COMPACTIFICATION OF A CLASS OF CONFORMALLY FLAT 4-MANIFOLD

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Abstract. In this paper we generalize Huber's result on complete surfaces of finite total curvature. For complete locally conformally flat 4-manifolds of positive scalar curvature with Q curvature integrable, where Q is a variant of the Chern-Gauss-Bonnet integrand; we first derive the Cohn-Vossen inequality. We then establish finiteness of the topology. This allows us to provide conformal compactification of such manifolds.

S0. Introduction

In this paper we study the ends of locally conformally flat 4-dimensional manifolds. Recall in the theory of complete surfaces of finite total curvature, Cohn-Vossen [CV] showed that if the Gauss curvature of a complete analytic metric is absolutely integrable then

$$\int K dA \leq 2\pi\chi, \tag{0.1}$$

where χ is the Euler number of the surface. Huber ([H]) extended this inequality to metrics with weaker regularity and proved that such surface can be conformally compactified by adjoining a finite number of points. For such surfaces the deficit in formula (0.1) has an interpretation as an isoperimetric constant. One may represent each end conformally as $R^2 \setminus D$ for some compact set D and consider the following isoperimetric ratio:

$$\nu = \lim_{r\to\infty} \frac{L^2(r)}{4\pi A(r)}$$

where $L(r)$ is the length of the boundary circle $\partial B_r = \{|x| = r\}$, and $A(r)$ the area of the annular region $B(r) \setminus D$. For a fairly large class of complete surfaces called surfaces with normal metrics, Finn ([F]) showed that,

$$\chi(M) - \frac{1}{2\pi}\int_M K dv_M = \sum \nu_j, \tag{0.2}$$

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where the sum is taken over each end of M.

In dimension four, Greene-Wu ([GW]) obtained a generaization of (0.1) to complete manifolds of positive sectional curvature outside a compact set. In a previous paper we considered a generaliztion of (0.2) in $\mathbb{R}^4$. In order to describe the result we briefly recall the fourth order curvature invariant Q. For conformal geometry in dimension four, the Paneitz operator

$$P = \Delta^2 + \delta(\frac{2}{3}Rg - 2Ric)d,$$

where δ denote the divergence, and d the differential, and R is the scalar curvature and Ric the Ricci tensor, plays the same role as the Laplacian in dimension two (cf. [P] [BCY] [CY], for example). Under conformal change of metric $g = e^{2w}g_0$, the Paneitz operator transforms by $P_g = e^{-4w}P_{g_0}$. The Paneitz operator defines a natural fourth order curvature invariant Q: for the conformal metric $g = e^{2w}g_0$,

$$P_{g_0}w + 2Q_{g_0} = 2Q_g e^{4w} \tag{0.3}$$

where

$$Q = \frac{1}{12}(-\Delta R + \frac{1}{4}R^2 - 3|E|^2) \tag{0.4}$$

and E is the traceless Ricci tensor. The Q curvature invariant is related to the Chern-Gauss-Bonnet integral in dimension four:

$$\chi(M) = \frac{1}{4\pi^2}\int_M (\frac{|W|^2}{8} + Q)dV \tag{0.5}$$

where W is the Weyl tensor and M is a compact, closed 4-manifold. We note here the conformal invariance of the integrand: $|W|^2 dV$ remains the same when the metric g undergoes a conformal change $g' = e^{2w}g$. More generally, when the manifold has a boundary, Chang and Qing [CQ] has defined a boundary operator P_3 and its associated boundary curvature invariant T:

$$P_3 w + T_{g_0} = T_g e^{3w}, \tag{0.6}$$

where

$$T = -\frac{1}{12}\partial_N R + \frac{1}{6}RH - R_{anbn}L_{ab} + \frac{1}{9}H^3 - \frac{1}{3}trL^3 + \frac{1}{3}\tilde{\Delta}H,$$

H is the mean curvature of the boundary, L_{ab} denotes the second fundamental form of the boundary, ∂_N denotes the unit inward normal derivative, $\tilde{\Delta}$ denotes the boundary Laplacian. Then the Chern-Gauss-Bonnet integral is supplemented by

$$\chi(M) = \frac{1}{4\pi^2}\int_M (\frac{|W|^2}{8} + Q)dV + \frac{1}{4\pi^2}\int_{\partial M}(L_4 + L_5 + T)d\sigma, \tag{0.7}$$

where

$$L_4 = -\frac{1}{3}R_{ijij}L_{aa} + R_{aNaN}L_{bb} - R_{aNbN}L_{ab} + R_{acbc}L_{ab}$$

and

$$L_5 = -\frac{2}{9}L_{aa}L_{bb}L_{cc} + L_{aa}L_{bc}L_{bc} - L_{ab}L_{bc}L_{ca}$$

where $i, j = 1, 2, 3, 4,\ a, b, c = 1, 2, 3,$ and N is the inward normal direction. In analogy with the Weyl term, $(L_4 + L_5)d\sigma$ is a pointwise conformal invariant.

Theorem [CQY]. *Suppose that $e^{2w}|dx|^2$ on R^4 is a complete metric with its Q-curvature absolutely integrable, and suppose that its scalar curvature is nonnegative at infinity. Then*

$$1 - \frac{1}{4\pi^2}\int_{R^4} Qe^{4w}dx = \lim_{r\to\infty}\frac{(vol(\partial B_r(0)))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\, vol(B_r(0))} \geq 0. \tag{0.8}$$

In this paper we extend this result to more general situations. First we localize arguments in ([CQY]) to an end corresponding to a puncture and obtain:

Theorem 1. *Suppose $(\mathbb{R}^4 \setminus B, e^{2w}|dx|^2)$ is a complete conformal metric with non-negative scalar curvature at infinity. If in addition*

$$\int |Q|dV < \infty,$$

then

$$\lim_{r\to\infty}\frac{(vol(\partial B_r))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\, vol(B_r\setminus B)} = \frac{1}{4\pi^2}\int_{\partial B} Te^{3w} - \frac{1}{4\pi^2}\int_{\mathbb{R}^4\setminus B} Qe^{4w}dx.$$

This formula makes it possible to extend the basic result (0.8) to allow the domain to have a finite number of punctures.

Corollary. *Suppose that (M, g) is a complete 4-manifold with only finitely many conformally flat simple ends. And suppose that the scalar curvature is nonnegative at each end, and the Q curvature is absolutely integrable. Then*

$$\chi(M) - \frac{1}{32\pi^2}\int_M \{|W|^2 + 8Q\}dv_M = \sum_{i=1}^{k}\nu_i,$$

where in the inverted coordinates centered at each end,

$$\nu_i = \lim_{r\to\infty}\frac{(vol(\partial B_r))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\, vol(B_r\setminus B)}.$$

We refer to definition 1.1 in section 1 for the definition of conformally flat simple end. We then use the Chern-Gauss-Bonnet formula to derive a compactification criteria in analogy with Huber's two dimensional result.

Theorem 2. *Let $(\Omega \subset S^4, g = e^{2w}g_0)$ be a complete conformal metric satisfying*
a) The scalar curvature is bounded between two positive constants, and $|\nabla_g R|$ is bounded,
(b) The Ricci curvature of the metric g has a lower bound,
(c) the Paneitz curvature is absolutely integrable, i.e.

$$\int_\Omega |Q|dv_g < \infty.$$

then $\Omega = S^4 \setminus \{p_1, ..., p_k\}$.

An essential ingredient in the above finiteness result is to view the boundary integral as measuring the growth of volume. The finiteness of the Q integral implies a control on the growth of volume, which can only accommodate the growth of a finite number of ends. As a consequence we can classify solutions of the equation $Q = constant$ in the following

Corollary. *If* $(\Omega \subset S^4, e^{2w} g_0)$ *is a complete conformal metric satisfying conditions (a) (b) (c) of Theorem 2 and in addition Q is constant , then there are only two possibilities:*

1. $(\Omega \subset S^4, e^{2w} g_0) = (S^4, g_0)$ *corresponding to* $Q = 3$;
2. $(\Omega \subset S^4, e^{2w} g_0) = (\mathbb{R}^4 \setminus \{0\}, \frac{|dx|^2}{|x|^2})$ *corresponding to* $Q = 0$.

To provide a more general context for this result, we recall that in a study of locally conformally flat structures with positive scalar curvature, Schoen-Yau ([SY]) proved that the holonomy cover of such manifolds can be conformally embedded as domains into spheres with boundary of small Hausdorff dimension. Thus our compactification criteria applies immediately to simply connected manifolds for which the conditions (a) (b) and (c) hold. In fact, since the argument localizes to each end, we can dispense with simple connectivity assumption provided we are willing to assume the natural notion of geometric finiteness in Kleinian groups. For convenience we will follow the terminology of ([Ra]). Recall that a discrete group of conformal transformations of the sphere also acts as hyperbolic isometrics on the interior of the sphere. There is a notion of geometric finiteness of a Kleinian group that we will state more precisely in section three. Thus we prove

Theorem 3. *Suppose* (M^4, g) *is a locally conformally flat complete 4-manifold satisfying conditions (a), (b), (c) and (d) the holonomy representation of the fundamental group* Γ *is a geometrically finite Kleinian group without torsion; then* $M = \bar{M} \setminus \{p_1, ..., p_k\}$ *where* $\bar{M}$ *is a compact locally conformally flat 4-manifold.*

We outline the rest of the paper. In section one we derive the generalized Chern-Gauss-Bonnet formula of Theorem 1 and its extensions to more general situations when we only require the ends be locally conformally flat. In section two we study the simply connected case and prove Theorem 2 and its corollary. Finally in section three we extend the argument to the non-simply connected situation and prove the compactification result in Theorem 3.

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§1. **Chern-Gauss-Bonnet Integral**

In this section, we will prove the generalized Chern-Gauss-Bonnet formula for complete 4-manifolds with only a finite number of conformally flat simple ends.

More precisely, we will establish Theorem 1.2 below, which generalizes Finn's result [F] in two dimension. We first define manifolds with conformally flat simple ends.

Definition 1.1. *Suppose that (M, g) is a complete noncompact 4-manifold such that*

$$M = N \bigcup \{ \bigcup_{i=1}^{k} E_i \}$$

where (N, g) is a compact Riemannian manifold with boundary

$$\partial N = \bigcup_{i=1}^{k} \partial E_i,$$

and each E_i is a conformally flat simple end of M; that is:

$$(E_i, g) = (R^4 \setminus B, e^{2w_i}|dx|^2),$$

for some function w_i, where B is the unit ball in R^4. Then we say (M, g) is a complete 4-manifold with a finite number of conformally flat simple ends.

There are many examples of complete 4-manifolds with only finite number of conformally flat simple ends. For example those constructed by Schoen [S] and Mazzeo-Pacard [MP] on the 4-sphere with a finite number of points deleted and having a complete, conformally flat metric with constant scalar curvature. Another large set of examples is given by Theorem 3.1 in this paper–they include all locally conformally flat metrics satisfying conditions (a), (b), (c) and (d) in the statement of Theorem 3.1. In this section, we will prove the following result:

Theorem 1.2. *Suppose that (M, g) is a complete 4-manifold with finite number of conformally flat simple ends. And suppose that*
(a) The scalar curvature is non-negative at infinity at each end.
(c) Its Q curvature is integrable; that is

$$\int_M |Q| dv_M < \infty.$$

Then

$$\chi(M) - \frac{1}{32\pi^2} \int_M \{|W|^2 + 8Q\} dv_M = \sum_{i=1}^{k} \nu_i, \tag{1.1}$$

where

$$\nu_i = \lim_{r \to \infty} \frac{(\int_{\partial B_r(0)} e^{3w_i} d\sigma(x))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}} \int_{B_r(0) \setminus B} e^{4w_i} dx}. \tag{1.2}$$

Corollary 1.3. *Under the assumptions of Theorem 1.2 we have in particular*

$$\chi(M) \geq \frac{1}{32\pi^2}\int_M (|W|^2 + 8Q)dv_M. \tag{1.3}$$

Combining the result in Corollary 1.3 and Theorem 3.1 in section 3, we have the following result.

Corollary 1.4. *Suppose (M,g) is a locally conformally flat 4-manifolds which satisfies conditions (a),(b),(c) and (d) as in the statement of Theorem 3.1; then (M,g) is conformally equivalent to $(\bar{M}\setminus\{p_i\}_{i=1}^k, e^{2w}|dx|^2)$ where $\bar{M}$ is a compact locally conformally flat 4-manifold and conclusions (1.1), (1.2) hold.*

The proof we shall present below is a modification and sharpened version of the proof in [CQY]. Thus we will sometimes refer to [CQY] for some analytic details.

There are three main steps in the proof. First we will establish the theorem (in Proposition 1.6) for manifold with a simple end which has an axis of symmetry. We then establish the theorem for metric with a "normal" end (as defined in definition 1.7 below) in Proposition 1.11 by comparing the metric with its averaged metric which is rotationally symmetric; finally we prove in Proposition 1.12 that under the assumptions (a) and (c) of Theorem 1.2, the metric is "normal".

To start with, consider a Riemannian manifold $E = (R^4\setminus B, e^{2w}|dx|^2)$, where B is the unit ball centered at the origin in R^4, with w a radial function on $R^4\setminus B$. We assume that the Paneitz curvature Q is absolutely integrable on E, i.e.$\int_{R^4\setminus B}|Q|e^{4w}dx < \infty$. In cylindrical coordinates $|x| = r = e^t$, we have

$$\left(\frac{\partial^2}{\partial t^2} - 2\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t}\right)v = 2Qe^{4v}, \quad 0 \leq t < \infty, \tag{1.4}$$

where $v = w + t$.

For convenience, we denote $2Qe^{4v}$ by F. Equation (1.4) is equivalent to the following ODE

$$v'''' - 4v'' = F, \quad 0 \leq t < \infty \tag{1.5}$$

with initial conditions $v(0), v'(0), v''(0), v'''(0)$ given, where $\int_0^\infty |F|dt < \infty$. By the method of variation of coefficients we first obtain one particular solution f to (1.5) as follows: denote $f'' = C(t)e^{-2t}$, then $C(t)$ satisfies

$$C''(t) - 4C'(t) = F(t)e^{2t},$$

or equivalently

$$(C'(t)e^{4t})' = F(t)e^{-2t}.$$

Thus we can solve for $C(t)$ as:

$$\begin{aligned} C(t) &= -\int_{-\infty}^t e^{4x}\int_x^\infty F(y)e^{-2y}dydx \\ &= -\frac{1}{4}e^{4t}\int_t^\infty F(x)e^{-2x}dx - \frac{1}{4}\int_{-\infty}^t F(x)e^{2x}dx. \end{aligned} \tag{1.6}$$

Therefore

$$f''(t) = -\frac{1}{4}e^{2t}\int_t^\infty F(x)e^{-2x}dx - \frac{1}{4}e^{-2t}\int_0^t F(x)e^{2x}dx, \tag{1.7}$$

and

$$\begin{aligned} f'(t) = \frac{1}{8}\{e^{-2t}\int_0^t F(x)e^{2x}dx - e^{2t}\int_t^\infty F(x)e^{-2x}dx \\ + \int_t^\infty F(x)dx - \int_0^t F(x)dx\}. \end{aligned} \tag{1.8}$$

One may determine f by choosing $f(0) = 0$. In general

$$v(t) = c_0 + c_1 t + c_2 e^{-2t} + c_3 e^{2t} + f(t) \tag{1.9}$$

for some constants c_0, c_1, c_2, c_3 depending on the given initial data of v.

The following simple facts are proved in [CQY-Lemma 2.1-2.5].

Lemma 1.5.

$$\lim_{t\to\infty} e^{-2t}\int_0^t F(x)e^{2x}dx = 0. \tag{1.10}$$

$$\begin{aligned} &\lim_{t\to\infty} f'(t) = -\frac{1}{8}\int_{-\infty}^\infty F(x)dx \\ &\lim_{t\to\infty} f''(t) = 0 \\ &\lim_{t\to\infty} f'''(t) = 0. \end{aligned} \tag{1.11}$$

Suppose, in addition, that either the scalar curvature is nonnegative at infinity or $v''(t) = O(1)$ as $t \to \infty$. Then $c_3 = 0$.

Thus we may conclude,

Proposition 1.6. *Suppose that w is a radial function on $R^4 \setminus B$ and $e^{2w}|dx|^2$ is a metric complete at infinity with $\int_{R^4\setminus B}|Q|e^{4w} < \infty$ and its scalar curvature nonnegative at infinity. Then*

$$\lim_{t\to\infty} v'(t) = \frac{1}{4\pi^2}\int_{\partial B} Te^{3w} - \frac{1}{8\pi^2}\int_{R^4\setminus B} 2Qe^{4w} \geq 0. \tag{1.12}$$

Moreover, we have

$$\lim_{r\to\infty}\frac{(vol(\partial B_r))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\,vol(B_r\setminus B)} = \frac{1}{4\pi^2}\int_{\partial B} Te^{3w} - \frac{1}{8\pi^2}\int_{R^4\setminus B} 2Qe^{4w}. \tag{1.13}$$

Proof. Under the assumptions, we have

$$v(t) = c_0 + c_1 t + c_2 e^{-2t} + f(t). \tag{1.14}$$

Thus, by Lemma 1.5, we have

$$\lim_{t\to\infty} v'''(t) = \lim_{t\to\infty} f'''(t) = 0.$$

On the other hand, the Chern-Gauss-Bonnet formula says:

$$\frac{1}{8\pi^2}\int_{0\le t\le s} 2Qe^{4v} + \frac{1}{4\pi^2}\int_{t=s} Te^{3v} - \frac{1}{4\pi^2}\int_{t=0} Te^{3v} = 0, \tag{1.15}$$

where, as defined in [CQ, remark 3.1], in the special case of the standard $S^3\times[0,s]$, we have

$$Te^{3v} = P_3 v = -\frac{1}{2}v''' + 2v'. \tag{1.16}$$

Therefore

$$-\frac{1}{8\pi^2}\int_{0\le t\le s} 2Qe^{4v} = \frac{1}{4}(-v'''(s) + 4v'(s)) - \frac{1}{4\pi^2}\int_{t=0} Te^{3v}. \tag{1.17}$$

Thus,

$$\lim_{t\to\infty} v'(t) = \frac{1}{4\pi^2}\int_{\partial B} Te^{3w} - \frac{1}{8\pi^2}\int_{R^4\setminus B} 2Qe^{4w}.$$

From the completeness of $e^{2w}g_0$ at infinity, we then conclude that $\lim_{t\to\infty} v'(t)\ge 0$. To see (1.13), we write

$$V_3(r) = \mathrm{vol}(\partial B_r) = \int_{\partial B_r} e^{3w(r)}d\sigma,$$

and

$$V_4(r) = \mathrm{vol}(B_r\setminus B) = \int_1^r\int_{\partial B_s} e^{4w(s)}d\sigma ds.$$

If the limit $\lim_{t\to\infty} v'(t)$ is strictly positive, then both $V_4(r)$ and $V_3(r)$ tend to infinity as r tends to infinity; then by L'Hospital's rule we have:

$$\begin{aligned}\lim_{r\to\infty}\frac{V_3^{\frac{4}{3}}}{V_4} &= \lim_{r\to\infty}\frac{4|\partial B|^{\frac{4}{3}}r^3e^{4w}(rw'+1)}{|\partial B|r^3e^{4w}}\\ &= 4|\partial B|^{\frac{1}{3}}\lim_{t\to\infty} v'(t),\end{aligned}$$

thus (1.13) holds. On the other hand, if $\lim_{t\to\infty} v'(t) = 0$ and $\lim_{r\to\infty} V_4(r)$ is finite, then $\lim_{t\to\infty} e^{4v} = 0$. Hence $\lim_{r\to\infty} V_3(r) = 0$. (1.13) still holds. We have thus finished the proof of the proposition.

We now restrict our attention to the study of a general conformally flat simple end, namely, $E = (R^4\setminus B, e^{2w}|dx|^2)$ where w is a smooth function on $R^4\setminus B$. Following Finn [F] we define the notion of a normal metric. We will show that, for a complete normal metric on $R^4\setminus B$, Proposition 1.6 still holds.

Suppose that $e^{2w}|dx|^2$ is a metric on $R^4\setminus B$ with its Paneitz curvature Q absolutely integrable on $(R^4\setminus B, e^{2w}g_0)$, i.e. condition (c) holds.

Definition 1.7. *A conformal metric $e^{2w}|dx|^2$ satisfying condition (c) is said to be normal on $E = (R^4 \setminus B, e^{2w}g_0)$ if*

$$w(x) = \frac{1}{8\pi^2}\int_{R^4\setminus B} \log\frac{|y|}{|x-y|} 2Q(y)e^{4w(y)}dy + \alpha\log|x| + h(x) \tag{1.18}$$

where α is some constant and $h(\frac{x}{|x|^2})$ is some biharmonic function on B.

Observe that in general if we call the integral in (1.18) the potential of $\Delta^2 w$, it differs from w by a biharmonic function. Thus the normal condition is meant to control the growth of the difference between w and the potential of $\Delta^2 w$.

We will show later in this section that a fairly large class of metrics on E are normal in particular those satisfies condition (c) and the condition that the scalar curvature is nonnegative at infinity We compare a normal metric to its logarithmic average over spheres. The following is a key technical lemma comparing the growth of $V_3(r)$ and $V_4(r)$ of a normal metric to those of its averaged metric. Since Lemma 1.8 below can be proved following the same outline as the corresponding lemma: Lemma 3.1 in [CQY], we will skip its proof here.

Lemma 1.8. *Suppose that the metric $e^{2w}|dx|^2$ on $R^4\setminus B$ is a normal metric. Then*

$$V_3(r) = |\partial B_r(0)|e^{3\bar{w}(r)}\cdot e^{o(1)} \tag{1.19}$$

and

$$\frac{d}{dr}V_4(r) = |\partial B_r(0)|e^{4\bar{w}(r)}\cdot e^{o(1)}, \tag{1.20}$$

where $\bar{w}(r) = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} w(y)d\sigma(y)$, and $o(1)\to 0$ as $|x|\to\infty$.

Next we discuss the metric $e^{2\bar{w}}|dx|^2$ on $R^4\setminus B$, where $\bar{w}(r) = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} w d\sigma$. For convenience, we will use the cylindrical coordinates again. Denote by $\bar{E} = (S^3\times[0,\infty), e^{2\bar{v}}g_c) = (R^4\setminus B, e^{2\bar{w}}|dx|^2)$ (where g_c is the standard metric of the cylinder $S^3\times R^1$), then, $\bar{v}$ satisfies

$$\bar{v}'''' - 4\bar{v}'' = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} 2Qe^{4w}d\sigma(y)e^{4t} = F(t), \quad 0\le t<\infty$$

with

$$\int_0^\infty |F(t)|dt = \frac{1}{|S^3|}\int_{R^4\setminus B} 2|Q|e^{4w}dx < \infty.$$

Lemma 1.9. *Suppose that $(R^4\setminus B, e^{2w}|dx|^2)$ is a complete normal metric. Then its averaged metric $(R^4\setminus B, e^{2\bar{w}}|dx|^2)$ is also a complete metric.*

Proof. This is basically a consequence of Lemma 1.8. Using the argument of Lemma 1.8 we have

$$\frac{1}{|S^3|}\int_{S^3} e^{w(r\sigma)}d\sigma = e^{\bar{w}(r)}\cdot e^{o(1)},$$

where $o(1)\to 0$ as $r\to\infty$. Hence

$$\frac{1}{|S^3|}\int_{S^3}\int_{r_0}^{r_1} e^{w(r\sigma)}drd\sigma = \int_{r_0}^{r_1}\frac{1}{|S^3|}\int_{S^3} e^{w(r\sigma)}d\sigma\, dr = \int_{r_0}^{r_1} e^{\bar{w}(r)}\cdot e^{o(1)}\, dr,$$

which proves the Lemma.

Lemma 1.10. *Suppose that $(R^4 \setminus B, e^{2w}|dx|^2)$ is a normal metric. Then*

$$|\Delta \bar{w}(r)| \leq \frac{C}{r^2}. \tag{1.21}$$

Proof. We compute

$$\Delta \bar{w}(r) = \Delta \frac{1}{|S^3|} \int_{S^3} w(r\sigma) d\sigma = \frac{1}{|S^3|} \int_{S^3} \Delta w(x)\, d\sigma(x).$$

Since the metric is normal,

$$\Delta \bar{w}(r) = \frac{1}{8\pi^2} \int_{R^4 \setminus B} \{\frac{1}{|S^3|} \int_{S^3} \frac{1}{|x-y|^2}\, d\sigma(x)\} 2Q(y) e^{4w(y)} dy + 2\alpha \frac{1}{|x|^2} + \overline{\Delta h}.$$

Following Finn in [F], we have

$$\frac{1}{|S^3|} \int_{S^3} \frac{1}{|r\sigma - y|^2}\, d\sigma = \begin{cases} \dfrac{1}{r^2} & \text{if } |y| < r \\ \dfrac{1}{|y|^2} & \text{if } |y| > r \end{cases} \tag{1.22}$$

since $\frac{1}{|x-y|^2}$ is the Green's function in 4-D. We also observe that $\lim_{r\to\infty} \overline{\Delta h} = \Delta h(\infty) = 0$, hence

$$|\Delta \bar{w}(r)| \leq \frac{1}{4\pi^2 r^2} \int_{R^4 \setminus B} |Q| e^{4w} dy + 2\alpha \frac{1}{r^2}.$$

This proves the lemma.

Proposition 1.11. *Suppose that $(R^4 \setminus B, e^{2w}|dx|^2)$ is a complete normal metric. Then*

$$\lim_{r\to\infty} \frac{(\int_{\partial B_r(0)} e^{3w} d\sigma(x))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}} \int_{B_r(0)\setminus B} e^{4w} dx} = \frac{1}{4\pi^2} \int_{\partial B} T e^{3w} - \frac{1}{4\pi^2} \int_{R^4 \setminus B} Q e^{4w} dx \geq 0.$$

Proof. Write the metric $e^{2\bar{w}(r)}|dx|^2$ in cylindrical coordinates $(S^3 \times [1,\infty), e^{2\bar{v}} g_c) = (R^4 \setminus B, e^{2\bar{w}} g_0)$. Then Proposition 1.6 gives

$$\lim_{t\to\infty} \bar{v}'(t) = \frac{1}{4\pi^2} \int_{\partial B} \bar{T} e^{3\bar{w}} - \frac{1}{4\pi^2} \int_{R^4 \setminus B} \bar{Q} e^{4\bar{w}} dx \geq 0,$$

where we have used Lemma 1.8 and Lemma 1.9. Notice that

$$2\bar{Q} e^{4\bar{w}}(r) = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} \Delta^2 \bar{w} = \frac{1}{|\partial B_r(0)|} \int_{\partial B_r(0)} \Delta^2 w = \frac{2}{|\partial B_r(0)|} \int_{\partial B_r(0)} Q e^{4w},$$

which implies

$$\frac{1}{8\pi^2}\int_{R^4\setminus B}\bar{Q}e^{4\bar{w}}dx = \frac{1}{8\pi^2}\int_{R^4\setminus B}Qe^{4w}dx;$$

and that

$$\bar{T}e^{3\bar{w}} = P_3\bar{w} = \overline{P_3 w} = \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)}Te^{3w},$$

which implies

$$\frac{1}{4\pi^2}\int_{\partial B}\bar{T}e^{3\bar{w}} = \frac{1}{4\pi^2}\int_{\partial B}Te^{3w}.$$

Therefore we may apply Lemma 1.8 and get

$$\lim_{r\to\infty}\frac{(\int_{\partial B_r(0)}e^{3w}d\sigma(x))^{\frac{4}{3}}}{\int_{B_r(0)\setminus B}e^{4w}dx} = \lim_{r\to\infty}\frac{|S^3|^{\frac{4}{3}}e^{4\bar{w}}r^4}{V_4(r)}.$$

According to (1.20) in Lemma 1.8, the volume of the cylindrical shell $B_r(0)\setminus B$ is bounded in the metric $e^{2w}|dx|^2$ if and only if it is bounded in the averaged metric $e^{2\bar{w}(r)}|dx|^2$. Thus we may apply argument similar to the proof at the end of Proposition 1.6 to finish the proof of Proposition 1.11 here.

Proposition 1.12. *Suppose that the Paneitz curvature Q of $(R^4\setminus B, e^{2w}|dx|^2)$ is absolutely integrable in $(R^4\setminus B, e^{2w}|dx|^2)$, and that its scalar curvature is nonnegative at infinity. Then it is a normal metric.*

Proof. Let

$$\phi(x) = \frac{3}{4\pi^2}\int_{R^4\setminus B}\log\frac{|y|}{|x-y|}2Qe^{4w}dy \tag{1.24}$$

and $\psi = w-\phi$. We will show that the biharmonic function ψ on $R^4\setminus B$ is of the form $\alpha\log|x| + h(x)$ for some constant α and some biharmonic function $h(\frac{x}{|x|^2})$ on B. Recall the transform formula for the scalar curvature function

$$\Delta w + |\nabla w|^2 = -Je^{2w}\text{where } J = frac16R.$$

And notice that $\Delta\psi$ is a harmonic function on $R^4\setminus B$. Thus

$$\begin{aligned}\Delta\psi(x_0) &= \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)}\Delta\psi d\sigma\\ &= -\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)}(|\nabla w|^2+J)d\sigma - \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)}\Delta\phi d\sigma\end{aligned}\tag{1.25}$$

as long as $B_r(x_0)\subset R^4\setminus B$. The first term on the right of (1.25) is certainly nonpositive since $J\geq 0$ when $|x_0|$ is large enough. For the second term, we have

$$\begin{aligned}\int_{\partial B_r(x_0)}\Delta\phi d\sigma =&\frac{3}{2\pi^2}\int_{R^4\setminus B}\{\frac{1}{|S^3|}\int_{S^3}\frac{1}{|r\sigma+x_0-y|^2}d\sigma\}2Q(y)e^{4w(y)}dy.\\ \leq&\frac{3}{2\pi^2r^2}\int_{R^4\setminus B}2|Q|e^{4w}dy.\end{aligned}$$

Therefore, taking $r = \frac{1}{2}|x_0|$ for $|x_0| > 2$, for instance, we have, for any $x_0 \in R^4 \setminus B$,

$$\Delta\psi(x_0) \leq \frac{C}{|x_0|^2}$$

for some constant C depending on ψ. Thus

$$\Delta(\psi + \frac{C}{2}\log|x|) \leq 0.$$

So, if we set $g(x) = -\psi - \frac{C}{2}\log|x|$, then, $\Delta g(\frac{x}{|x|^2}) \geq 0$ and is harmonic on $B \setminus \{0\}$. By Bôcher's theorem (Theorem 3.9 in [ABR]), we have

$$\Delta g(\frac{x}{|x|^2}) = \beta\frac{1}{|x|^2} + b(x),$$

for some positive constant β and some harmonic function $b(x)$ on B. Hence $g(\frac{x}{|x|^2}) + \frac{1}{2}\beta\log|\frac{x}{|x|^2}|$ is a biharmonic function on B. Define $h(x) = \psi + \frac{C}{2}\log|x| - \frac{1}{2}\beta\log|x|$, $h(\frac{x}{|x|^2})$ is biharmonic on B and

$$\psi(x) = \frac{1}{2}(\beta - C)\log|x| + h(x) \quad \text{on } R^4 \setminus B.$$

We have thus finished the proof of the Proposition.

We combine the results in Proposition 1.7, Proposition 1.11 and Proposition 1.12 to conclude:

Corollary 1.12. *Suppose that $(R^4 \setminus B, e^{2w}|dx|^2)$ is a complete metric with its Paneitz curvature absolutely integrable in $(R^4 \setminus B, e^{2w}|dx|^2)$, and that its scalar curvature is nonnegative at infinity. Then*

$$\lim_{r\to\infty} \frac{(\int_{\partial B_r(0)} e^{3w} d\sigma(x))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\int_{B_r(0)\setminus B} e^{4w}dx} = \frac{1}{4\pi^2}\int_{\partial B} Te^{3u} - \frac{1}{4\pi^2}\int_{R^4\setminus B} Qe^{4u}dx \geq 0.$$

Proof of Theorem 1.2. Suppose

$$M = N\bigcup\{\bigcup_{i=1}^k E_i\}$$

where (N, g) is a compact Riemannian manifold with boundary

$$\partial N = \bigcup_{i=1}^k \partial E_i.$$

Recall the Chern-Gauss-Bonnet formula on the compact manifold N

$$\chi(N) = \frac{1}{32\pi^2}\int_N (|W|^2 + 8Q) dv_N + \sum_{i=1}^{k} \frac{1}{4\pi^2}\int_{\partial E_i} (L+T) d\sigma_N.$$

Since each end E_i is conformal to $R^4 \setminus B$, we have $L|_{\partial E_i} = 0$ (see [CQ]). Then, apply Corollary 1.12 to each end, we obtain

$$\frac{1}{4\pi^2}\int_{\partial E_i} T d\sigma = \nu_i + \frac{1}{32\pi^2}\int_{E_i} 8Q dv_M. \tag{1.26}$$

We also observe that the Weyl curvature W vanishes on each end E_i, thus from (1.26) we have

$$\chi(M) = \chi(N) = \frac{1}{32\pi^2}\int_M (|W|^2 + 8Q) dv_M + \sum_{i=1}^{k} \nu_i.$$

This establishes (1.1). (1.2) is also a direct consequence of Corollary 1.13. We have thus finished the proof of the theorem.

§2. **Simply connected case**

In this section, we will prove the following:

Theorem 2.1. *Suppose that M is a a subdomain in S^4 and $g = \nu^2 g_c$ is a complete conformal metric on M, where g_c is the standard metric on S^4; satisfying:*
(a) The scalar curvature is bounded between two positive constants, and $|\nabla_g R|$ is bounded,
(b) The Ricci curvature of the metric g has a lower bound,
(c) the Paneitz curvature is absolutely integrable, i.e.

$$\int_M |Q| dv_g < \infty.$$

Then $M = S^4 \setminus \{p_i\}_{i=1}^k$.

It is easy to produce a large number of such metrics. The basic example is the infinite cylinder $(R \times S^3, dt^2 + d\sigma^2)$ which is conformally equivalent to $(R^4 \setminus \{0\}, |dx|^2)$. The cylinder metric has $R = 6, |Ric|^2 = 12, Q = 0$. Thus given a domain $\Omega = S^4 \setminus \{p_i\}_{i=1}^k$, one simply glues the cylinder metrics on $B_{\epsilon r}(p_i) \setminus \{p_i\}$ to the spherical metric on $S^4 \setminus \bigcup B_r(p_i)$. To make the scalar curvature positive over the glueing regions $B_r(p_i) \setminus B_{\epsilon r}(p_i)$, one has to take ϵ sufficiently small.

We remark that given any simply connected, locally conformally flat, complete manifold M of dimension $n \geq 3$, there always exists an immersion $\Phi : M \to S^n$ such that the locally conformally flat structure of M is induced by Φ. This immersion Φ is called the developing map of M. Under the further assumption that the scalar curvature is positive, Schoen-Yau proved that the developing map is injective. Therefore, any such manifold can be considered as a subdomain of S^n with a complete metric $g = \nu^{\frac{4}{n-2}} g_c$ where g_c is the standard metric on the sphere S^n. Combining this result with Theorem 2.1 above, we obtain:

Corollary 2.2. *Suppose M is a simply connected, locally conformally flat, complete 4-manifold satisfying conditions (a), (b) and (c) as in the statement of Theorem 1.1 above, then M is conformally equivalent to $S^4 \setminus \{p_i\}_{i=1}^k$.*

In the remainder of the section, we will prove Theorem 2.1. Suppose M is a subdomain of S^n, for convenience, we choose a point P in M and use stereographic projection which maps S^n to R^n and P to infinity; then we may identify M as $M = (\Omega, u^{\frac{4}{n-2}}|dx|^2)$, where $\Omega \subset R^n$ and $|dx|^2$ is the standard metric of R^n. In the following we will estimate the size of the conformal factor $u(x)$ for $x \in \Omega$ in terms of the Euclidean distance $d(x) = distance(x, \partial\Omega)$. First we have the following lower bound estimate from ([SY, Theorem 2.12, Chapter VI]):

Lemma 2.3. *Suppose $M = (\Omega, g = u^{\frac{4}{n-2}}|dx|^2)$, where $\Omega \subset R^n$; and suppose*
(a) $|R|$ and $|\nabla_g R|$ both bounded,
(b) the Ricci curvature has a lower bound.
Then there exists a constant $C > 0$ such that

$$u(x) \geq C d(x)^{-\frac{n-2}{2}} \quad \textit{for all } x \in \Omega \ . \tag{2.1}$$

We remark that in the statement of Theorem 2.12, Chapter VI in [SY], a stronger assumption that M has bounded curvature is listed for the above result. But it is clear from the proof (e.g. applying method of gradient estimate), that assumptions (a) and (b) are sufficient for the conclusion.

In the case when $M = (\Omega, u^2|dx|^2)$ where $\Omega \subset R^4$ and satisfies the assumptions (a) and (c) in Theorem 1.1, we can also establish the upper bound of u in terms of the distance function d.

Lemma 2.4. *Suppose $M = (\Omega, u^2|dx|^2)$ is a complete manifold such that*
(a) its scalar curvature R satisfies $0 < R_0 \leq R \leq R_1$ and $|\nabla R|_g \leq \bar{C}$, where $R_0, R_1, \bar{C}$ are constants, and
(c) $\int_\Omega |Q| u^4 dx < \infty$.
Then there exists some constant C so that

$$u(x) \leq C d(x)^{-1} \quad \textit{for all } x \in \Omega. \tag{2.2}$$

Our proof of the lemma uses a blow-up argument which was used by Schoen to obtain the upper bound for conformal metrics with constant scalar curvature. Thus what we have done here is to replace the constant scalar curvature condition by the integral bound of the Paneitz curvature and the condition (a).

The proof we give below for Lemma 2.4 depends on the following simple result, which is a special case of Theorem 4.1 in [CQY]. We present the proof here to make the paper self-contained.

Lemma 2.5. *On $(R^4, u^2|dx|^2)$, the only metric with $Q \equiv 0$ and $R \geq 0$ at infinity is isometric to $(R^4, |dx|^2)$.*

Proof of Lemma 2.5. Recall the transformation formula for the scalar curvature

$$\Delta w + |\nabla w|^2 = -Je^{2w} \tag{2.3}$$

where $J = \frac{R}{6}$ and R is the scalar curvature for the metric $(R^4, e^{2w}|dx|^2)$. Notice that when $Q \equiv 0$, then $(\Delta)^2 w = Qe^{4w} = 0$, thus Δw is a harmonic function , hence

$$\begin{aligned}\Delta w(x_0) =& \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} \Delta w d\sigma \\ =& -\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} (|\nabla w|^2 + J) d\sigma.\end{aligned}$$

Thus by our assumption that J is non-negative, we have by taking $r \to \infty$, for each $x_0 \in R^4$,

$$\Delta w(x_0) \leq 0.$$

Applying the Liouville theorem for bounded harmonic functions to Δw we conclude $\Delta w = c_0$. It follows that any partial derivative of w is harmonic, i.e.

$$\Delta w_{x_i} = 0.$$

Apply the mean value theorem again, we have

$$|w_{x_i}(x_0)|^2 = |\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} w_{x_i} d\sigma|^2 \leq \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} |\nabla w|^2 d\sigma.$$

But from (2.3) above we have

$$|\nabla w|^2 = -C_0 - Je^{2w},$$

hence we conclude similarly as before that for each $x_0 \in R^4$,

$$|w_{x_i}(x_0)|^2 \leq -C_0.$$

Hence all partial derivatives of w are constants. It follows that $\Delta w \equiv C_0 \equiv 0$. Hence all partial derivatives of w vanish. Thus w is a constant.

Proof of Lemma 2.4. Assume (2.2) does not hold, then there exists a sequence of points x_i in Ω with $d(x_i) \to 0$, and

$$u(x_i)d(x_i) = A_i \to \infty, \quad \text{as } i \to \infty.$$

By passing to a subsequence of x_i, we may also assume that $B(x_i, \frac{1}{2}d(x_i))$ are disjoint and it follows from our assumption (c) on the integrability of the Paneitz curvature Q that

$$\int_{B(x_i, \frac{1}{2}d(x_i))} |Q|u^4 dx \to 0, \quad \text{as } i \to \infty. \tag{2.4}$$

We will now re-scale u at each point x_i. To do so, at each point x_i, denote $\sigma_i = \frac{1}{2}d(x_i)$ and define

$$f_i(y) = (\sigma_i - d(y, x_i))u(y).$$

Then $f_i(x_i) = \sigma_i u(x_i) = \frac{1}{2}A_i$, and $f_i(y) = 0$ for all $y \in \partial B(x_i, \sigma_i)$. Thus there exists some point y_i in $B(x_i, \sigma_i)$ for which

$$f(y_i) = \max\{f(x) : x \in B(x_i, \sigma_i)\}.$$

We let

$$\lambda_i = u(y_i),$$

and set

$$v_i(x) = \lambda_i^{-1}u(\lambda_i^{-1}x + y_i).$$

Denote $r_i = \frac{1}{2}(\sigma_i - d(x_i, y_i))$ and $R_i = r_i u(y_i)$; then $x \in B(0, R_i)$ if and only if the corresponding point y defined as $y = \lambda_i^{-1}x + y_i$ is in $B(x_i, r_i)$. We claim the following properties (2.5) to (2.7) hold for v_i:

$$v_i(0) = \lambda_i^{-1}u(y_i) = 1, \tag{2.5}$$

$$0 < v_i(x) \leq 2 \quad \text{for } x \in B(0, R_i), \text{and } R_i \to \infty \text{ as } i \to \infty, \tag{2.6}$$

and

$$-\Delta v_i = J_i v_i^3, \quad \text{with } |\nabla J_i(x)| \; \text{ uniformly bounded for } x \in B(0, R_i), \tag{2.7}$$

where $J_i(x) = J(\lambda_i^{-1}x + y_i)$.

To verify (2.6), first we have $2R_i = (\sigma_i - d(x_i, y_i))u(y_i) \geq \sigma_i u(x_i) = \frac{1}{2}A_i \to \infty$ as $i \to \infty$ by the maximality of the choice of y_i and (2.3). Furthermore we have for $y = \lambda_i^{-1}x + y_i$

$$\begin{aligned} v_i(x) = \frac{u(y)}{u(y_i)} &= \frac{u(y)(\sigma_i - d(y, x_i))}{u(y_i)(\sigma_i - d(y, x_i))} \\ &\leq \frac{u(y_i)(\sigma_i - d(y_i, x_i))}{u(y_i)(\sigma_i - r_i)} \\ &\leq \frac{\sigma_i}{\sigma_i - r_i} \leq 2 \quad \text{for x} \in B(0, R_i). \end{aligned}$$

The equation in (2.7) is a direct re-scaling of the scalar curvature equation of the metric $g = u^2|dx|^2$. The gradient estimate of J_i follows from our assumption (a) on the gradient bound of R with respect to g and (2.6); we may see this as:

$$|\nabla_x J_i(x)| = |\nabla_x(J(\lambda_i^{-1}x + y_i))| = |\lambda_i^{-1}\nabla_y J(y)| \leq \frac{\bar{C}}{6}\lambda_i^{-1}u(y) \leq \frac{\bar{C}}{3},$$

where $y = \lambda_i^{-1}x + y_i$.

It follows from (2.7) that, by taking subsequence if necessary, we have

$$J_i \to J_\infty \text{ in } C^\alpha_{loc}(R^4)$$

for some $J_\infty \in C^\alpha(R^4)$ and $J_\infty \geq \frac{1}{6}R_0 > 0$. Hence some subsequence of v_i converges uniformly on compact in $C^{1,\alpha}(\mathbb{R}^4)$. Thus it follows from (2.7) that by taking another subsequence, we also have

$$v_i \to u_\infty \text{ in } C^{2,\alpha}_{loc}(R^4) \tag{2.8}$$

where $u_\infty \in C^{2,\alpha}(R^4)$, $u_\infty(0) = 1$, and

$$-\Delta u_\infty = J_\infty u_\infty^3 \quad \text{in } R^4. \tag{2.9}$$

By the maximum principle we know that $u_\infty(x) > 0$ for all $x \in R^4$, which implies that, if we let $w_i = \log v_i$ and $w_\infty = \log u_\infty$, and passing to a subsequence we have

$$w_i \to w_\infty \quad \text{in } C^{2,\alpha}_{loc}(R^4). \tag{2.10}$$

We now claim that

$$\Delta^2 w_\infty = 0 \quad \text{in } R^4. \tag{2.11}$$

To see this, for any $\phi \in C^\infty(R^4)$ with compact support, we have

$$\begin{aligned}
|\int \Delta w_\infty \Delta \phi| &= |\lim_{i\to\infty} \int \Delta w_i \Delta \phi| \\
&= |\lim_{i\to\infty} \int \Delta^2 w_i \phi| \\
&\leq C_\phi \lim_{i\to\infty} \int_K |\Delta^2 w_i| \text{ where } K \text{ is the support of } \phi \\
&\leq C_\phi \lim_{i\to\infty} \int_{K_i} |\Delta^2 w| \ \text{ where } K_i = \{y : y - y_i \in \lambda_i^{-1} K\} \\
&\leq C_\phi \lim_{i\to\infty} \int_{K_i} 2|Q|e^{4w} \\
&= 0.
\end{aligned} \tag{2.12}$$

for any $\phi \in C^\infty(R^4)$ with compact support. The last step in above argument follows from (2.4), which is a consequence of our assumption (c) in the Lemma.

Hence we have $u_\infty, w_\infty \in C^\infty(R^4)$, and the metric ${u_\infty}^2|dx|^2$ is a metric on R^4 with $Q \equiv 0$ and $R \geq 0$. It then follows from Lemma 2.5 that u_∞ is a constant function, in contradiction with equation (2.9). We have thus finished the proof of Lemma 2.4.

We will now estimate the size of the integral of Q over suitable subset of Ω in terms of the integral of the boundary curvature T (as defined in the introduction) via the Chern-Gauss-Bonnet formula. It will be advantegeous to consider domains formed by level sets of the conformal factor $u = e^w$. We will derive a formula for the boundary integral. We consider $M = (\Omega, u^2|dx|^2)$ where $\Omega \subset R^4$, and denote the level set for the conformal factor $u = e^w$ by

$$U_\lambda = \{x : 1 \leq u \leq \lambda\}, \quad \text{and} \quad S_\lambda = \{x : u = \lambda\}. \tag{2.13}$$

Also, ∂_n denotes the normal derivative (chosen so that $\frac{\partial w}{\partial n} \geq 0$) .

Lemma 2.6. *Suppose that $M = (\Omega, u^2 g_0)$ is a complete Riemannian manifold. Then on the level set S_λ where λ is a regular value for u, we have:*

$$\begin{aligned}-\int_{S_\lambda} \partial_n \Delta w d\sigma =&\lambda \frac{d}{d\lambda}\left(\int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J\partial_n w e^{2w} d\sigma + 2\int_{U_\lambda} J|\nabla u|^2 dx\right)\\ &+ \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J^2 \frac{e^{4w}}{\partial_n w} d\sigma.\end{aligned} \tag{2.14}$$

Proof. Observe that in terms of $w = \log u$, the scalar curvature equation becomes

$$\Delta w + |\nabla w|^2 = -Je^{2w}$$

as in (2.3). Thus when restricted to the set S_λ, we have

$$\partial_n \partial_n w + H\partial_n w + (\partial_n w)^2 = -Je^{2w}, \tag{2.15}$$

where H is the mean curvature on S_λ.

To calculate the derivative $\frac{d}{d\lambda}\int_{S_\lambda} f d\sigma$ we use the first variation formula $\partial_n d\sigma = H d\sigma$, and the chain rule $\frac{d}{d\lambda} = \frac{1}{e^w \partial_n w}\partial_n$. Thus

$$\frac{d}{d\lambda}\int_{S_\lambda} f d\sigma = \int_{S_\lambda} \frac{df}{d\lambda} d\sigma + \int_{S_\lambda} \frac{f}{e^w \partial_n w} H d\sigma.$$

In particular we have

$$\begin{aligned}\frac{d}{d\lambda}\int_{S_\lambda} (\partial_n w)^3 e^w d\sigma =&3\int_{S_\lambda} \partial_n w \partial_n \partial_n w d\sigma + \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} (\partial_n w)^2 H d\sigma\\ =&2\int_{S_\lambda} \partial_n w \partial_n \partial_n w d\sigma - \int_{S_\lambda} J(\partial_n w)e^{2w} d\sigma.\end{aligned} \tag{2.16}$$

On the other hand,

$$\begin{aligned}-\int_{S_\lambda} \partial_n \Delta w d\sigma &= \int_{S_\lambda} \partial_n((\partial_n w)^2 + Je^{2w}) d\sigma\\ &= 2\int_{S_\lambda} \partial_n w \partial_n \partial_n w d\sigma + \int_{S_\lambda} (\partial_n J)e^{2w} d\sigma + 2\int_{S_\lambda} J(\partial_n w)e^{2w} d\sigma.\end{aligned} \tag{2.17}$$

Combine (2.16) and (2.17), we obtain

$$-\int_{S_\lambda} \partial_n \Delta w d\sigma = \frac{d}{d\lambda}\int_{S_\lambda} (\partial_n w)^3 e^w d\sigma + 3\int_{S_\lambda} J(\partial_n w)e^{2w} d\sigma + \int_{S_\lambda} (\partial_n J)e^{2w} d\sigma. \tag{2.18}$$

Next we compute

$$\begin{aligned}\frac{d}{d\lambda}\int_{S_\lambda} J(\partial_n w)e^{3w}d\sigma =& \int_{S_\lambda}(\partial_n J)e^{2w}d\sigma + \int_{S_\lambda} J\frac{\partial_n\partial_n w}{\partial_n w}e^{2w}d\sigma\\ &+3\int_{S_\lambda} J(\partial_n w)e^{2w}d\sigma + \int_{S_\lambda} Je^{2w}Hd\sigma\\ &= \int_{S_\lambda}\partial_n Je^{2w}d\sigma - \int_{S_\lambda} J^2\frac{e^{4w}}{\partial_n w}d\sigma + 2\int_{S_\lambda} J(\partial_n w)e^{2w}d\sigma.\end{aligned} \tag{2.19}$$

Thus

$$\begin{aligned}-\int_{S_\lambda}\partial_n\Delta w d\sigma =& \frac{d}{d\lambda}\int_{S_\lambda}(\partial_n w)^3 e^w d\sigma + \frac{d}{d\lambda}\int_{S_\lambda} J(\partial_n w)e^{3w}d\sigma\\ &+\int_{S_\lambda} J(\partial_n w)e^{2w}d\sigma + \int_{S_\lambda} J^2\frac{e^{4w}}{\partial_n w}d\sigma.\end{aligned} \tag{2.20}$$

Or, equivalently, since $e^w = \lambda$,

$$\begin{aligned}-\int_{S_\lambda}\partial_n\Delta w d\sigma =& \lambda\frac{d}{d\lambda}\left(\int_{S_\lambda}(\partial_n w)^3 d\sigma + \int_{S_\lambda} J(\partial_n w)e^{2w}d\sigma\right)\\ &+\int_{S_\lambda}(\partial_n w)^3 d\sigma + 2\int_{S_\lambda} J(\partial_n w)e^{2w}d\sigma + \int_{S_\lambda} J^2\frac{e^{4w}}{\partial_n w}d\sigma.\end{aligned} \tag{2.21}$$

Finally we notice that by the co-area formula, we have

$$\int_{S_\lambda} J(\partial_n w)e^{2w}d\sigma = \lambda\int_{S_\lambda} J|\nabla u|d\sigma = \lambda\frac{d}{d\lambda}\int_{U_\lambda} J|\nabla u|^2 dx. \tag{2.22}$$

Substituting (2.22) to (2.21), we obtain (2.14) and finish the proof of the lemma.

We now state a simple covering lemma which we will use later in the proof of Theorem 2.1.

Lemma 2.7. *Suppose that Λ is a compact subset of $\mathbb{R}^4$. Then*

$$|\{x : dist(x,\Lambda) = s\}| \geq \begin{cases} Ns^3, & \text{for any } N > 0 \text{ if } \dim(\Lambda) = 0 \text{ and } H^0(\Lambda) = \infty\\ Cs^{3-\alpha}, & \text{for } \alpha = \frac{3}{4}\beta \quad \text{if } \dim(\Lambda) = \beta > 0.\end{cases} \tag{2.23}$$

Where $dim(\Lambda)$ denotes the Hausdorff dimension of the set Λ, and H^β denotes the Hausdorff β measure of the set.

Proof. By a standard covering lemma, we have $K = K(r)$ balls $B(z_i, r)$ covering for Λ such that $B(z_i, \frac{1}{5}r)\cap B(z_j,\frac{1}{5}r) = \emptyset$ for $i \neq j$. And by the definition of Hausdorff measure we also know

$$CKr^\beta \geq H^\beta(\Lambda).$$

Now notice that

$$\bigcup_i B(z_i, \frac{1}{5}r) \subset \{x : \mathrm{dist}(x, \Lambda) \le r\}.$$

we therefore have

$$|\{x : \mathrm{dist}(x, \Lambda) \le r\}| \ge CKr^4 \ge CH^\beta(\Lambda) r^{4-\beta},$$

which, by the isoperimetric inequality, implies

$$|\{x : \mathrm{dist}(x, \Lambda) = r\}| \ge C(H^\beta(\Lambda))^{\frac{3}{4}} r^{3-\frac{3}{4}\beta}.$$

This is easily seen to imply (2.23). So we have proved the lemma.

Proof of Theorem 2.1. We identify (M, g) as $(\Omega, u^2|dx|^2)$ for some subset Ω of R^4 as before. Notice that all ends of M is in bounded region inside Ω. Apply integration by parts, we get

$$\int_{U_\lambda} 2Qe^{4w} dx = \int_{U_\lambda} \Delta^2 w dx = -\int_{S_1} \partial_n \Delta w d\sigma + \int_{S_\lambda} \partial_n \Delta w d\sigma. \tag{2.24}$$

Apply formula (2.15) in Lemma 2.4, we obtain

$$\begin{aligned}
&-\int_{U_\lambda} 2Qe^{4w} dx - \int_{S_1} \partial_n \Delta w d\sigma \\
=&-\int_{S_\lambda} \partial_n \Delta w d\sigma \\
=&\lambda \frac{d}{d\lambda}\Big(\int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J(\partial_n w) e^{2w} d\sigma + 2\int_{U_\lambda} J|\nabla u|^2 dx\Big) \\
&+\int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J^2 \frac{e^{4w}}{\partial_n w} d\sigma \\
\ge&\lambda \frac{d}{d\lambda} V(\lambda),
\end{aligned} \tag{2.25}$$

where $V(\lambda)$ is defined as:

$$V(\lambda) = \int_{S_\lambda} (\partial_n w)^3 d\sigma + \int_{S_\lambda} J(\partial_n w) e^{2w} d\sigma + 2\int_{U_\lambda} J|\nabla u|^2 dx. \tag{2.26}$$

We recall the scalar curvature equation

$$-\Delta u = Ju^3 \quad \text{in } \Omega.$$

Thus

$$\begin{aligned}
\int_{U_\lambda} J|\nabla u|^2 dx &\ge J_0 \int_{U_\lambda} |\nabla u|^2 dx \\
&\ge J_0\Big(\int_{U_\lambda} Ju^4 dx - \int_{S_1} u\partial_n u d\sigma + \int_{S_\lambda} u\partial_n u d\sigma\Big) \\
&\ge J_0^2 \int_{U_\lambda} u^4 dx,
\end{aligned} \tag{2.27}$$

where $J \geq J_0 > 0$ as assumed in (a). Consequently,

$$V(\lambda) \geq 2J_0^2 \int_{U_\lambda} u^4 dx. \tag{2.28}$$

To estimate the growth of V we use the lower and upper estimates of the conformal factor u as in Lemma 2.3 and Lemma 2.4; thus we may replace the region U_λ by

$$D_\lambda = \{x : C_1 \geq d(x, \partial\Omega) \geq C_2\lambda^{-1}\} \subset U_\lambda. \tag{2.29}$$

Therefore we have

$$V(\lambda) \geq \int_{D_\lambda} u^4 = \int_{\frac{C_2}{\lambda}}^{C_1} \int_{\{x:d(x,\partial\Omega)=s\}} u^4 d\sigma ds$$

by the co-area formula. Hence

$$V(\lambda) \geq C \int_{\frac{C_2}{\lambda}}^{C_1} |\{x : d(x, \partial\Omega) = s\}| s^{-4} ds. \tag{2.30}$$

We now estimate the size of the set $\partial\Omega$ by Lemma 2.7. In the case $\dim(\partial\Omega) = \beta$ is positive, we have from (2.23) and (2.30) that

$$V(\lambda) \geq \frac{C}{\alpha}((\frac{\lambda}{C_2})^\alpha - \frac{1}{C_1^\alpha}), \tag{2.31}$$

for $\alpha = \frac{3}{4}\beta$ which is positive. In the case when $\dim(\partial\Omega)$ is zero, we have either the zero Hausdorff measure of the set (i.e. number of points in the set) is finite; then we have proved the theorem; or we have

$$|\{x : d(x, \partial\Omega) = s\}| \geq Ns^3 \tag{2.32}$$

for any number $N > 0$. Hence

$$V(\lambda) \geq N \int_{\frac{C_2}{\lambda}}^{C_1} \frac{1}{s} ds = N \log \lambda - C. \tag{2.33}$$

In either case of (2.31) or (2.33), we conclude that there exists at least a sequence of $\lambda_i \to \infty$ as $i \to \infty$ such that λ_i are all regular values (due to Sard's theorem) and

$$\lambda_i \frac{d}{d\lambda} V(\lambda_i) \geq N \tag{2.34}$$

for any number $N > 0$. But in view of the equality (2.25), this contradicts with our assumption (c) that Q is integrable. We have thus finished the proof of the theorem.

Corollary 2.7. *If $(\Omega \subset S^4, e^{2w}g_0)$ is a complete conformal metric satisfying (a) (b) and (c) and in addition Q is a constant, then: either*
1. $(\Omega \subset S^4, e^{2w}g_0) = (S^4, g_0)$*; or*
2. $(\Omega \subset S^4, e^{2w}g_0) = (\mathbb{R}^4 \setminus \{0\}, \frac{|dx|^2}{|x|^2})$.

Proof of Corollary 2.3: According to Theorems 1 and 2, $\Omega = S^4 \setminus \{p_1, ..., p_k\}$; and if $k \geq 1$ the inequality (2.1) shows that the metric has infinite volume. Thus in case $k = 0$, the standard metric on S^4 is the unique solution of the equation $Q = 3$ up to conformal transformations. In case $k \geq 1$ we must have $Q = 0$. Hence the euler number can only have two possibilities: $\chi = 1$ or $\chi = 0$. In the former we have Ω is conformally R^4, then the metric must be standard according to Lemma 2.5, but the scalar curvature is zero hence this case cannot occur. The remaining possibility is the case $\chi = 0$ so that Ω is conformally $R^4 \setminus \{0\}$, and $Q = 0$. To determine the metric $e^{2w}|dx|^2$, we use Proposition 1.12 to conclude that w is of the form

$$w(x) = \alpha log|x| + h(x), \text{ for } |x| > 1$$

where h is a biharmonic function which extends over infinity and $\alpha \leq -1$ due to completeness of the metric. On the other hand in the punctured ball we have

$$w(x) = -\beta log|x| + k(x), \text{ for } |x| < 1$$

where k is a biharmonic function on $|x| < 1$ and $\beta \leq 1$ due to the completeness requirement. On the other hand the generalized Chern-Gauss-Bonnet formula (0.7) requires $\alpha + \beta \geq 0$. Hence $\alpha = -1, \beta = 1$ and we conclude that $h = k = constant$, and therefore $e^{2w}|dx|^2 = \frac{|dx|^2}{|x|^2}$ up to a homothety as claimed.

§3. **Non-simply connected manifolds**

In this section we will prove Theorem 3.1. The main idea is to localize the argument in the proof of Theorem 2.1 in the previous section and appeal to some well known elementary facts in the theory of Kleinian groups. Suppose (M, g) is a locally conformally flat manifold with positive scalar curvature, then by the result of Schoen-Yau, the universal cover $\tilde{M}$ can be embedded as a domain in the 4-sphere. Hence the fundamental group Γ acts on S^4 as a discrete group of conformal transformations with a domain of discontinuity $\Omega(\Gamma)$ which contains $\tilde{M}$. (Here as in the rest of this section, we refer to [Ra] for standard notations and definitions of Kleinian groups.) The limit set L consists of accumulation points of orbits of Γ. The discrete group Γ also acts as hyperbolic isometrics on the interior B of S^4. We recall the following definitions of limit points. A point $p \in S^4$ is called a conical limit point of the group Γ if there is a point $x \in B$, a sequence $\{g_i\} \subset \Gamma$, a hyperbolic ray γ in B ending at p and a positive number r such that $\{g_i(x)\}$ converges to p within hyperbolic distance r from the geodesic γ. A point p is a cusped limit point of a discrete group Γ if it is a fixed point of a parabolic element of Γ that has a cusped region. To explain the notion of a cusped region, we identify S^4 as R^4 and conjugate the point p to infinity in the upper half space $\mathbb{R}^5_+$ in $\mathbb{R}^5$, and consider the stabilizer Γ_∞ of ∞. Γ_∞ is a discrete subgroup of isometrics of $\mathbb{R}^4$ of rank $0 < m \leq 4$. Let E be the maximal Γ_∞-invariant subspace such that E/Γ_∞ is compact. Denote by N a neighborhood of E in $\mathbb{R}^5_+$ and set $U = \bar{\mathbb{R}}^5_+ \setminus \bar{N}$. Then U is an open Γ_∞ invariant subset of $\bar{\mathbb{R}}^5_+$. The set U is said to be a cusped region for Γ based at ∞ if and only if for all g in $\Gamma \setminus \Gamma_\infty$, we have $U \cap gU = \emptyset$.

Definition. *A convex polyhedron P in the hyperbolic space is called geometrically finite if and only if for each point $x \in \bar{P} \cap S^4$ there is an open Euclidean neighborhood of x that only meets the faces of P incident with x. A discrete group of conformal transformations of S^4 is called geometrically finite if and only if Γ has a geometrically finite convex fundamental polyhedra.*

An important aspect of geometrically finite Kleinian groups is that its limit set consists only of conical limit points and cusped limit points ([Ra], Theorem 12.3.5, p.512) Now we are ready to state and prove our main theorem in this section.

Theorem 3.1. *Suppose M is a locally conformally flat complete 4-manifold which satisfies:*
(a) The scalar curvature is bounded between two positive constants, and $|\nabla_g R|$ is bounded,
(b) The Ricci curvature has a lower bound ,
(c) The Paneitz curvature is absolutely integrable, i.e.

$$\int_M |Q| dv_g < \infty,$$

(d) The fundamental group of M acting as deck transformation group is a geometrically finite Kleinian group without torsion.
Then $M = \bar{M} \setminus \{p_i\}_{i=1}^k$ where $\bar{M}$ is compact manifold with a locally conformally flat structure.

Remarks

1. It is worthwhile to point out that $\bar{M}$ is sometimes called the conformal compactification of M.

2. Suppose M satisfies all assumptions in Theorem 3.1 except its fundamental group being torsion free. Instead, its fundamental group has a finite index subgroup without torsion. Then, by passing to a finite covering M' of M, we may apply Theorem 3.1 to M'. For example, we have Selberg's lemma ([Ra, p. 327]) which states that any finitely generated discrete matrix group contains a finite index subgroup without torsion. So, in the cases when the fundamental group of M is finitely generated then Theorem 3.1 applies to the torsion free subgroup, this means that the ends are conformally the quotient of punctured 4-ball.

Proof of Theorem 3.1. Denote

$$\begin{aligned} \Phi :& \tilde{M} \hookrightarrow S^4 \\ \pi :& \downarrow \\ & M \end{aligned}$$

where Φ is the developing map from the universal covering $\tilde{M}$ of M into S^4, which is an embedding. The holonomy representation Γ of the fundamental group of M then becomes a Kleinian group in the conformal transformation group of S^4, which is assumed to be geometrically finite and torsion free. Let $\Omega = \Phi(\tilde{M})$. Clearly any point in Ω is a ordinary point for Γ, that is $\Omega \subset \Omega(\Gamma)$: the domain of discontinuity of Γ. We are interested in the set $\partial\Omega = S^4 \setminus \Omega$. If M is compact, then $\partial\Omega = L(\Gamma)$ is the set of all limit points of Γ and $\Omega = \Omega(\Gamma)$, therefore $M = \Omega(\Gamma)/\Gamma$. But, in our case, we have

$$\partial\Omega = (\partial\Omega \cap \Omega(\Gamma)) \bigcup L(\Gamma),$$

and $L(\Gamma)$ consists of only conical limit points and cusped limit points. We claim:

Claim 1. Every point in $\partial\Omega \cap \Omega(\Gamma)$ is an isolated point in S^4, and

Claim 2. There is no cusped limit points except possibly cusped limit points of rank four in which case the closure of the fundamental region for the cusp does not meet the limit set.

First we apply the proof of Theorem 2.1 in previous section to prove Claim 1. According to Theorem 2.9 and Theorem 2.11 in Chapter VI of [SY], $\dim(\partial\Omega) \leq d(M) < 1$, hence is a totally disconnected set (cf Lemma 4.1, Chapter 4 in [Fa]). Therefore, for any $x \in \partial\Omega \cap \Omega(\Gamma)$, there exists a ball $B(r, x)$ such that $\gamma B(r, x) \bigcap B(r, x) = \emptyset$ for all $\gamma \in \Gamma$ and $\partial \bar{B}(r, x) \bigcap \partial\Omega = \emptyset$. Since the Q curvature is absolutely integrable over $\Omega \cap B(r, x)$, we can restrict the conformal metric to $B(r, x) \cap \Omega$ and apply the argument in the proof of Theorem 2.1 to conclude that $\partial\Omega \cap B(r, x)$ consists of at most finite number of points including x, thus in particular x is an isolated boundary point.

To prove Claim 2, we recall, from the definition of the cusped limit points, a cusped limit point p is a fixed point of a parabolic element γ_p in Γ and there is a cusped region U for Γ based at p. The cusped region based at p restricted to the 4-sphere gives a conformal coordinates chart for M at the end E_p around p of the following form (cf: Chapter 12 in [Ra]):

$$(M, g) \supset (E_p, g) = (S_m, \phi^2 g_m)$$

for some $S_m, 1 \leq m \leq 4$, where $S_1 = T^1 \times \{x \in R^3 : |x| \geq K\}$, $S_2 = T^2 \times \{x \in R^2 : |x| \geq K\}$, $S_3 = T^3 \times \{x \in R : |x| \geq K\}$, $S_4 = T^4$, where T^k is a flat torus of dimension k, $K > 0$ is a large positive number, g_m is the product metric on each S_m, and ϕ is a positive smooth function. Now we will show that such ends E_p can not exist in our case.

Lemma 3.2. *There is no complete conformal metric $\phi^2 g_m$ on (S_m) such that its scalar curvature bounded from below by a positive number, for $m = 1, 2, 3$.*

Proof.
Case (i): When $m = 1$. Recall the scalar curvature equation on S_1:

$$-(\partial_r\partial_r\phi + \frac{2}{r}\partial_r\phi + \frac{1}{r^2}\Delta_{S^2}\phi + \Delta_{T^1}\phi) = J\phi^3$$

for $r = |x|$, $r \in [K, \infty)$ the norm of a point x in R^3. We take the average of ϕ over $S^2 \times T^1$ for each r and get

$$-(\partial_r \partial_r \bar{\phi} + \frac{2}{r} \partial_r \bar{\phi}) \geq J_0 (\bar{\phi})^3$$

where $J \geq J_0$ as assumed. Now take a change of variables

$$\begin{cases} e^t = r \\ \psi = r\bar{\phi} \end{cases}, \tag{3.1}$$

therefore

$$-\psi_{tt} + \psi_t \geq J_0 \psi^3 \tag{3.2}$$

We will show that ψ attains zero or infinity at some finite t, which will be a contradiction. First we observed that, if $\psi_t(t_0) \leq 0$ at certain t_0, then $\psi_{tt}(t_0) < 0$. Therefore we have $\psi_t \leq -\alpha < 0$ and $\psi_{tt} \leq 0$ for all $t \geq t_1$ for some $t_1 > t_0$, which implies ψ has to be zero at some finite t. Thus, we may assume $\psi_t > 0$ for all t. Next we observed that, if $\psi_t - J_0 \psi^3 \leq 0$ at some t_0, then $\psi_{tt}(t_0) \leq 0$ and therefore $\psi_{tt}(t) \leq -\beta < 0$ for all $t \geq t_2$ for some $t_2 > t_0$; then ψ_t can not be positive for all t. Thus, we may assume

$$\psi_t - J_0 \psi^3 > 0$$

for all t. But this implies

$$\frac{d}{dt}(\frac{1}{\psi^2}) < -2J_0,$$

which is impossible unless ψ goes to infinity at some finite t. This finished the proof of case (i).

Case (ii): When $m = 2$, then the scalar curvature equation for S_2 is:

$$-(\partial_r \partial_r \phi + \frac{1}{r} \partial_r \phi + \frac{1}{r^2} \Delta_{S^1} \phi + \Delta_{T^2} \phi) = J\phi^3$$

for $r \in [K, \infty)$ the norm of a point in R^2. Then we take the average over $S^1 \times T^2$ and the change of variables as in (3.1) to get

$$-\psi_{tt} + 2\psi_t \geq J_0 \psi^3 + \psi. \tag{3.3}$$

A similar argument as in the above proves the lemma for S_2.

Case (iii): For $m = 3$, then the scalar curvature equation on S_3 becomes:

$$-(\partial_r \partial_r \phi + \Delta_{T^3} \phi) = J\phi^3$$

for $r \in [K, \infty)$. Then again we take the average and the change of variables, and get

$$-\psi_{tt} + 3\psi_t \geq J_0 \psi^3 + 2\psi. \tag{3.4}$$

Once again, a similar argument as in the previous cases establishes the lemma for S_3.
Case (iv): When $m = 4$, in this case $S_4 = T^4$ is compact, hence its fundamental domain in the domain of discontinuity can be chosen to have its closure bounded away from the limit set.

We have thus finished the proof of the lemma.

To continue the proof of Theorem 3.1, we apply Lemma 3.2 to conclude that the limit set consists of either conical limit points or cusp of rank four.

Since fixed points of either hyperbolic or parabolic elements in Γ are all in the limit set $Ł(\Gamma)$ of Γ, Γ acts on $\Omega(\Gamma)$ has no fixed points. Thus $\Omega(\Gamma)/\Gamma = \bar{M}$ is a manifold with a locally conformally flat structure. We consider a fundamental domain F which satisfies:

$$\Omega(\Gamma) = \bigcup_{\gamma\in\Gamma} \gamma F \quad \text{and} \quad F \bigcap \gamma F = \emptyset, \ \forall \gamma \in \Gamma.$$

And let $C = \bar{F}$ the closure of F in S^4. Since all points in $L(\Gamma)$ are either conical or cusps of rank four, the closure of F does not meet the limit set, hence $C \subset \Omega(\Gamma)$. This proves that $\bar{M} = C/\Gamma$ is compact since C is compact. In the mean while, we have $M = (C \setminus (\partial\Omega \cap \Omega(\Gamma))$. Now since $\partial\Omega \cap \Omega(\Gamma)$ are all isolated points without accumulation points in $\Omega(\Gamma)$, $C \cap (\partial\Omega \cap \Omega(\Gamma))$ must be a finite point set. Therefore

$$M = \bar{M} \setminus \{p_i\}_{i=1}^k$$

for some $k < \infty$. We have thus finished the proof of the theorem.

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