REGULARITY OF HARMONIC MAPS

Sun-Yung A. Chang, Lihe Wang and Paul C. Yang

Princeton University and UCLA, UCLA and University of Iowa, USC

Abstract. We present an elementary argument of the regularity of weak harmonic maps of a surface into the spheres, as well as the partial regularity of stationary harmonic maps of a higher dimensional domain into the spheres. The arguement does not make use of the structure of Hardy spaces.

§0. Introduction and Statement of Results

In a recent work we have made some progress in understanding the regularity theory of biharmonic maps. Since the techniques that we use are based on a simplified treatment of the regularity theory for harmonic map, we present here the argument for regularity of harmonic maps as an introduction to our work for the more complicated situation of biharmonic maps. To orient the reader, we briefly recall the basic references to the subject. For harmonic maps of surfaces, Morrey ([M]), Schoen ([S]), and Helein ([H1], [H2]) provided the classic regularity results. In higher dimensions, the corresponding regularity results are due to Hildebrandt-Kaul-Widman ([HKW]), Schoen-Uhlenbeck ([SU]), Evans ([E]) and Bethuel([B]).

First we will present an elementary proof of Helein's [H1] regularity theory for weakly harmonic maps from compact surface to spheres. Our proof is more elementary because it does not rely on the structure theory of the Hardy spaces ([CMLS]). We will first derive the continuity for harmonic maps of surfaces, we will then indicate how our method can be adapted to study stationary harmonic maps when the dimension of the manifold of the domain is greater than two, this again simplifies an earlier result of Evans [E]. At the end of the paper we will also give an easy proof of the $C^{1,\alpha}$ regularity of harmonic maps once the solution is continuous.

We would like to remark that, regularity results of Helein [H1] and Evans [E] have also been extended to arbitrary target manifolds (in [H2] and [B], cf also the excellent book of Helein [H3] for a complete treatment of the theory). At this moment, it is not clear how to extend our method to treat the case of general target manifolds, we hope to extend this type of argument to cover the general case.

Research supported in part by NSF grant DMS-9706864

Research supported in part by NSF grant DMS-9801374 and a Sloan Foundation Fellowship

Research supported in part by NSF grant DMS-9706507

§1 Weakly Harmonic Maps

Suppose M^m, N^n are Riemannian manifolds we assume N^n to be isometrically embedded in Euclidean space R^k and let $u: M \to N$ be a smooth map. Denote its differential by $du: TM \to TN$. In local coordinates $\{x^i\}$ on M and ambient Euclidean coordinates $\{u^\alpha\}$, the Riemannian metric g on M is represented by $g = g_{ij}dx^i dx^j$; and we denote du by the matrix $\left(\frac{\partial u^\alpha}{\partial x^i}\right)$.

Definition 1.1. The energy density of u at $x \in M$ is defined by

$$e(u) = \frac{1}{2}|du|^2 = \frac{1}{2}\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\alpha}{\partial x^j}g^{ij}$$

where g^{ij} is the inverse matrix of g_{ij}, i.e. $g^{ij}g_{jk} = \delta^i_k$.

Definition 1.2. The energy of u is defined by

$$E(u) = \int e(u)\, d\mathrm{Vol}$$

where

$$d\mathrm{Vol} = \sqrt{\det(g_{ij})}dx^1 \dots dx^m \ .$$

Definition 1.3. A map $u: M \to N$ a.e. and $du \in L^2(M, \mathbb{R}^k)$ is harmonic if it is a critical point of the energy in the sense of calculus of variations, i.e., if for each smooth deformation of u_t such that $u_0 = u$, we have

$$\frac{d}{dt}E(u_t)\Big|_{t=0} = 0 \ .$$

Theorem 1.4. *(Helein [H1], [H2]) Any harmonic map from a surface is Hölder continuous.*

In the following we shall give a simple proof of above theorem in the case the target manifold is the standard sphere.

Let us recall the following standard estimates from the linear theory. Denote by B_1 the unit ball in $\mathbb{R}^n$. Denote by $⨍_{B_1} f$ the average integral of f over B_1.

Lemma 1.5. *Suppose u is a scalar weak solution of*

$$\begin{cases} div(A(x)du) & = div(F) = \sum_{i=1}^m \frac{\partial F^i}{\partial x^i} \text{ on } B_1, \\ u & = 0 \text{ on } \partial B_1 \end{cases}$$

with $\lambda I \le A(x) \le \Lambda I$ and that $A(x)$ is Hölder continuous in $\bar{B}_1$, then for any $1 < q < \infty$ there is a constant C depending only on q, the dimension m, two elliptic constants λ, Λ and $\|A\|_{C^\alpha(B_1)}$ such that

$$\|du\|_{L^q(B_1)} \le C\|F\|_{L^q(B_1)}.$$

Theorem 1.6. *Any harmonic map from a two dimensional disk to sphere S^n is Hölder continuous.*

Proof. We first fix some $1 < q < 2$ and denote $p = \frac{2q}{2-q}$. We want to show that if $E_1(u) = \frac{1}{2}\int_{B_1} |\nabla u|^2$ is small enough, then for some fixed $s < 1/2$ we have

$$(1.1) \qquad (⨍_{B_s} |u - A_1|^p)^{\frac{1}{p}} \le \frac{1}{2}(⨍_{B_1} |u - A_0|^p)^{\frac{1}{p}},$$

for some constant vectors A_0, A_1 satisfying

$$(1.2) \qquad |A_0 - A_1|^p \le C ⨍_{B_1} |u - A_0|^p.$$

Then a rescaling yields,

$$(1.3) \qquad (⨍_{B_{s^k}} |u - A_k|^p)^{\frac{1}{p}} \le \frac{1}{2}(⨍_{B_{s^{k-1}}} |u - A_{k-1}|^p)^{\frac{1}{p}},$$

with

$$(1.4) \qquad |A_k - A_{k-1}|^p \le C ⨍_{B_{s^{k-1}}} |u - A_{k-1}|^p.$$

It then follows from (1.3) and (1.4) that

$$(1.5) \qquad |A_k - A_{k-1}| \le C_p(\frac{1}{2})^k(⨍_{B_1} |u - A_0|^p)^{1/p} \le C_p(\frac{1}{2})^k E_1[u]$$

for some constant C_p depending only on C and p. Thus the sequence A_k converges exponentially to a vector A, with

$$(1.6) \qquad |A_k - A| \le C_p(\frac{1}{2})^k E_1[u].$$

We have also from (1.3)

$$(1.7) \qquad (⨍_{B_{s^k}} |u - A_k|^p)^{\frac{1}{p}} \le (\frac{1}{2})^k(⨍_{B_1} |u - A_0|^p)^{\frac{1}{p}}.$$

The Hölder continuity of u follows from the decay estimates (1.6) and (1.7).

Let us recall the equation of harmonic map to a sphere $u = (u^1, \ldots, u^{n+1})$ as

$$(1.8) \qquad -\Delta u^\alpha = u^\alpha |du|^2.$$

An important observation of Helein is to rewrite this equation when the target space is S^n:

$$(1.9) \qquad -\Delta u^\alpha = u^\alpha |du|^2 = \sum_{\beta=1}^{n+1}\sum_{k=1}^{m}(u^\alpha u_k^\beta - u^\beta u_k^\alpha)u_k^\beta.$$

Let us assume $\int_{B_1} |\nabla u|^2 \le \epsilon$ for some ϵ small. Let $\frac{1}{2} \le r \le 1$ be such that $\int_{\partial B_r} |u - A_0|^p \le 8 \int_{B_1} |u - A_0|^p$ for some constant vector A_0; it turns out the proof below works for any constant vector A_0. Let h be the harmonic function such that $u = h$ on ∂B_r. We have

$$|dh(x)|^p \le C \int_{\partial B_r} |h - A_0|^p \le C(\int_{B_1} |u - A_0|^p) \tag{1.10}$$

for $|x| \le \frac{r}{4}$. From (1.9) we have,

$$-\Delta(u^\alpha - h^\alpha) = \sum_{\beta=1}^{n+1} \sum_{k=1}^{m} \partial_k[(u^\alpha u_k^\beta - u^\beta u_k^\alpha)(u^\beta - A_0^\beta)]. \tag{1.11}$$

denote $E_r = \frac{1}{2} \int_{B_r} |Du|^2$ as before, then for any fixed $1 < q < 2$, $p = \frac{2q}{2-q}$ we have from Lemma 1.2 and the Hölder inequality that

$$\begin{aligned}
\int_{B_r} |d(u-h)|^q \le & C \int_{B_r} \sum_{\alpha,\beta,k} [(u^\alpha u_k^\beta - u^\beta u_k^\alpha)(u^\beta - A_0^\beta)]^q \\
\le & C \sum_{\alpha,\beta,k} (\int_{B_r} [(u^\alpha u_k^\beta - u^\beta u_k^\alpha)]^2)^{\frac{q}{2}} (\int_{B_r} |u^\beta - A_0^\beta|^p)^{\frac{2-q}{2}} \\
\le & C E_r^{q/2} (\int_{B_r} \sum_\beta |u^\beta - A_0^\beta|^p)^{\frac{2-q}{2}} \\
\le & C E_r^{q/2} (\int_{B_1} |u - A_0|^p)^{\frac{q}{p}}. \qquad (1.12)
\end{aligned}$$

Now for any $s < \frac{1}{4} < r$ we have, via Sobolev inequality,

$$\begin{aligned}
\frac{1}{s^2} \int_{B_s} |u - h(0)|^p dx \le & \frac{2^{p-1}}{s^2} \int_{B_s} |u - h|^p dx + \frac{2^{p-1}}{s^2} \int_{B_s} |h - h(0)|^p dx \\
& \le \frac{2^{p-1}}{s^2} (\int_{B_r} |u - h|^p) + \frac{2^{p-1}}{s^2} \int_{B_s} |h - h(0)|^p dx \\
& \le C \frac{2^{p-1}}{s^2} (\int_{B_r} |d(u-h)|^q dx)^{\frac{p}{q}} + 2^{p-1} C s^p \sup_{B_{\frac{1}{4}}} |dh|^p \\
& \le \frac{C E_r^{p/2}}{s^2} (\int_{B_1} |u - A_0|^p) + C s^p (\int_{B_1} |u - A_0|^p). \qquad (1.13)
\end{aligned}$$

Now, taking s small such that the second term is less than

$$\frac{1}{2^{p+2}} \int_{B_1} |u - A_0|^p$$

and then taking $E = E_1$ small such that the first term is less than

$$\frac{1}{2^{p+2}} \fint_{B_1} |u - A_0|^p$$

so we have by letting $A_1 = h(0)$

$$\fint_{B_s} |u - A_1|^p dx \leq \frac{1}{2^p} \fint_{B_1} |u - A_0|^p.$$

This proves (1.1). We then observe that for $A_1 = h(0) = \fint_{\partial B_r} h = \fint_{\partial B_r} u$,

$$\begin{aligned} |A_1 - A_0| = |h(0) - A_0| &= | \fint_{\partial B_r} (u - A_0)| \\ &\leq (\fint_{\partial B_r} |u - A_0|^p)^{\frac{1}{p}} \\ &\leq C(\fint_{B_1} |u - A_0|^p)^{\frac{1}{p}}. \end{aligned}$$

Thus (1.2) is satisfied for any constant vector A_0. From (1.1) and (1.2), we conclude that u is Hölder continuous by the arguments in (1.3) to (1.7).

Remark The main point in the above argument is that the right hand side of (1.11) is the divergence of a quadratic oscillation of u while the left hand side of (1.11) is the divergence of the linear oscillation of u. The indices in L^p norms in (1.13) are from different sources. The index p on the left is from the Sobolev embedding and the index p on the right hand side is from the Hölder inequality (1.12). These two indices match only in dimension 2. In our later work on biharmonic maps we again observe a similar matching of indices in dimension 4.

We will use BMO semi-norm to study the case when these indices do not match in Section 3, where we will have $\frac{mq}{m-q}$ on the left hand side and $\frac{2q}{2-q}$ on the right of the inequality (1.13).

§2. Stationary Harmonic Maps

In this section we modify the argument in section 1 to give an alternative proof of Evans' theorem that the singular set of a stationary harmonic map from an m-manifold to a sphere has $m - 2$ Hausdorff measure zero.

Definition 2.1. (Stationary Harmonic Maps) Let u be a harmonic map from a manifold M (possibly with boundary) to another compact manifold N. We say that u is stationary if

$$\frac{d}{dt} E(u(\varphi(t))) = 0 \text{ at } t = 0,$$

where $\varphi(t) : M \to M$ is a smooth one parameter family of diffeomorphism such that $\varphi(0) = identity$.

Definition 2.2. A function f defined on a smooth domain $\Omega \subset R^m$ is in BMO, i.e. function of bounded mean oscillation, if

$$(2.1) \qquad ||u||_{\mathrm{BMO}(\Omega)} = \sup_B ⨍_B |u - u_B| dx < +\infty \ ,$$

where $u_B = ⨍_B u$, the supremum is taken over all balls with $B \subset \Omega$.

We will use the following classical result of John and Nirenberg([JN], cf also Chapter IV of Stein [St]).

Theorem 2.3. *For any $1 < p < \infty$, there exists an C_p (which depends only on p and dimension m) such that if $u \in BMO(\Omega)$, then*

$$\frac{1}{C_p} ||u||_{\mathrm{BMO}(\Omega)} \leq \sup_B (⨍_B |u - u_B|^p)^{\frac{1}{p}} \leq C_p ||u||_{\mathrm{BMO}(\Omega)}$$

where the supremum is taken over all balls with $B \subset \Omega$.

The following monotonicity formula is proved by Schoen-Uhlenbeck ([SU]) in the case of minimizing harmonic maps, and by P. Price ([P]) for stationary harmonic maps.

Theorem 2.4. *A stationary harmonic map from the m-dimensional Euclidean disk satisfies the following monotonicity formula: The scaling invariant energy*

$$E(r) = r^2 ⨍_{B_r} |du|^2 = ⨍_{B_1} |du(rx)|^2 \ .$$

is monotonically increasing in r for all concentric balls $B_r = B(x, r) \subset B(x, 1) = B_1$.

The following regularity result is due to Evans ([E]):

Theorem 2.5. *A stationary harmonic map from the m-dimensional Euclidean disk to sphere S^n is Hölder continuous except a set of m-2 dimensional Hausdorff measure zero.*

Our proof below is patterned after the two dimensional argument. In higher dimensions the exponents resulting from the two inequalities (1.12) and (1.13) do not match so we show instead that the BMO norm of the map decays. In fact we have to show the decay of the map in every scale. The monotonicity formula makes the control in every scale possible.

We will show that when $E_1(u)$ is small enough, we can choose s small, so that

$$(2.2) \qquad ||u||_{BMO(B_s)} \leq \frac{1}{2} ||u||_{BMO(B_1)},$$

Then an iteration of (2.2) yields,

$$(2.3) \qquad ||u||_{BMO(B_{s^k})} \leq \frac{1}{2^k} ||u||_{BMO(B_1)}.$$

We start with the equation of a harmonic map $u = (u^1, \ldots, u^{n+1})$ with target S^n as in (1.8). We also assume $\int_{B_1} |du|^2 \leq \epsilon$ for some ϵ small here B_1 is the unit ball in R^m. The monotonicity formula asserts $E_r = r^{2-m} \int_{B_r(0)} |du|^2 \leq 2\epsilon$ for all $0 < r < 1$. A little reflection will show that for any $B_r(x) \subset B_{\frac{1}{2}}(0)$ we also have for some constant C_m:

$$r^2 \fint_{B_r(x)} |du|^2 \leq C_m \epsilon. \tag{2.4}$$

Now fix r and x_0 with $B_r(x_0) \subset B_{\frac{1}{2}}(0)$; we abbreviate B_r for $B_r(x_0)$; and choose $r/2 \leq r_1 \leq r$ so that

$$\int_{\partial B_{r_1}} |u - A_0| \leq 8 \int_{B_r} |u - A_0| \tag{2.5}$$

here $A_0 = \fint_{B_r} u$.

Let h be the harmonic function such that $u = h$ on ∂B_{r_1}. We have from (2.5):

$$|dh(x)|^p \leq C_p r_1^{-p} \fint_{B_r} |u - A_0|^p$$

for $|x| \leq \frac{r_1}{4}$. Then from equation (1.11) we have with $E_r = r^2 \fint_{B_r} |du|^2$, that exactly as (1.12) that

$$\fint_{B_{r_1}} |d(u-h)|^q \leq C E_{r_1}^{q/2} \left(\fint_{B_r} |u - A_0|^{\frac{2q}{2-q}} \right)^{\frac{2-q}{2}} r_1{}^{-q}. \tag{2.6}$$

Now taking $p = \frac{mq}{m-q}$, for any $s < r_1/4$, we have, via Sobolev inequality and the John-Nirenberg's inequality as in Theorem 2,1.

$$\begin{aligned}
\frac{1}{s^m} \int_{B_s} |u - h(x_0)|^p dx &\leq \frac{2^{p-1}}{s^m} \int_{B_{r_1}} |u - h|^p dx + \frac{2^{p-1}}{s^m} \int_{B_s} |h - h(x_0)|^p dx \\
&\leq C \frac{2^{p-1} r_1^{p+m}}{s^m} \left(\fint_{B_{r_1}} |d(u-h)|^q \right)^{\frac{p}{q}} + \frac{2^{p-1}}{s^m} \int_{B_s} |h - h(x_0)|^p dx \\
&\leq C \frac{2^{p-1} r_1^{p+m}}{s^m} \left(\fint_{B_{r_1}} |d(u-h)|^q dx \right)^{\frac{p}{q}} + C 2^{p-1} s^p \sup_{B_{r_1/4}} |dh|^p \\
&\leq C \frac{2^{p-1} r_1^m}{s^m} E_{r_1}^{p/2} \left(\fint_{B_{r_1}} |u - A_0|^{\frac{2q}{2-q}} \right)^{\frac{p(2-q)}{2q}} + C \frac{s^p}{r_1^p} \left(\fint_{B_r} |u - A_0|^p \right) \\
&\leq C \frac{r_1^m}{s^m} E_{r_1}^{p/2} (||u||_{BMO(B_r)})^p + C \frac{s^p}{r_1^p} (||u||_{BMO(B_r)})^p \ . \qquad (2.7)
\end{aligned}$$

Now, taking $s/r_1 = r_0$ small such that the second term is less than

$$\frac{1}{2^{2p+1}} (\|u\|_{BMO(B_r)})^p$$

and then taking E small depending only on r_0 such that the first term is less than

$$\frac{1}{2^{2p+1}}(\|u\|_{BMO(B_r)})^p.$$

Thus,

$$\fint_{B_s}|u-h(x_0)|^p \le \frac{1}{2^{2p}}\|u\|^p_{BMO(B_r)}. \tag{2.8}$$

Since

$$\fint_{B_s}|u-\fint_{B_s}u|^p \le 2^p\fint_{B_s}|u-h(x_0)|^p,$$

we have

$$\fint_{B_s}|u-\fint_{B_s}u|^p \le \frac{1}{2^{p}}\|u\|^p_{BMO(B_r)}, \tag{2.9}$$

where $s=s_0=r_1r_0$, $B_s=B_s(x_0)\subset B_r(x_0)=B_r$. But for concentric balls B_s for $s\le s_0$, a rescaling argument shows that (2.9) still holds.

We then vary r and x_0 with $B_r=B_r(x_0)\subset B_{\frac{1}{2}}(0)$. As B_r runs over all balls in $B_{\frac{1}{2}}(0)$, B_s runs over all balls in $B_{s_0}(0)$, From this we conclude that (2.2) holds. This finishes the proof of the theorem.

§3 $C^{1,\gamma}$ Regularity

We will prove the following regularity result in this section.

Theorem 3.1. *If u is a weakly harmonic map from M^m to N^n for $m\ge 2$ and u is continuous in an open set in M^m, then u is locally smooth there.*

We use compactness argument to establish $C^{1,\gamma}$ regularity. Then the higher order regularity of the map follows from the harmonic equation and standard elliptic theory.

Suppose the u is a weak solution of

$$\Delta u^\alpha = f^\alpha(x,\nabla u), \tag{3.1}$$

We would assume also that

$$|f^\alpha(x,P)|\le A(1+|P|^2), \tag{3.2}$$

for some constant A.

Our iteration scheme depends on a finer structure of the right hand side of (3.2) defined as

$$|f^\alpha(x,P)|\le A(1+\mu|P|^2), \tag{3.3}$$

for some constant A sufficiently small and with some constant μ suitably small.

Remarks 1. We introduce (3.3) in order to trace the different decay rate of the constant term and the quadratic term in f^α.
2. Once we know u is continuous and satisfies a system of equations (3.1) and (3.2), then we can define

$$u_1(x) = \frac{u(rx) - u(0)}{c(r)}, \tag{3.4}$$

where $c(r) = r + \sup_{B_1} |u(rx) - u(0)|$. Then u_1 satisfies equation of same type as that of (3.1) with

$$f_1^\alpha(x, P) = \frac{r^2}{c(r)} f^\alpha(rx, \frac{c(r)}{r} P).$$

Thus f_1 satisfies equation of type (3.3) with

$$|f_1(x, P)| \le A_1(1 + \mu|P|^2)$$

with $A_1 = c(r)^{\frac{1}{2}} A$ and $\mu = c(r)^{\frac{1}{2}}$, both can be made arbitrarily small when r is sufficiently small. Also, the $C^{1,\gamma}$ estimates of u follows from that of u_1. Thus we may assume without loss of generality that the harmonic map in our proof of Theorem 3.1 below satisfies both (3.1) and (3.3). This will be the only place in our proof where we will use the assumption that u is continuous.
3. We also remark that the $C^{1,\gamma}$ regularity theory holds if we replace the Δu^α by any elliptic systems, in particular it covers the case when Δ is the Laplacian operator on a manifold.

Lemma 3.2. *Suppose w is a solution of (3.1) satisfying (3.3) in B_1 with $\mu|w| \le C_1$, and $AC_1 < 1$ then*

$$\int_{B_{\frac{1}{2}}} |\nabla w|^2 \le C \int_{B_1} (w^2 \ + 1). \tag{3.5}$$

where $C = C(C_1, A)$.

Proof. This is a Caccioppoli type inequality. Choose a smooth cut-off function η with $\eta(x) = \eta(|x|)$ of compact support in B_1 and $\eta = 1$ on $B_{\frac{1}{2}}$. Multiplying the equation (3.1) by $\eta^2 w$ and integrate by parts, we have

$$\int \eta^2 w(-\Delta w) = \int \eta^2 w f \ . \tag{3.6}$$

As before, we have

$$\begin{aligned}\int \eta^2 |\nabla w|^2 &= \int \eta^2 w f - 2 \int \eta \nabla \eta \cdot w \nabla w \\ &\le \int A|w|\eta^2 + \int AC_1 \eta^2 |\nabla w|^2 + \epsilon \int \eta^2 |\nabla w|^2 + \frac{1}{\epsilon} |\nabla \eta|^2 w^2 \\ &\le \int A\eta^2(|w|^2 + 1) + \frac{1}{\epsilon}|\nabla \eta|^2 w^2 + (C_1 A + \epsilon) \int |\eta|^2 |\nabla w|^2 \ .\end{aligned}$$

Now taking ϵ small so that $C_1 A + \epsilon < 1$, we obtain the inequality immediately.

Lemma 3.3. *For any given $\epsilon > 0$, there is a $\delta > 0$ such that if $A \le \delta$ then for any solution w of (3.1) with (3.3) satisfying $\mu|w| \le C_1$ and $\fint_{B_1} |w|^2 \le 1$, there exists some harmonic function h defined in $B_{\frac{1}{2}}$ which approximates w in the sense that:*

$$\int_{B_{\frac{1}{2}}} |w - h|^2 \le \epsilon^2. \tag{3.7}$$

Proof. We prove the result by contradiction. Suppose there exist some $\epsilon > 0$, and sequences of w_n and f_n satisfying

$$-\Delta w_n = f_n(x, \nabla w_n)$$
$$\fint_{B_1} w_n^2 \le 1 \ ,$$
$$|f_n| \le \frac{1}{n}(1 + \mu|\nabla w_n|^2).$$

But for any harmonic function v in $B_{\frac{1}{2}}$, we have

$$\int |w_n - v|^2 dx \ge \epsilon^2 \ .$$

We then have by (3.6) in the previous lemma,

$$\int_{B_{\frac{3}{4}}} |\nabla w_n|^2 \le C \ .$$

Hence $\{w_n\}$ has a convergent subsequence, which we still denoted as w_n, such that

$$w_n \to w \quad \text{weakly in} \quad H^1 \quad \text{and} \quad w_n \to w \quad \text{strongly in} \quad L^2(B_{\frac{3}{4}}) \ .$$

We will show that w itself is harmonic in $B_{\frac{3}{4}}$, which is a contradiction. Actually, for any test function $\varphi \in C_0^\infty(B_{\frac{3}{4}})$,

$$\int \nabla\varphi \nabla w_n = \int \varphi f_n dx \ .$$

Thus if we let n go to infinity, we have

$$\int \nabla\varphi \nabla w = 0 \ .$$

Hence w is actually harmonic. We have thus proved the assertion (3.7).

Corollary 3.4. *For any $0 < \gamma < 1$ and C_1, there exist some $\epsilon > 0$ and $0 < \lambda < \frac{1}{2}$ such if $A \leq \epsilon$ and w is a solution of (3.1) and (3.3) with $\mu|w| \leq C_1$ and $⨍_{B_1} |w|^2 \leq 1$, then there is a linear function $l(x) = Bx + C$ such that*

$$⨍_{B_\lambda} |w - l|^2 \leq \lambda^{2(1+\gamma)} \tag{3.8}$$

and $|B| + |C| \leq C_0$ a universal constant.

Proof. Let h be the harmonic function such that

$$\int_{B_{\frac{1}{2}}} |w - h|^2 \leq \epsilon^2, \tag{3.9}$$

as in the statement of Lemma 3.2. By the triangle inequality, we have

$$\int_{B_{\frac{1}{2}}} |h|^2 \leq 2 \int_{B_1} (|w|^2 + 1) \leq C_0 \ .$$

Since h is harmonic, we have

$$|\nabla^2 h(x)|^2 \leq C \int_{B_{\frac{1}{2}}} |h|^2 \quad \leq CC_0 \qquad \text{for} \quad |x| \leq \frac{1}{4} \ , \tag{3.10}$$

for some constant C=C(m). If we now denote $l(x)$ be the first order Taylor polynomial of h at 0, we have for $\lambda \leq \frac{1}{4}$,

$$\begin{aligned} ⨍_{B_\lambda} |w - l|^2 dx &\leq 2 ⨍_{B_\lambda} |w - h|^2 dx + 2 ⨍_{B_\lambda} |h - l|^2 dx \\ &\leq C\lambda^{-m}\epsilon^2 + C\lambda^4. \end{aligned} \tag{3.11}$$

by (3.9) and (3.10), where $C = C(m)$.
Thus we can take λ small so that the second term in (3.11) is less than

$$\frac{1}{2}\lambda^{2(1+\gamma)} \ ,$$

and then taking ϵ sufficiently small so that the first term in (3.11) is less than

$$\frac{1}{2}\lambda^{2(1+\gamma)},$$

we obtain (3.8) in the corollary.

We now prove Theorem 3.1.

Proof of Theorem 3.1. We first assert that by Remark(2) at the beginning of this section, we may assume that the continuous harmonic map we have also satisfy that $|u(x)| \leq 1$ for $x \in B_1$ and system (3.1) with conditions (3.3).

We will prove by induction the following statement (*):

There exist some constants C_0, $0<\lambda<\frac{1}{2}$ and $\epsilon>0$ such that for $|u|\le 1$ and u is a solution of (3.1) with (3.3) with $A\le\epsilon$, there are linear functions $l_k(x)=B_k\cdot x+C_k$ such that

$$\fint_{B_{\lambda^k}}|u-l_k|^2\le\lambda^{2(1+\gamma)k} \tag{3.12}$$

and the constants B_k and C_k satisfy

$$\lambda^k|B_k-B_{k+1}|+|C_k-C_{k+1}|\le C_0\lambda^{(\gamma+1)k}, \tag{3.13}$$

with C_0 a universal constant.

Assuming statement (*), we can then argue as in the proof of Theorem 1.3 of section 1 that both B_k and C_k converge in an exponentially decay rate to B and C respectively, and u can be approximated by a linear function $l(x)=B\cdot x+C$ satisfying

$$\fint_{B_{\lambda^k}}|u-l|^2\le 2\lambda^{2(1+\gamma)k} \tag{3.14}$$

for each k. Thus u is in $C^{1,\gamma}$ by the usual Morrey estimates.

To prove the statement (*), we notice that the statement for $k=0$ follows from our assumption on u and the statement $k=1$ follows from Corollary 3.3. Thus we need only to establish the inductive step.

We assume statement (*) for up to k. We first observe that from (3.13) that $|B_k|=|\nabla l_k|\le\frac{C_0}{1-\lambda^\gamma}\le\frac{C_0}{1-2^{-\gamma}}$, similarly $|C_k|\le\frac{C_0}{1-2^{-\gamma}}$.

Define $w(x)=\frac{(u-l_k)(\lambda^k x)}{\lambda^{(1+\gamma)k}}$. Then w satisfies,

$$\Delta w^\alpha=f_1^\alpha(x,\nabla w),$$

where

$$f_1^\alpha(x,P)=\lambda^{(1-\gamma)k}f^\alpha(\lambda^k x,\lambda^{\gamma k}P+\nabla l_k).$$

By our assumption that u satisfies (3.2), we have

$$|f_1(x,P)|\le A\lambda^{(1-\gamma)k}(1+2|\nabla l_k|^2)+2\lambda^{(1+\gamma)k}A|P|^2. \tag{3.15}$$

We verify that w satisfies the conditions of Corollary 3.3. To see this, we have w satisfy (3.1) and (3.15), with

$$2|w|\lambda^{(1+\gamma)k}=2(u-l_k)(\lambda^k x)\le 2(1+|\nabla l_k|+|C_k|)\le 4(1+\frac{C_0}{1-2^{-\gamma}})$$

and

$$\fint_{B_1}|w|^2=\lambda^{-2(1+\gamma)k}\fint_{B_{\lambda^k}}|u-l_k|^2\le 1.$$

Thus we may apply Corollary 3.3 to conclude that there exist some linear function $l(x) = Bx + C$ with $|B| + |C| \leq C_0$ some universal bound, and

$$\fint_{B_\lambda} |w - l|^2 \leq \lambda^{2(1+\gamma)}. \tag{3.16}$$

We now define $l_{k+1} = l_k + \lambda^{(1+\gamma)k} l(\frac{x}{\lambda^k})$ and check that inequalities (3.12) and (3.13) in the statement (*) for $k+1$ follow directly from (3.16) and the bounds on $l(x)$. We have thus finished the proof of the inductive step in statement (*) and hence the proof of Theorem 3.1.

References

[B] F. Bethuel; "On the singular set of stationary harmonic maps", Manuscripta Math., 78 (1993), pp 417-443.

[CLMS] R. Coifman, P.L. Lions, Y. Meyer and S. Semmes; " Compensated compactness and Hardy spaces", J. Math. Pures Appl. 72 (1993), pp 247-286

[E] C. Evans; "Partial regularity for stationary harmonic maps into spheres", Arch. Rat. Mech. Anal. 116 (1991), 101-163.

[H1] F. Helein; "Regularity des applications faiblement harmoniques entre une surface et une sphere", C.R. Acad. Sci. Paris 311 (1990), 519-524.

[H2] F. Helein; " Regularite des applications faiblement harmoniques entre unne surface et une variete riemannienne", C. R. Acad. Sci. Paris 312 (1991), pp 591-596.

[H3] F. Helein; "Harmonic maps, conservation laws and moving frames", Diderot Publishers, Paris, 1997.

[HKW] S. Hildebrandt, H. Kaul and K.O. Widman: "An existence theorem for harmonic mappings of Riemanian manifolds", Acta Math. 138(1970), 550-569.

[JN] F. John and L. Nirenberg; "On functions of bounded mean oscillations", Comm. Pure and Applied Math. 14 (1961), 415-426.

[M] C. B. Jr. Morrey; "The problem of Plateau on a Riemannian manifold", Ann. of Math. 49 (1948), 807-851.

[P] P. Price; " A Monotonicity formula for Yang-Mills fields", Manuscripta Math. 43 (1983), pp 131-166.

[SU] R.Schoen and K. Uhlenbeck; " A regularity theory for harmonic maps", JDG 17 (1982), pp 307-335.

[S] R. Schoen, " Analytic aspects of the harmonic map problem", Math. Sci. Res. Inst. Publ. 2, Springer, Berlin, 1984.

[St] E.Stein; " Harmonic Analysis; real variable methods, orthogonality and oscillatory integrals ", Princeton University press, 1993.

Address of the authors:

Sun-Yung A. Chang, Dept. of Math., UCLA, Los Angeles, CA 90095-1555.

Lihe Wang, Dept. of Math., UCLA, Los Angeles, CA 90095-1555 and Dept. of Math., Univ. of Iowa, Iowa City, Iowa 52240.

Paul C. Yang, Dept. of Math., USC, Los Angles, CA 90089-1113.