^n is a fourth order curvature invariants. i.e. under dilation $\delta_t g = t^{-2} g$,

$$(Q_4^n)(\delta_t g) = t^4 (Q_4^n)(g).$$

Fefferman-Graham (1985) systematically construct (pointwise) conformal inviariants:

Example: The Riemann curvature tensor has the decomposition

$$R_{ijkl} = W_{ijkl} + [A_{jk}g_{il} + A_{il}g_{jk} - A_{jl}g_{ik} - A_{ik}g_{jl}]$$

where

$$A = \frac{1}{n-2}[R_{ij} - \frac{R}{2(n-1)}g_{ij}]$$

is called the Schouten tensor. The Weyl tensor satisfies $W_{g_w} = e^{-2w}W_g$.

Graham-Jenne-Mason-Sparling (1992) applied method of construction to existence results of general conformal covariant operators P_{2k}^n for n even.

§2. Conformally compact Einstein manifold

Given (M^n, g), denote $[g]$ class of conformal metrics $g_w = e^{2w} g$ for $w \in C^\infty(M^n)$.

Definition: Given (X^{n+1}, M^n, g^+) with smooth boundary $\partial X = M^n$. Let r be a defining function for M^n in X^{n+1} as follows:

$$\begin{aligned} r &> 0 \quad \text{in } X; \\ r &= 0 \quad \text{on } M; \\ dr &\neq 0 \quad \text{on } M. \end{aligned}$$

• We say g^+ is conformally compact, if there exists some r so that $(X^{n+1}, r^2 g^+)$ is a compact manifold.
• (X^{n+1}, M^n, g^+) is conformally compact Einstein if g^+ is Einstein (i.e. $Ric_{g^+} = c g^+$).
• We call g^+ a Poincare metric if $Ric_{g^+} = -n g^+$.

Example:

On (H^{n+1}, S^n, g_H)

$$(H^{n+1}, (\frac{2}{1-|y|^2})^2|dy|^2).$$

We can then view $(S^n, [g])$ as the compactification of H^{n+1} using the defining function

$$r = 2\frac{1-|y|}{1+|y|}$$

$$g_H = g^+ = r^{-2}\left(dr^2 + (1-\frac{r^2}{4})^2 g\right).$$

Given (M^n, g), consider $M^+ = M^n \times [0,1]$ and metric g^+ with
(i) g^+ *has* $[g]$ as conformal infinity,
(ii) $Ric(g^+) = -ng^+$.
In an appropriate coordinate system (ξ, r), where $\xi \in M$ with
(iii) $g^+ = r^{-2}\left(dr^2 + \sum_{i,j=1}^{n} g_{ij}^+(\xi, r) d\xi_i d\xi_j\right)$,
and g_{ij}^+ is even in r.

Theorem: (C. Fefferman- R. Graham, '85)
(a) In case n is odd, up to a diffeomorphism fixing M, there is a unique formal power series solution of g^+ to (i)–(iii).
(b) In case n is even, there exists a formal power series solution for g^+ for which the components of $Ric(g^+) + ng^+$ vanish to order $n-2$ in power series of r.

Remarks:

• Conformally compact Einstein manifold is of current interest in the physics literature. The Ads/CFT correspondence proposed by Maldacena involves string theory and super-gravity on such X.

• The construction of the Poincare metric is actually accomplished via the construction of a Ricci flat metric, called the ambient metric on the manifold $\tilde{G}$, where $\tilde{G} = G \times (-1, 1)$ of dimension $n+2$ and G is the metric bundle

$$G = \left\{(\xi, t^2 g(\xi)) : \ \xi \in M^n, \ t > 0\right\}$$

of the bundle of symmetric 2 tensors S^2T^*M on M. The conformal invariants are then contractions of $(\tilde{\nabla}^{k_1}\tilde{R}\otimes\tilde{\nabla}^{k_2}\tilde{R}\otimes......\tilde{\nabla}^{k_l}\tilde{R})$ restricted to TM where $\tilde{R}$ denotes the curvature tensor of the ambient metric.

A model example is given by the standard sphere (S^n, g). Denote $S^n = \left\{ \sum_1^{n+1} \xi_k^2 = 1 \right\}$.

$$G = \left\{ \sum_1^{n+1} p_k^2 - p_0^2 = 0 \right\}$$

under $\xi_k = p_k/p_0$ $(1 \le k \le n+1)$. Then the ambient space $\tilde{G}$ is Minkowski space

$$R^{n+1,1} = \left\{ (p, p_0), |p \in \mathbb{R}^{n+1}, p_0 \in \mathbb{R} \right\}$$

with the Lorentz metric

$$\tilde{g} = |dp|^2 \, - \, dp_0^2,$$

The standard hyberbolic space is realized as the quadric $H^{n+1} = \left\{ |p|^2 - p_0^2 = -1 \right\} \subset R^{n+1,1}$.

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Graham, Jenne, Mason and Sparling (1992)
The existence of conformal covariant operator P^n_{2k} on (M^n, g) with:

• Order $2k$ with leading symbol $(-\Delta)^k$

• Conformal covariant of bi-degree $(\frac{n-2k}{2}, \frac{n+2k}{2})$; where $k \in \mathbb{N}$ when n is odd, but $2k \leq n$ when n is even.

• In general,the operators P^n_{2k} is not unique, e.g. add $|W|^k$ to P^n_{2k} , where W is the Weyl tensor, when k is even.

• On $\mathbb{R}^n$, the operator is unique and is equal to $(-\Delta)^k$. Hence the formula for P^n_{2k} on the standard sphere and on Einstein metric.

Q curvature associated with P^n_{2k}.

• When $2k \neq n$, then $P^n_{2k}(1) = c(n,k)Q^n_{2k}$, e.g. when $k = 1$, $2 < n$, $P^n_2 = -c_n\Delta + R = L$, and $Q^n_2 = R = P^n_2(1)$.

• When $2k = n$, n even Branson ('93) justified the existence of Q^n_n by a dimension continuation (in n) argument from Q^n_{2k}.
e.g. When $k = 1$ and $n = 2k = 2$, $Q^2_2 = K$ the Gaussian curvature.
When $k = 2$ and $n = 4$, $Q^4_4 = 2Q_4$.

• Graham and Zworski ('02) Existence of Q^n_n when n even, the analytic continuation of a spectral parameter in scattering theory.

Spectral Theory on (X^{n+1}, M^n, g^+), with g^+ Poincare metric and $(M^n, [g])$ as conformal infinity.

• A basic fact is (Mazzeo, Melrose-Mazzeo)

$$\sigma(-\Delta_{g^+}) = [(\frac{n}{2})^2, \infty) \ \cup \ \sigma_{pp}(-\Delta_{g^+})$$

the pure point spectrum $\sigma_{pp}(-\Delta_{g^+})$ (L^2 eigenvalues), is finite.

• For $s(n-s) \notin \sigma_{pp}$, consider

$$(-\Delta_{g+} - s(n-s))u = 0.$$

Given $f \in C^\infty(M)$, then there is a meromorphic family of solutions $u(s) = \wp(s)f$

$$\wp(s)f = Fr^{n-s} + Gr^s \quad \text{if } s \notin n/2 + \mathbb{N}$$

with $F|_M = f$

Define Scattering matrix to be

$$S(s)f = G|M$$

The relation of f to $S(s)f$ is like that of the Dirichlet to Neumann data.

Theorem: (Graham-Zworski 2002)
Let (X^{n+1}, M^n, g^+) be a Poincare metric with $(M^n, [g])$ as conformal infinity. Suppose n is even, and $k \in \mathbb{N}$, $k \leq \frac{n}{2}$ and $s(n-s)$ not in $\sigma_{pp}(-\Delta_{g^+})$. Then the scattering matrix $S(s)$ has a simple pole at $s = \frac{n}{2} + k$ and

$$c_k P^n_{2k} = -Res_{s=\frac{n}{2}+k} S(s)$$

When $2k \neq n$, $P^n_{2k}(1) = c(n,k) Q^n_{2k}$

When $2k = n$, $c_{\frac{n}{2}} Q_n = S(n)1$.

§3. Known facts for Q_n, n even:

• Q_n is a conformal density of weight -n ; i.e. with respect to the dilation δ_t of metric g given by $\delta_t(g) = t^2 g$, we have $(Q_n)_{\delta_t g} = t^{-n}(Q_n)_g$.

• $\int_{M^n} (Q_n)_g dv_g$ is conformally invariant.

• For $g_w = e^{2w} g$, we have

$$(P_n)_g w + (Q_n)_g = (Q_n)_{g_w} e^{nw}.$$

• When (M^n, g) is locally conformally flat, then $(Q_n)_g = c_n \sigma_{\frac{n}{2}}(A_g) +$ divergence terms, e.g. $Q_4 = \sigma_2(A_g) - \frac{1}{6}\Delta_g R$.

• Alexakis

$$Q_n = c_n \text{Pfaffian} + J + \text{div}(T_n).$$

where Pfaffian is the Euler class density, which is the integrand in the Gauss-Bonnet formula, J is a pointwise conformal invariant, and div(T_n) is a divergence term.

• Alexakis (also Fefferman-Hirachi) has extended the existence of conformal covariant operator to conformal densities of weight γ , where $\gamma \neq (-\frac{n}{2}) + k$ where k is a positive integer and γ not a nonnegative integer. An example of such operator is:

$$2P(f) = \nabla^i(||W||^2\nabla^i f) + \frac{n-6}{n-2}||W||^2\Delta f.$$

with corresponding Q-curvature explicit.

• Fefferman and Hirachi have also extended the construction of conformal covariant operator and Q curvature to CR manifolds.

• Branson, Eastwood-Gover survey articles, AIM meeting August 2003.

§4. Renormalized Volume (Witten, Gubser-Klebanov -Polyakov, Henningson-Skenderis, Graham)

On conformal compact (X^{n+1}, M^n, g^+) with defining function r, For n odd,

$$\begin{aligned}\mathrm{Vol}_{g^+}(\{r > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots\cdots \\ + c_{n-1}\epsilon^{-1} + V + o(1)\end{aligned}$$

For n even,

$$\begin{aligned}\mathrm{Vol}_{g^+}(\{r > \epsilon\}) = c_0\epsilon^{-n} + c_2\epsilon^{-n+2} + \cdots \\ + c_{n-2}\epsilon^{-2} + L\log\frac{1}{\epsilon} + V + o(1)\end{aligned}$$

• For n odd, V is independent of $g \in [g]$, and for n even, L is independent of $g \in [g]$, and hence are conformal invariants.

Theorem: (Graham-Zworski) When n is even,

$$L = -2\int_M S(n)1 = 2c_{\frac{n}{2}}\int_M Q_n dv_g.$$

Theorem: (Fefferman-Graham '02) Consider $v = \frac{d}{ds}|_{s=n}S(s)1$ then v is a smooth function defined on X solving

$$-\Delta_{g^+}(v) = n$$

and with the asymptotic

$$v = \begin{cases} \log x + A + Bx^n logx & \text{for n even} \\ \log x + A + Bx^n & \text{for n odd} \end{cases}$$

where $A, B \in C^\infty(X)$ are even mod O(x^∞) and $A|_M = 0$. Moreover

(i) If n is even, then

$$B|_M = -2S(n)1 = -2c_{\frac{n}{2}}Q_n$$

hence $L = 2c_{\frac{n}{2}}\int_M Q_n$.

(ii) If n is odd, then

$$B|_M = -\frac{d}{ds}|_{s=n} S(s)1,$$

and if one defines $Q_n(g^+, [g])$ to be

$$Q_n(g^+, [g]) = k_n B|_M$$

then

$$k_n V = \int_M Q_n(g^+, [g]) dv_g.$$

Remark: when n is odd, the Q curvature thus defined is not intrinsic, it depends not only on the boundary metric g on M but also on the extension of g^+ on X.

On compact Riemannian 4-manifold (X^4, M^3, g^+) with boundary, Chang-Qing introduced

$$(P_b)_{g_w} = e^{-3w}(P_b)_g, \text{ on M and}$$
$$(P_b)_g w + T_g = T_{g_w} e^{3w} \text{on M.}$$

$$8\pi^2\chi(X) = \int_{X^4}(\frac{1}{4}|W|^2 + Q_4)dv + 2\int_{M^3}(\mathcal{L}+T)d\sigma,$$

where $\mathcal{L}$ is a point-wise conformal invariant term on the boundary of the manifold.

On conformally compact Einstein (X^4, M^3, g^+):

$$(P_b)_g = -\frac{1}{2}\frac{\partial}{\partial n}\Delta_{g^+}|_M, \;\; T_g = \frac{1}{12}\frac{\partial R}{\partial n}|_M,$$

and in this case $\mathcal{L}$ vanishes.

§ When $n = 3$, on (X^4, M^3, g^+), conformally compact Einstein

Theorem: (Chang-Qing-Yang)
On (X^4, M^3, g^+)
(i) $(Q_4)_{e^{2v}g^+} = 0$,

Proof: Recall

$$Q_4 = \frac{1}{6}(-\Delta R + R^2 - 3|Ric|^2).$$

Thus for g^+ a Poincare metric with $Ric\ g^+ = -3g^+$, we have $(Q_4)_{g^+} = 6$ and

$$(P_4)_{g^+} = (\Delta)^2{}_{g^+} + 2\Delta_{g^+}.$$

We then use the equations $-\Delta_{g^+}(v) = n = 3$ and

$$(P_4)_{g^+}(v) + (Q_4)_{g^+} = (Q_4)_{e^{2v}g^+}$$

to conclude the proof.

(i) $(Q_4)_{e^{2v}g^+} = 0$,
(ii) $Q_3(e^{2v}g^+, [e^{2v}g]) = 3B|_{x=0} = T_{e^{2v}g}$.

As a consequence we have

$$6V = \int_{X^4} (Q_4)_{e^{2v}g^+} + 2\int_{M^3} T_{e^{2v}g}$$
$$= \int_{X^4} \sigma_2(A_{e^{2v}g^+}).$$

Hence (M. Anderson)

$$8\pi^2\chi(X^4) = \frac{1}{4}\int_{X^4} |W|^2 dv_{\bar{g}} + \int_{X^4} \sigma_2(A_{\bar{g}})$$
$$= \frac{1}{4}\int_{X^4} |W|^2 dv_{\bar{g}} + 6V,$$

for $\bar{g} = e^{2v}g^+$ or any conformal compact $\bar{g}$.

Conformal Sphere Theorem:
(Chang-Gursky-Yang)
On (M^4, g) with $Y(M^4, g) > 0$. If

$$\int_{M^4} |W_g|^2 dv_g < 16\pi^2 \chi(M^4),$$

or equivalently

$$\int_{M^4} \sigma_2(A_g) dv_g > 4\pi^2 \chi(M^4)$$

then M^4 is diffeomorphic to S^4 or $\mathbb{R}^4$.

Note that on (M^4, g), with $Y(M^4, g) > 0$.

$$\int_{M^4} \sigma_2(A_g) dv_g \leq 16\pi^2$$

with equality if and only if M^4 is diffeomorphic to S^4.

Theorem: (Chang-Qing-Yang)
Suppose (X^4, M^3, g^+) is a conformal compact Einstein manifold, and $(M^3, [g])$ has positive Yamabe constant, then
(i) $V \leq \frac{4\pi^2}{3}$, with equality holds if and only if (X^4, g^+) is the hyperbolic space (H^4, g_H), and therefore (M^3, g) is the standard 3-sphere.
(ii) If

$$V > \frac{1}{3}(\frac{4\pi^2}{3}\chi(X)),$$

then X is homeomorphic to the 4-ball B^4 up to a finite cover.
(iii) If

$$V > \frac{1}{2}(\frac{4\pi^2}{3}\chi(X)),$$

then X is diffeomorphic to B^4 and M is diffeomorphic to S^3.

A crucial step in the proof of the theorem above is an earlier result:

Theorem: (Qing '02)
Suppose (X^{n+1}, M^n, g^+) is a conformal compact Einstein manifold, with $Y(M^n, [g])$ positive, then there is a positive eigenfunction u satisfying

$$-\Delta_{g^+} u = (n+1)u \text{ on } X^{n+1},$$

so that $(X^{n+1}, u^{-2}g^+)$ is a compact manifold with totally geodesic boundary and the scalar curvature is greater than or equal to $\frac{n+1}{n-1}R_g$, where $g \in [g]$ is the Yamabe metric.

PICTURE

Theorem:

(Chang-Qing-Yang, Epstein)

On conformally compact Einstein (X^{n+1}, M^n, g^+), when n is odd,

$$\int_{X^{n+1}} W_{n+1} dv_g + c_n V(X^{n+1}, g) = \chi(X^{n+1})$$

for some curvature invariant W_{n+1}, which is a sum of contractions of Weyl curvatures and/or its covariant derivatives in an Einstein metric.

Proof:

Use structure equation of Q_n; in particular, the result of Alexakis that

$$Q_n = a_n \text{Pfaffian} + J + \text{div}(T_n).$$

§5. Renormalized volume when n is even.

The renormalized volume can also be defined via the scattering matrix:

$$V(X^3, [g], g^+) = -\int_{M^2} \frac{d}{ds}|_{s=2} S(s) 1 dv_g, \text{ for } n = 2$$

$$V(X^5, [g], g^+) = -\int_{M^4} \frac{d}{ds}|_{s=4} S(s) 1 dv_g - \frac{1}{32 \cdot 36} \int_{M^4} R^2[g] dv_g, \text{ for } n = 4$$

$$V(X^{n+1}, [g], g^+) = -\int_{M^n} \frac{d}{ds}|_{s=n} S(s) 1 dv_g + \text{ correction terms, for } n \text{ even}$$

Definition: We call a functional $\mathcal{F}$ defined on (M^n, g) a conformal primitive of a curvature tensor $\mathcal{A}$ if

$$\frac{d}{d\alpha}|_{\alpha=0}\mathcal{F}[e^{2\alpha w}g] = -2c_{\frac{n}{2}}\int_M w\,\mathcal{A}_g dv_g.$$

Theorem: On (X^{n+1}, M^n, g^+), n even, the scattering term $\mathcal{S}(g, g^+) = \frac{d}{ds}|_{s=n}\mathcal{S}(s)1(g, g^+)$ is the conformal primitive of $(Q_n)_g$.

Corollary: (Henningson-Skenderis, Graham)
On (X^3, M^2, g^+), V is the conformal primitive of K, the Gaussian curvature.
On (X^5, M^4, g^+), V is the conformal primitive of $\frac{1}{16}\sigma_2$, where $\sigma_2 = \frac{1}{6}(R^2 - 3|Ric|^2)$.

• Qing established the rigidity result that any conformal compact Einstein manifold with conformal infinity the standard n-sphere is the hyperbolic $n+1$ space extending prior results of L. Andersson.

• X. Wang proved that on (X^{n+1}, M^n, g^+) with $\lambda_0(g^+) > n-1$, then $H_n(X, \mathbb{Z}) = 0$. In particular, the conformal infinity M is connected; thus extending an earlier result of Witten-Yau.

Given $(M^n, [g])$ in general, both the existence and uniqueness problem of a conformal compact Einstein manifold with $(M^n, [g])$ as conformal infinity remain open.