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A Riemannian metric $g = \sum g_{ij} dx^i dx^j$ gives rise to measurement of angles between vectors, and a conformally equivalent metric $\bar{g} = \rho g$ for some positive function ρ gives the same angle measurements. Two spaces X, X' are said to be conformally equivalent if there is a map $T : X \to X'$ which preserves the angle measurements.

• We are interested to study conformal invariants, i.e. terms which are invariant under conformal change of metrics. This includes

• local or pointwise conformal invariants. Examples are curvature tensors, e.g. Weyl tensor W_g, which satisfies $W_{\bar{g}} = \rho^{-1} W_g$ and measures the deviation of the metric from a conformally flat metric.

• Global or integral conformal invariants. Examples are integral of curvature invariants, e.g. integral of Gaussian curvature over a Riemann surface.

• We are also interested in conformal covariant operators, i.e. operators which transform by simple rules under conformal change of metrics; such operators are usually closely associated with local conformal invariants. e.g.
• Δ_g on compact surface,
• The conformal Laplace operator $L_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$ on (M^n, g) for $n \geq 3$ where R_g is the scalar curvature.

Important aspects of the theory includes:

- Existence and construction of local conformal invariants:

E. Cartan's theory of differential invariants.
T. Thomas's theory of tractor calculus.
C. Fefferman and R. Graham introduced the ambient metric construction.

• Construction and properties of conformally covariant operators and their associated Q curvatures:

Paneitz introduced 4-th order operators.

Graham-Jenne-Mason-Sparling introduce the n-th order operator on n-manifolds (n even).

Branson relates the operators to Q-curvature.

Fefferman-Graham, Zworski relates the n-th order operators on n-manifolds to the scattering theory of conformally compact Einstein spaces.

Alexakis's result on the structure of Q curvatures

- Nonlinear PDE's associated with the conformally covariant operators:
Work on the Gauss curvature equation.
Work on the Yamabe equation
Work on the Q-curvature equation and the related fully nonlinear PDE's.

- Connection to spectral theory.

- Applications to 4-dimensional conformal geometry and higher dimensional Kleinian groups. An existence theorem for conformal metrics of positive Ricci curvature. A conformal sphere theorem.

- Conformally compact Einstein structures.

Review of the Riemann curvature tensor:

• The Riemann tensor $Rm = R_{ijkl}$ is defined in terms of a nonlinear expression involving up to two derivatives of the metric.

• The sectional curvature of the plane $v \wedge w$ is given by $K(v \wedge w) = \sum R_{ijkl} v^i v^k w^j w^l$ when v, w are orthonormal.

• Ricci curvature in the direction $v = v_1$ is given as a trace $Ric(v, v) = \sum_{i=2}^{n} K(v, v^i)$.

• The scalar curvature $R = \sum_{i=1}^{n} Ric(v_i, v_i)$.

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Decomposition of the curvature tensor:

The Riemann curvature tensor has the decomposition

$$Rm = W \bigoplus A \owedge g$$

where

$$A = \frac{1}{n-2}[R_{ij} - \frac{R}{2(n-1)} g_{ij}]$$

is called the Schouten tensor which is determined by the Ricci tensor; and $\owedge$ is the Nomizu-Kulkarni product of the symmetric two tensors.

• The Weyl tensor satisfies $W_{\bar{g}} = \rho^{-1} W_g$.

• The Ricci tensor controls the growth of volume of balls, and hence the topology of the underlying space.

§ Analytic aspects: A blow up sequence of functions

Sobolev Embedding Theorem:
For all $v \in C_0^\infty(\mathbb{R}^n)$, $n \geq 3$

$$(*) \quad \Lambda \, (\int_{\mathbb{R}^n} |v|^p dx)^{\frac{2}{p}} \; \leq \; \int_{\mathbb{R}^n} |\nabla v|^2 dx.$$

• We say that $W_0^{1.2}(\mathbb{R}^n)$ embeds into $L^p(\mathbb{R}^n)$.

• By a dilation of $v(x)$ to $v(\lambda x)$, we see p in (*) is $p = \frac{2n}{n-2}$.

The best constant Λ and the extremal functions v for (*): Assume $v(x) = v(|x|) = v(r)$,

$$\begin{cases} v'' + \frac{n-1}{r} v' + \Lambda \, v^{\frac{n+2}{n-2}} = 0, \\ v(0) = a, \;\; v'(0) = 0. \end{cases}$$

One solution is

$$\begin{cases} v(x) = (\frac{2}{1+|x|^2})^{\frac{n-2}{2}} \\ \Lambda = \frac{n(n-2)}{4}\omega_n^{2/n}, \end{cases}$$

where ω_n is the surface area of the unit sphere S^n. We then observe that the inequality is invariant under:

$$v \to v_\epsilon(x) = \epsilon^{\frac{2-n}{2}} v(\frac{x - x_0}{\epsilon}),$$

where $\epsilon > 0$ and x_0 is any point in $\mathbb{R}^n$. In other words, we have

$$v_\epsilon(x) = (\frac{2\epsilon}{\epsilon^2 + |x - x_0|^2})^{\frac{n-2}{2}}$$

are all extremals for the Sobolev embedding (*), we have the following remarkable theorem.

Theorem: (Bliss, Talenti, T. Aubin)
The best constant in the Sobolev inequality

$$(*) \quad \Lambda \, (\int_{\mathbb{R}^n} |v|^p dx)^{\frac{2}{p}} \; \leq \; \int_{\mathbb{R}^n} |\nabla v|^2 dx.$$

for $p = \frac{2n}{n-2}$ is $\Lambda = \frac{n(n-2)}{4}\omega_n^{2/n}$. It is only realized by the functions v_ϵ.

Properties of v_ϵ: (fix $x_0 = 0$, $\epsilon > 0$,)

$$v_\epsilon(x) = (\frac{2\epsilon}{\epsilon^2 + |x|^2})^{\frac{n-2}{2}}$$

(i) $v_\epsilon(0) = (\frac{2}{\epsilon})^{\frac{n-2}{2}} \longrightarrow \infty$ as $\epsilon \longrightarrow 0$,

(ii) $v_\epsilon(x) \longrightarrow 0$, for all $x \neq 0$, as $\epsilon \longrightarrow 0$,

(iii) $\int_{\mathbb{R}^n} |v_\epsilon(x)|^{\frac{2n}{n-2}} dx = \int_{\mathbb{R}^n} |v_1(x)|^{\frac{2n}{n-2}} dx$,

(iv) $\int_{\mathbb{R}^n} |\nabla v_\epsilon(x)|^2 dx = \int_{\mathbb{R}^n} |\nabla v_1(x)|^2 dx$.
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Thus v_ϵ is a sequence of functions

- bounded in $W^{1,2}(\mathbb{R}^n)$,

- The weak limit as $\epsilon \to 0$ is the zero function;

Hence it does not have a convergent subsequence in $L^{\frac{2n}{n-2}}$.

- The embedding of the Sobolev space $W^{1,2}(\mathbb{R}^n)$ into $L^{\frac{2n}{n-2}}$ is not compact. This lack of compactness due to the non-compact group of translations and dilations of $\mathbb{R}^n$ is the heart of the problem.

The Euler Lagrange equation for the extremal function satisfies:

$$-\Delta v = \frac{n(n-2)}{4} v^{\frac{n+2}{n-2}} \ on \ \mathbb{R}^n.$$

Thus functions v_ϵ above are solutions.

Theorem: (Caffarelli-Gidas-Spruck)
v_ϵ are the only positive solutions of above equation.

We conculde:

- All critcal points of the Sobolev embedding are minimal points.

- The positive solutions are unique up to dilations and translations.

§ Blow up sequence on the unit sphere S^n

Consider stereographic projection.

$$\pi : (S^n - \text{ north pole }) \to \mathbb{R}^n$$

$$\xi \overset{\pi}{\longmapsto} x(\xi)$$

Sending the north pole on S^n to ∞;
$\xi = (\xi_1, \xi_2, ..., \xi_{n+1})$ is a point in $S^n \subset \mathbb{R}^{n+1}$, $x = (x_1, x_2, ..., x_n)$, then $\xi_i = \frac{2x_i}{1+|x|^2}$ for $1 \le i \le n$; $\xi_{n+1} = \frac{|x|^2-1}{|x|^2+1}$.

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Suppose u is a smooth function defined on S^n, note that the Jacobian of π^{-1} as

$$J_{\pi^{-1}} = (\frac{2}{1+|x|^2})\, I$$

$$v(x) = u(\xi(x))(\frac{2}{1+|x|^2})^{\frac{n-2}{2}},$$

Sobolev inequality on S^n:

$$\Lambda\left(\int_{S^n} |u(\xi)|^{\frac{2n}{n-2}} d\sigma(\xi)\right)^{\frac{n-2}{2}} \leq \int_{S^n} |\nabla u(\xi)|^2 d\sigma(\xi) + \frac{n(n-2)}{4} \int_{S^n} |u(\xi)|^2 d\sigma(\xi),$$

where $d\sigma(\xi) = \left(\frac{2}{1+|x|^2}\right)^n$ is the standard area form on the unit sphere S^n.

The transformed function $u(\xi)$ satisfies:

$$-\Delta_g u + \frac{n(n-2)}{4} u = \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} \text{ on } S^n,$$

where $\Delta_g = \left(\frac{2}{1+|x|^2}\right)^2 \Delta_x$.

• Uniqueness Functions u_ϵ obtained from v_ϵ are the only positive solutions.

On (M^n, g), the conformal Laplacian L_g

$$L_g = -\Delta_g + c_n R_g$$

where $c_n = \frac{n-2}{4(n-1)}$, and R_g denotes the scalar curvature of the metric g.

Euler equation for Sobolev inequality on (M^n, g): Yamabe equation:

$$L_g u \ = \ c_n \ R_{\bar{g}} \ u^{\frac{n+2}{n-2}}.$$

where the conformal metric $\bar{g} = u^{\frac{4}{n-2}} g$ for some positive function u.

- Yamabe problem:

Given (M^n, g), find positive function u so that $R_{\bar{g}}$ a constant.
(Yamabe, Trudinger, Aubin, Schoen).

Variational method:
Find the extremals for the inequality:

$$\Lambda_g \left(\int_M |u|^{\frac{2n}{n-2}} dv_g\right)^{\frac{n-2}{2}} \le \int_M |\nabla_g u|^2 dv_g + c_n \int_M R_g |u|^2 dv_g,$$

for some constant $\Lambda_g \le \Lambda$.

• This constant Λ_g is called the Yamabe constant, and is conformally invariant.

A crucial ingredient in the proof: to establish some criteria for compactness of the minimizing sequence. That is to distinguish the manifold from the standard sphere by establishing $\Lambda_g < \Lambda_{g_c}$.

• In the solution by Aubin, the non-vanishing of the Weyl tensor in high dimensions plays this crucial role.

• Schoen uses the positive mass theorem to differentiate the conformal structure from the standard n-sphere.

Mass associated to a point p is defined as the finite part A in the asymptotic expansion of the Green's function of the conformal Laplacian with pole at p : in a geodesic coordinate system x whose origin is the given pole p, the Green's function G is the solution of the equation

$$L_g G = (n-2)\omega_{n-1}\delta_p \,.$$

Near the point p there is an expansion:

$$G(x) = |x|^{2-n} + A + O(|x|)\,.$$

• $A \geq 0$, $A = 0$ if and only if $(M^n, g) = (S^n, g_c)$.

§ Moser-Trudinger Inequality

Sobolev embedding Theroem:

$$W_0^{1,q}(D) \hookrightarrow L^q \text{ with } \frac{1}{p} = \frac{1}{q} - \frac{1}{n}.$$

When $q = 2$, $p = \frac{2n}{n-2}$ for $n \geq 3$.
When $q = 2$, $n = 2$ $0 < p < \infty$, but $p \neq \infty$.
Example: Take D to be the unit ball B in R^2, $w(x) = \log|\log(e - 1 + \frac{1}{|x|})|$.

Theorem: (Moser, Trudinger)
Suppose D is a smooth domain in $\mathbb{R}^2$, then there is a constant C, for all functions $w \in W_0^{1,2}(D)$ with $\int_D |\nabla w(x)|^2 dx \leq 1$, we have

$$\int_D e^{\alpha w^2}(x) dx \leq C|D|,$$

for any $\alpha \leq 4\pi$, with 4π being the best constant.

• Existence of extremal functions for Moser's inequality. (Carleson-Chang)

• Linearized form of the inequality is useful:

$$log\frac{1}{|D|}\int_D e^{2w}dx \leq \frac{1}{4\pi}\int_D |\nabla w|^2 dx.$$

• (W.Chen and C. Li)
Suppose w is in $C^2(\mathbb{R}^2)$, with $e^{2w} \in L^1(\mathbb{R}^2)$, and satisfies the equation

$$-\Delta w = e^{2w} \ \ on \ \ \mathbb{R}^2.$$

Then

$$w(x) = log\frac{2\epsilon}{\epsilon^2 + |x - x_0|^2}$$

for some $\epsilon > 0$ and some $x_0 \in \mathbb{R}^2$.

§ Gaussian curvature on compact surface

• Recall on (M^2, g) a compact surface, we have $\Delta = \Delta_g$ and the Gaussian curvature $K = K_g$.

• Under the conformal change $g_w = e^{2w} g$,

$$(1) \quad -\Delta_g w + K_g = K_w e^{2w} \ on \ M$$

K_w denotes the Gaussian curvature of (M, g_w).

• The Gauss-Bonnet Theorem:

$$2\pi \, \chi(M) = \int_M K_w \, dv_{g_w}$$

where $\chi(M)$ is the Euler characteristic of M.

• Uniformization Theorem to classify compact closed surfaces can be viewed as finding solutions with $K_w \equiv -1$, 0, or 1 according to the sign of $\int K dv_g$.

$$(1) \quad -\Delta_g w + K_g = K_w e^{2w} \text{ on } M$$

Variational Functional:

$$J[w] = \int_M |\nabla w|^2 dv_g + 2\int_M K_g w dv_g \\ - (\int_M K_g dv_g) \log \frac{\int_M dv_{g_w}}{\int_M dv_g}.$$

Nirenberg problem: Which functions can be the Gaussian curvature function K_w, in particular on (S^2, g_c).

• Kazdan-Warner

On $S^2 = \{(\xi_1, \xi_2, \xi_3) | \sum_{i=1}^3 \xi_i^2 = 1\}$, there is an obstruction for the problem:

$$\int_{S^2} \nabla K_w \cdot \nabla \xi \; e^{2w} dv_{g_c} = 0.$$

Theorem: (Moser)
Any positive C^2 even function f (i.e. $f(\xi) = f(-\xi)$ for all $\xi \in S^2$) can be a Gaussian curvature function on (S^2, g_c).

Theorem: (Onofri; T.Aubin) $J[w] \geq 0$
and $J[w] = 0$ precisely for conformal factors w of the form $e^{2w}g = T^*g$ where T is a Mobius transformation of the 2-sphere.

Leray-Schauder degree theory for (1):
(Chang-Yang, Chang-Gursky-Yang)
(C.C. Chen and C.S. Lin)

Assume f is a Morse function satisfying the (non-degenerate condition) $\Delta f(\xi) \neq 0$ at the critical points ξ of f,

$$\text{degree} = \sum_{\nabla f(q)=0, \Delta f(q)<0} (-1)^{ind(q)} - 1.$$

§ Geometric content of the functional $J[w]$

Polyakov-Ray-Singer Formula

On (M^2, g)

$$J[w] = 12\pi \ log \ (\frac{det(-\Delta_g)}{det(-\Delta_{g_w})})$$

where the determinant of the Laplacian $det(-\Delta_g)$ is defined by Ray-Singer as :

$$\log \det(-\Delta_g) := -\zeta'(0).$$

Definition

On compact Riemannian manifold (M^n, g), consider eigenvalue of $-\Delta_g$

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$$

and the zeta function

$$\zeta(s) := \sum_{\lambda_k \neq 0} \lambda_k^{-s},$$

Formal differentiation leads to

$$\zeta'(s) = \sum_{\lambda_k \neq 0} -(\log \lambda_k)\lambda_k^{-s}, \text{ i.e.}$$

$$\zeta'(0) = -\sum_{\lambda_k \neq 0} \log \lambda_k = -\log \prod_{k=1}^{\infty} \lambda_k.$$

Apply Mellin transform for all $x > 0$,

$$x^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-xt} t^{s-1}\, dt.$$

We can rewrite $\zeta(s)$ in terms of the Gamma function:

$$\begin{aligned}\zeta(s) &= \frac{1}{\Gamma(s)} \int_0^\infty \sum_{j=1}^{\infty} e^{-\lambda_j t} t^{s-1}\, dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty (Z(t) - 1) t^{s-1}\, dt,\end{aligned}$$

where $Z(t)$ denotes the Heat kernel. The existence of $\zeta'(0)$ can be justified via Weyl's asymptotic formula of the heat kernel.

• Onofri's inequality is equivalent to the statement $det(-\Delta_{g_c})$ is maximal among all metrics g on S^2.

• Osgood-Phillips-Sarnak independently derived Onofri's inequality and established the C^∞ compactness of isospectral metrics on compact surfaces.

• Chang-Yang, Brooks-Perry-Peterson: Partial results for isospectral compactness for 3-manifolds.

• Okikiolu: Among all metrics with the same volume as the standard metric on the 3-sphere, the standard canonical metric is a local maximum for the functional $det(-\Delta_g)$.