# Q-curvature on 4-manifolds

Sun-Yung Alice Chang
Princeton University

On $(M^n, g)$, compact Riemannian manifold
A metric $\bar{g}$ is conformal to $g$, if $\bar{g} = \rho g$ for some $\rho > 0$. Denote $\rho = e^{2w}$, and $g_w = e^{2w} g$.

An operator $A$ is a conformally covariant operator of bidegree $(a, b)$, if under $g_w = e^{2w} g$,

$$A_{g_w}(\phi) = e^{-bw} A(e^{aw}\phi) \quad \forall \phi \in \mathcal{C}^\infty(M^n) .$$

Examples:

1. when $n = 2$, $\Delta$ of bidegree $(0, 2)$.

Properties of $\Delta_g$ on $(M^2, g)$

a. $\Delta_{g_w} = e^{-2w} \Delta_g$.

b. $-\Delta_g w + K_g = K_{g_w} e^w$,

Gauss-Bonnet formula:

$$2\pi\chi(M) = \int_M K_g dv_g$$

2. On $(M^n, g)$, $n \geq 3$, denote

$$L_g = -c_n \Delta_g + R_g$$

where $c_n = \frac{4(n-1)}{n-2}$, $R_g$ the scalar curvature.
Under $\bar{g} = u^{\frac{4}{n-2}} g$,

$$L_{\bar{g}}(\phi) = u^{-\frac{n+2}{n-2}} L_g(u\phi),$$

for all $\phi \in C^\infty(M)$.
Yamabe equation

$$L_g(u) = R_{\bar{g}} u^{\frac{n+2}{n-2}}$$

Using the notion $g_w = e^{2w} g$,

$$L_{g_w} = e^{-\frac{n+2}{2}w} L_g(e^{\frac{n-2}{2}w} \cdot)$$

i.e. $L$ of bidegree $(\frac{n-2}{2}, \frac{n+2}{2})$.

3. When $n = 4$, Paneitz operator: (1983)

$$P\varphi \equiv \Delta^2\varphi + \delta[\left(\frac{2}{3}Rg - 2Ric\right) d\varphi]$$

where $\delta$ denotes the divergence, $d$ the deRham differential and $Ric$ the Ricci tensor.

For example:
On $(R^4, |dx|^2)$, $P = \Delta^2$,
On $(S^4, g_c)$, $P = \Delta^2 - 2\Delta$,
On $(M^4, g)$, $g$ Einstein, $P = (-\Delta) \circ (L)$.

P bidegree $(0, 4)$ on 4-manifolds, i.e.

$$P_{g_w}(\phi) = e^{-4\omega} P_g(\phi) \;\; \forall \phi \in \mathcal{C}^\infty(M^4)\,.$$

Properties of Paneitz operator on $(M^4, g)$

1. $P_{g_w} = e^{-4w} P_g$
2. $$P_g w + 2Q_g \ = \ 2Q_{g_w} e^{4w}$$

$$Q \ = \ \frac{1}{12}(-\Delta R + R^2 - 3|Ric|^2)$$

Gauss-Bonnet-Chern Formula:

$$4\pi^2 \chi(M^4) = \int (Q_g + \frac{1}{8}|W_g|^2) \ dv,$$

where $W$ denotes the Weyl tensor.

• $W_{g_w} = e^{-2w} W_g$ implies $|W_g|^2 dv_g = |W_{g_w}|^2 dv_{g_w}$ Thus $|W_g|^2 dv_g$ is a pointwise conformal invariant; and the curvature integral $\int Q_g dv_g$ is a conformal invariant.

3. (Paneitz, Branson)

$\exists$ 4-th order Conformal Paneitz operator $P_4^n$, on $(M^n, g)$

$$P_4^n = \Delta^2 + \delta\,(a_n R g + b_n \text{Ric})\, d + \frac{n-4}{2} Q_4^n,$$

for $\bar{g} = u^{\frac{4}{n-4}} g$:

$$P_4^n u \,=\, \bar{Q}_4^n u^{\frac{n+4}{n-4}}.$$

$P_4^n$ is conformal covariant of bidegree $(\frac{n-4}{2}, \frac{n+4}{2})$.

§ Q-curvature, PDE aspect:

$$(**) \quad P_g w + 2Q_g = 2Q_{g_w} e^{4w}$$

Theorem: (Gursky, Chang-Yang)
(i) If $\lambda_1(L_g) > 0$ and $\int Q_g dv_g > 0$ then $P_g \geq 0$ with $Ker P = \{\text{constants}\}$.
(ii) Under assumptions in (i), (**) can be solved with $Q_{g_w}$ given by a constant.

Remark: Yamabe constant

$$Y(M^n, g) \equiv \inf_w \frac{\int R_{g_w} dv_{g_w}}{vol(g_w)^{\frac{n-2}{n}}}$$

then $Y(M^n, g) > 0$ iff $\lambda_1(L_g) > 0$.

Above existence result based on Branson-Orsted Formula on $(M^4, g)$ for

$$F[w] = log\frac{det\, A_g}{det A_{g_w}},$$

for $A_g$ conformal covariant operators.

Chang-Yang: Existence of extremal metrics of $F$ over $g_w$:

$$\gamma_1 |W|^2 + \gamma_2 Q - \gamma_3 \Delta R = \bar{k} \cdot Vol^{-1}$$

for constants $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_2\gamma_3 > 0$ and $\bar{k}$.

Regularity: Chang-Gursky-Yang

§ Fully non-linear equations on $(M^n, g)$

$$Rm = W \ \oplus \ \frac{1}{n-2} A \bigcirc\!\!\!\!\wedge\, g,$$

where

$$A \ = \ Ric - \frac{R}{2(n-1)} g$$

$A$ is called the Schouten tensor.

Under $g_w = e^{2w} g$, $W_{g_w} = e^{-2w} W_g$.

$$A_{g_w} = A_g + (n-2)\{-\nabla^2 w + dw \otimes dw - \frac{|\nabla w|^2}{2} g\}.$$

Denote $\sigma_k(A_g)$ = k-th elementary function of eigenvalues of $A_g$.

Examples:

$$\begin{aligned}
\sigma_1(A_g) &= \sum_i \lambda_i = \frac{n-2}{2(n-1)} R_g, \\
\sigma_2(A_g) &= \sum_{i<j} \lambda_i \lambda_j \\
&= \frac{1}{2}(|Tr\ A_g|^2 - |A_g|^2) \\
&= \frac{n}{8(n-1)} R^2 - \frac{1}{2}|Ric|^2, \\
\sigma_n(A_g) &= det(A_g)
\end{aligned}$$

Equation of Monge-Ampere type:

$$\sigma_k(\nabla^2 u) = f > 0$$

Dirichlet problem for $u$ defined on convex domain $\Omega \subset \mathbb{R}^n$

• Existence of functions $u$ for which the right hand side is some positive function is guaranteed by the convexity of the domain.

Caffarelli-Nirenberg-Spruck
Krylov, Evans
Pogorolev, Cheng-Yau, Caffarelli

Comparison of the two equations:

Fully Non-linear PDE: for $n = 4$, $k = 2$,

$$\sigma_2(\nabla^2 u) = \frac{1}{2}[(\Delta u)^2 - |\nabla^2 u|^2]$$

While for $A_g$:

$$\begin{aligned}\sigma_2(A_{g_w})e^{4w} &= \sigma_2(A_g) + 2[(\Delta w)^2 - |\nabla^2 w|^2 \\ &+ (\nabla w, \nabla|\nabla w|^2) + \Delta w|\nabla w|^2)] \\ &+\text{lower order terms.}\end{aligned}$$

Regularity properties of $\sigma_k(A_{g_w})$ appears to be much better than that of $\sigma_k(\nabla^2 v)$; much more to be explored.

Geometric content of sign of $\sigma_k(A_g)$:
For $k = 1$, $\sigma_1(A_g) = c_n R_g$.

Some Algebraic fact:

- When $n = 3$ and $\sigma_2(A_g) > 0$, then either $R_g > 0$ and the sectional curvature of g is positive or $R_g < 0$ and the sectional curvature of g is negative on $M$.

- When $n = 4$ and $\sigma_2(A_g) > 0$, then either $R_g > 0$ and $Ric_g > 0$ on $M$ or $R_g < 0$ and $Ric_g < 0$ on $M$.

- For general $n$ and $\sigma_i(A_g) > 0$ $\forall 1 \leq i \leq k$ (i.e. $A_g \in \Gamma_k^+$) for some $k \geq \frac{n}{2}$, then $Ric_g > 0$.

Study of $\sigma_k(A_g) = constant$

For $k = 1$, Yamabe equation

For $k = 2$, denote

$$\mathcal{F}_2[g] = \int \sigma_2(A_g) dv_g.$$

Theorem: (Gursky-Viaclovsky)
For 3-manifolds, a metric $g$ with $\mathcal{F}_2[g] \geq 0$ is critical for the functional $\mathcal{F}_2$ restricted to class of metrics with volume one if and only if $g$ has constant sectional curvature.

For $k = 2$ on $(M^n, g)$, $n = 4$,

$$\sigma_2(A_g) = \frac{1}{6}R^2 - \frac{1}{2}|Ric|^2$$

$$Q_g = \frac{-1}{12}\Delta R_g + \frac{1}{2}\sigma_2(A_g).$$

Gauss-Bonnet-Chern formula

$$4\pi^2\chi(M) = \int_M (Q_g + \frac{1}{8}|W_g|^2) dv_g.$$

$$8\pi^2\chi(M) = \int_M (\sigma_2(A_g) + \frac{1}{4}|W_g|^2) dv_g.$$

Hence

$$\int_M \sigma_2(A_g) dv_g$$

is a conformal invariant.

Theorem: (Chang-Gursky-Yang)
On $(M^4, g)$, assume
(i) $Y(M^4, g) > 0$;
(ii) $\int \sigma_2(A_g) dv_g > 0$;
then $\exists\, w \in \mathcal{C}^\infty(M)$, with $\sigma_2(A_{g_w}) \equiv 1$.

Corollary: Under (i),(ii), on $(M^4, g)$, $\exists\, g_w = e^{2w} g$ with $Ric_{g_w} > 0$; hence $\pi_1(M^4)$ is finite.

Examples:

• Gauss-Bonnet-Chern and Signature formulae imply that

$$2\chi + 3\tau > 0$$

as well as

$$2\chi - 3\tau > 0$$

precisely the same conditions satisfied by the class of positive Einstein 4-manifolds.

• Results of Donaldson and Freedman $\longrightarrow$ the possible homeomorphic classes of simply connected 4-manifolds appear in the two lists:
In the non-spin case:

$$k(\mathbb{CP}^2)\# l(\bar{\mathbb{C}}\bar{\mathbb{P}}^2), \text{ where } k > l \text{ and } 4 + 5l > k;$$

In the spin case: $k(S^2 \times S^2)$.

Proof of Theorem:

Part I: existence part: Under (i) and (ii) solve

$$\sigma_2(A_{g_w}) = f, \qquad \text{for some } f > 0.$$

Part II: regularity part:
Deform f to constant by method of continuity and degree theory.

Part I: Difficulty is lack of ellipticity. Recall

$$Q_g = \frac{-1}{12}\Delta R_g + \frac{1}{2}\sigma_2(A_g),$$
$$P_g w + 2Q_g \;=\; 2Q_{g_w} e^{4w}$$

Solve

$$(*)_\delta : \qquad \sigma_2(A_g) = \frac{\delta}{4}\Delta_g R_g + f$$

Suitable $f$ is $c|W_g|^2$; condition (ii) $\Rightarrow c > 0$
$\delta = 1$ extremal of $F[w] = log\frac{det\ L_g}{det\ L_{g_w}}$
$\delta = \frac{2}{3}$, solution of $Q = \frac{c}{2}|W|^2$,
$\delta \to 0$, apriori estimate, using integral form of Pogorolev estimates

Analytically: Regularized equation:

$$(*)_\delta : \quad \delta(-\Delta)^2 w \;=\; \sigma_2(A_{g_w}) \;-\; c|W_{g_w}|^2.$$

• Gursky-Viaclovsky: A different proof of above theorem. Based on Harnack estimates of solutions of

$$A_g^t := Ric - \frac{tR}{2(n-1)} g$$

Start with t at $-\infty$ and let t tends to 1.

• Chang-Gursky-Yang, A. Li-Y. Li:
Uniqueness Theorem: If

$$\sigma_k(A_{g_w}) = 1 \text{ on } \mathbb{R}^n$$

for $g_w = e^{2w}dx$, then

$$w(x) = \log \frac{2\epsilon}{(\epsilon^2 + |x - x_0|^2)} + c_n$$

for some $\epsilon > 0$, and $x_0 \in \mathbb{R}^n$.

• P.F.Guan-Wang, A. Li-Y. Li: A priori estimates for solutions of $\sigma_k(A_{g_w}) = f > 0$ for the special case when $(M^n, g)$ is locally conformally flat.

Conformal Sphere Theorem:
(Chang-Gursky-Yang)

On $(M^4, g)$ with $Y(M^4, g) > 0$.
(a) If

$$\int_{M^4} |W_g|^2 dv_g < 16\pi^2 \chi(M^4),$$

then $M^4$ is diffeomorphic to $S^4$ or $\mathbb{R}P^4$.
(b) If

$$\int_{M^4} |W_g|^2 dv_g = 16\pi^2 \chi(M^4),$$

and $M^4$ not diffeomorphic to $S^4$ or $\mathbb{R}P^4$ then either
(1) $(M^4, g)$ is conformal to $(\mathbb{C}P^2, g_{FS})$, or
(2) $(M^4, g)$ is conformal to $((S^3 \times S^1)/\Gamma, g_{prod})$.

Gauss-Bonnet:

$$8\pi^2\chi(M^4) = \int(\sigma_2(A_g) + \frac{1}{4}|W_g|^2)\ dv_g,$$

$$(a) \leftrightarrow (a)' : \ \int_{M^4} \sigma_2(A_g) > \frac{1}{4}\int_{M^4} |W_g|^2 dv_g$$

$$(b) \leftrightarrow (b)' : \ \int_{M^4} \sigma_2(A_g) = \frac{1}{4}\int_{M^4} |W_g|^2 dv_g$$

Margerin: Weak Pinching:

$$WP \equiv \frac{|W|^2 + 2|E|^2}{R^2}$$

where $E$ denotes traceless Ricci.
$WP < \frac{1}{6}$ iff (a)": $\sigma_2 > \frac{1}{4}|W|^2$ and
$WP \equiv \frac{1}{6}$ iff (b)": $\sigma_2 = \frac{1}{4}|W|^2$.

Remarks:

- Margerin established previous Theorem under (a)" and (b)".
- Hamilton: assume curvature operator positive.

Proof of Theorem:

(a) In the conformal class, solve

$$\sigma_2(A_{g_w}) = \frac{1}{4}|W_{g_w}|^2 + c$$

with $c > 0$.

(b) and not (a) happens at minimal points of $\int_{M^4} |W|^2 dv$, thus the Bach tensor vanishes. Bach tensor:

$$B_{ij} = \nabla^k \nabla^l W_{kijl} + \frac{1}{2} R^{kl} W_{kijl}.$$

In this case we solve for

$$\sigma_2(A_{g_w}) = \frac{1-\epsilon}{4}|W_{g_w}|^2 + C_\epsilon$$

where $C_\epsilon \to 0$ as $\epsilon \to 0$. Degenerate elliptic happens at $x$ where $|W|^2(x) = 0$.

§ Boundary operator, Cohn-Vossen inequality

Suppose $(N^2, M^1, g)$ is a compact surface with boundary; Gauss-Bonnet formula

$$2\pi\chi(N) = \int_N K\, dv + \oint_M k\, d\sigma,$$

where $k$ is the geodesic curvature on $M$. Under conformal change of metric $g_w$ on $N$, we have

$$\frac{\partial}{\partial n} w + k = k_w e^w \text{ on M.}$$

• On $(N^2, M^1, g)$, we have $(\Delta_g, K_g)$, with the corresponding boundary operator and curvature $(\frac{\partial}{\partial n}, k)$.

• On $(N^{n+1}, M^n, g)$, $n \geq 2$, we have $(L_g, R_g)$ with the corresponding Robin boundary operator $B_g = \frac{\partial}{\partial n} + c_n H_g$ and the mean curvature $H_g$.

Theorem : (Cohn-Vossen, Huber)
Suppose $(M^2, g)$ a complete surface with $K_g$ integrable, then

$$\int_M K_g dv_g \leq 2\pi\chi(M).$$

Furthermore $M$ has a conformal compactification $\bar{M} = M \cup \{q_1, ..., q_l\}$ as a compact Riemann surface and

$$2\pi\chi(M) = \int_M K_g dv_g + \sum_{k=1}^{l} \nu_k,$$

where at each end $q_k$, take a conformal coordinate disk $\{|z| < r_0\}$ with $q_k$ at its center, then

$$\nu_k = \lim_{r\to 0} \frac{Length(\{|z| = r\})^2}{2Area(\{r < |z| < r_0\})}.$$

PICTURE

On a four manifold with boundary $(N^4, M^3, g)$; Chang-Qing Existence of boundary operator $P_3$ along with boundary curvature invariant $T$.

$$(P_3)_{g_w} = e^{-3w}(P_3)_g, \text{ on M and}$$
$$(P_3)_g w + T_g = T_{g_w} e^{3w} \text{on M.}$$

Special cases:

• On $(B^4, S^3, dx)$, where $B^4$ is the unit ball in $\mathbb{R}^4$, we have

$$P_4 = (-\Delta)^2,$$

$$P_3 = -\Big(\frac{1}{2}\,\frac{\partial}{\partial n}\,\Delta + \tilde{\Delta}\frac{\partial}{\partial n} + \tilde{\Delta}\Big)$$

and $T = 2$. where $\tilde{\Delta}$ the intrinsic boundary Laplacian on $M$.

The pair $(P_4, Q_4)$ together with $(P_3, T)$ satisfy:

Gauss-Bonnet-Chern formula:

$$8\pi^2\chi(N) = \int_N (\frac{1}{4}|W|^2 + Q_4)\ dv + \oint_M (\mathcal{L} + T)\ d\sigma.$$

where $\mathcal{L}$ is a third order boundary curvature (pointwise) conformal invariant.

Generalization of Cohn-Vossen Theorem to 4-manifold:

Theorem (Chang-Qing-Yang)
Suppose $(M^4, g)$ is a complete conformally flat manifold, satisfying the conditions:
(i) The scalar curvature $R_g$ is bounded between two positive constants and $|\nabla_g R_g|$ is also bounded;
(ii) The Ricci curvature is bounded below;
(iii) $\int_M |Q_g| dv_g < \infty$;
then
Assume $M$ is simply connected, it is conformally equivalent to $S^4 - \{q_1, ..., q_l\}$ and we have

$$4\pi^2 \chi(M) = \int_M Q_g \, dv_g + 4\pi^2 l \ ;$$

• Schoen-Yau: For $(M^n, g)$ simply connected, locally conformally flat, with $R_g \geq 0$,

$$(M^n, g) \hookrightarrow (S^n, e^{2w} g_c).$$

Say $M = S^n - \Lambda$, then Hausdorff dimension $\Lambda \leq \frac{n-2}{2}$.

PICTURE

Key estimates:
Assume the conformal metrics $e^{2w} g_c$ defined over domains $\Omega = S^4 - \Lambda$, then

$$e^{w(x)} \cong \frac{1}{d(x, \partial\Omega)}.$$

Generalizations:

• (Hao Fang) On $(N^4, g)$, with $Q$ integrable replaced by $\sigma_2(A_g)$ integrable; also on $(N^n, g)$ under further pinching conditions of curvatures.

• (Carron-Herzlich) On $(N^n, g)$ with conformal structures which are not necessarily locally comformally flat.

• (M. González, Chang-Hang-Yang) On $(M^n, g)$ locally conformally flat, assume $g \in \Gamma_k^+$. (i.e. $\sigma_i(A_g) > 0$ for all $i \leq k$.), then $(M, g) \hookrightarrow (S^n, e^{2w} g_c)$, Say $M = S^n - \Lambda$, then Hausdorff dimension $\Lambda \leq \frac{n-2k}{2}$

Corollary:(M. González, Izeki) A compact conformally flat manifold $(M^n, g)$ with $g \in \Gamma_k^+$ for $2k > n - 2$ is a quotient of Schottky group.

• The main open question in dimension four is the classification of conformal structures whose Schouten tensor belongs to the cone $\Gamma_2^+$.

• So far the results in dimension four relies on explicit form of $P_4, Q_4$ on $M^4$. To extend this theory to general dimensions, it will be important to find structural properties of the operators $P$ and the $Q$ curvatures.