ON THE CHERN-GAUSS-BONNET INTEGRAL FOR CONFORMAL METRICS ON R^4

Sun-Yung A. Chang, Jie Qing and Paul C. Yang

Abstract. In this paper we discuss a generalized Chern-Gauss-Bonnet formula for complete conformal metrics on R^4. We show that, for a natural class of conformal metrics $e^{2w}|dx|^2$ called normal metrics on R^4 (cf. see the Definition 3.1 in this paper), one has

$$\lim_{r\to\infty} \frac{(\mathrm{vol}(\partial B_r(0)))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\,\mathrm{vol}(B_r(0))} = 1 - \frac{1}{4\pi^2}\int_{R^4} Qe^{4w}dx.$$

This generalizes a result of Cohn-Vossen, Huber and Finn for two dimensional surfaces. We also show that, if the Paneitz curvature Q of a metric $e^{2w}|dx|^2$ on R^4 is absolutely integrable, and that the scalar curvature is nonnegative at infinity, then the metric is normal.

1. Introduction

An important landmark in the theory of surfaces is the introduction of the notion of complete open surface by Hopf and Rinow ([HR]). Subsequently, Cohn-Vossen [CV] studied the Gauss-Bonnet integral for such a surface M with analytic metrics and showed that if the Gaussian curvature K is absolutely integrable, then

$$\int_M K dv_M \leq 2\pi\chi(M), \tag{1.1}$$

where $\chi(M)$ is the Euler number of M. Later, Huber [Hu] extended this inequality to metrics with much weaker regularity. More importantly, he proved that such surface M is conformally equivalent to a closed surface with finitely many punctures. This result has been generalized to dimension four four complete manifolds whose sectional curvature is positive outside a compact set by Greene-Wu ([GW]).

The deficit in formula (1.1) has an interpretation as an isoperimetric constant. On a complete and open surface with Gaussian curvature absolutely integrable, one

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may represent each end conformally as $R^2 \setminus K$ for some compact set K and consider the following isoperimetric ratio:

$$\nu = \lim_{r \to \infty} \frac{L^2(r)}{4\pi A(r)}$$

where $L(r)$ is the length of the boundary circle $\partial B_r = \{|x| = r\}$, and $A(r)$ the area of the annular region $B(r) \setminus K$. For a fairly large class of complete surfaces Finn ([F]) showed that,

$$\chi(M) - \frac{1}{2\pi} \int_M K dv_M = \sum \nu_j, \tag{1.2}$$

where the sum is taken over each end of M. For more recent development on the subject of complete surface of finite total curvature the reader is referred to the work of Li-Tam ([LT]). Except for the work of Cheeger-Gromov ([CG]) on the Chern-Gauss-Bonnet formula for manifolds with bounded geometry, there is very little known about the situation in higher dimensions.

In this paper we consider a generalization of (1.2) in dimension four. For conformal geometry in dimension four, the Paneitz operator

$$P = \Delta^2 + \delta(\frac{2}{3} RI - 2Ric)d,$$

where δ denote the divergence, and d the differential, and R is the scalar curvature and Ric the Ricci tensor, plays the same role as the Laplacian in dimension two (cf. [P] [BCY] [CQ], for example). The Paneitz operator enjoys the following invariance property under conformal change of metric $g = e^{2w} g_0$; the Paneitz operator transforms by $P_g = e^{-4w} P_{g_0}$. For the conformal metric $g = e^{2w} g_0$, the Paneitz operator applied to the conformal factor w calculates a fourth order curvature invariant Q:

$$P_{g_0} w + 2Q_{g_0} = 2Q_g e^{4w} \tag{1.3}$$

where

$$Q = \frac{1}{12}\{-\Delta R + \frac{1}{4} R^2 - 3|E|^2\} \tag{1.4}$$

and E is the traceless Ricci tensor. The Q curvature invariant is related to the Chern-Gauss-Bonnet integral in dimension four:

$$\chi(M) = \frac{1}{4\pi^2} \int_M (\frac{|W|^2}{8} + Q) dV \tag{1.5}$$

where W is the Weyl tensor and M is a compact, closed 4-manifold. More generally, when the manifold has a boundary, Chang and Qing [CQ] has defined a boundary operator P_3 and its associated boundary curvature invariant T:

$$P_3 w + T_{g_0} = T_g e^{3w}. \tag{1.6}$$

Then the Chern-Gauss-Bonnet integral is supplemented by

$$\chi(M) = \frac{1}{4\pi^2}\int_M (\frac{|W|^2}{8} + Q)dV + \frac{1}{4\pi^2}\int_{\partial M}(L+T)d\Sigma, \tag{1.7}$$

where $Ld\Sigma$ is a pointwise conformal invariant.

Our motivations are twofold. Firstly we are searching for the analogue of the above mentioned result of Finn. Secondly we would like to find a geometric interpretation of the fourth order curvature invariant Q (which we call Paneitz curvature after the relation in equation (1.3)). We are fortunate to find both in one formula. Here we take the initial analytic step and study complete conformal metrics $e^{2w}|dx|^2$ on R^4. For interesting applications in global geometry it would be desirable to establish some criteria for conformal compactification, at least for locally conformally flat 4-manifolds. We will continue to study this question in a forthcoming article.

We now state our results. By solving an ODE, we first verify that

Theorem 2.7. *Suppose that w is a radial function on R^4 and $e^{2w}|dx|^2$ is a complete metric with $\int_{R^4}|Q|e^{4w}dx < \infty$. And suppose that its scalar curvature is nonnegative at infinity. Then*

$$\int_{R^4} Qe^{4w}dx \leq 4\pi^2\chi(R^4) = 4\pi^2, \tag{1.8}$$

and

$$1 - \frac{1}{4\pi^2}\int_{R^4} Qe^{4w}dx = \lim_{r\to\infty}\frac{(vol(\partial B_r(0)))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}vol(B_r(0))}. \tag{1.9}$$

Remark 1.1

1. By non-negativity of scalar curvature at infinity we mean that the scalar curvature $R(x)$ is nonnegative when $|x|$ is large enough. It is easy to see that without the additional condition on the scalar curvature Theorem 2.7 does not hold. For instance, suppose that $e^{2v}|dx|^2$ is the standard metric for the 4-sphere S^4, then the metric $e^{2(v+r^2)}|dx|^2$ is a complete metric on R^4 but

$$\begin{aligned}\frac{1}{4\pi^2}\int_{R^4} Q_{v+r^2}e^{4(v+r^2)}dx =& \frac{1}{4\pi^2}\int_{R^4}\Delta^2(v+r^2)dx \\ = \frac{1}{4\pi^2}\int_{R^4}\Delta^2 v = \frac{1}{4\pi^2}&\int_{R^4} Q_{S^4}e^{4v}dx = \frac{3|S^4|}{4\pi^2} = 2 > 1.\end{aligned}$$

2. For domains in R^n for $n > 2$, it turns out that there are more than one isoperimetric constants to be considered. These isoperimetric constants can be defined as the ratio of pairs of "mixed volume" ([BZ] [T]) of the domain. It turns out that in our setting, after suitable normalization, all these isoperimetric ratios tend to the same limit as in (1.9), and we have established a generalized form of Theorem 2.7 above (and also Theorems 3.8 and 4.1 as stated below) with respect to all these isoperimetric ratios.

In his attempt to prove (1.2) for any complete open surface with absolutely integrable Gaussian curvature, Finn [F] has found an interesting intermediate class of metrics for which (1.2) holds. That is the class of normal metrics, which are the conformal metric $e^{2w}|dx|^2$ where w is a constant plus a potential (see Definition 3.1). We found the same phenomenon in four dimension.

Theorem 3.8. *Suppose that the metric $e^{2w}|dx|^2$ is a complete and normal metric on R^4, then both (1.8) and (1.9) hold.*

The notion of normal metrics arises naturally. In the recent work of several authors ([CL], [CLi], [CY], [Lin] and [Xu]), they have all pointed out that it is essential to verify that a metric is normal in order to control the asymptotic behavior for metrics at infinity. Fortunately we are able to show that non-negativity of the scalar curvature (at infinity) ensures that the metric is normal.

Theorem 4.1. *Suppose that the Paneitz curvature of a metric $e^{2w}|dx|^2$ on R^4 is absolutely integrable and suppose that its scalar curvature is nonnegative at infinity, then the metric is normal.*

We believe that one also can verify a metric to be normal if one assumes some decay conditions of the Paneitz curvature such as those given in [CLi]. In the meanwhile our Theorem 4.1 sheds some light on estimating asymptotic behavior for a metric $e^{2w}|dx|^2$ on R^4 under appropriate geometric conditions.

Finally, combining Theorem 3.8 and Theorem 4.1 we have our main theorem:

Main Theorem. *Suppose that $e^{2w}|dx|^2$ on R^4 is a complete metric with its Paneitz curvature Q absolutely integrable, and suppose that its scalar curvature is nonnegative at infinity. Then both (1.8) and (1.9) hold.*

The paper is organized as follows: in Section 2, we will study the radially symmetric case. Our approach is to first find a particular and good solution to the ODE, then to control the asymptotic behavior of the given metric. In Section 3, we will prove Theorem 3.1. Our strategy is to reduce the problem to a radially symmetric one, then to apply the result in Section 2. In Section 4, we will prove Theorem 4.1 and hence our main theorem.

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2. Symmetric Case

Consider a conformal metric $(R^4, e^{2w}|dx|^2)$. We assume that the Paneitz curvature Q of the metric:

$$\Delta^2 w = 2Qe^{4w} \quad \text{on } R^4 \tag{2.1}$$

satisfies

$$\int_{R^4} |Q|e^{4w} dx < \infty. \tag{2.2}$$

For convenience, we often use cylindrical coordinates $|x| = r = e^t$ to rewrite the equation (2.1) as:

$$(\frac{\partial^2}{\partial t^2} - 2\frac{\partial}{\partial t})(\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t})(w+t) = 2Qe^{4(w+t)} \tag{2.3}$$

i.e.

$$(\frac{\partial^2}{\partial t^2} - 2\frac{\partial}{\partial t})(\frac{\partial^2}{\partial t^2} + 2\frac{\partial}{\partial t})v = 2Qe^{4v}, \quad -\infty < t < \infty, \tag{2.4}$$

when we denote $w+t$ by v.

In contrast to the situation on surfaces, where there is one isoperimetric ratio to consider, there exist a family of isoperimetric inequalities for mixed volumes when the dimension of the manifold is greater than 2 (c.f [BZ],[T]). In the case when Ω is a convex domain in R^n, following Trudinger [T], we define for each $1 \le m < n$, the mixed volume

$$V_m(\Omega) = \frac{1}{n\binom{n-1}{m}} \int_{\partial\Omega} H_{n-1-m}[\partial\Omega] dH^{n-1},$$

where $H_l[\partial\Omega]$ is the l-th symmetric form of the principle curvatures of $\partial\Omega$ and H^{n-1} is the $n-1$ dimensional Hausdorff measure. For $m = n$, define $V_n(\Omega)$ = volume of Ω. Denote $\zeta_m(\Omega) = V_m^{1/m}(\Omega)$, then the isoperimetric inequalities for mixed volumes

$$\zeta_m \le \zeta_l \tag{2.5}$$

hold for all $1 \le l \le m \le n$. Motivated by these inequalities, we may similarly define in our case (when $n = 4$),

$$V_4 = \mathrm{vol}(\{s \le t\} = |S^3| \int_{-\infty}^{t} e^{4v} ds, \tag{2.6}$$

$$V_3 = \frac{1}{4} \int_{s=t} e^{3v} ds, \tag{2.7}$$

$$V_2 = \frac{1}{12} \int_{s=t} H_1 e^{3v} ds = \frac{1}{4} |S^3| v'(t) e^{2v(t)}, \tag{2.8}$$

$$V_1 = \frac{1}{12} \int_{s=t} H_2 e^{3v} ds = \frac{1}{24} \int_{s=t} (H_1^2 - \mathrm{tr} L^2) e^{3v} ds = \frac{1}{4} |S^3| (v'(t))^2 e^{v(t)}. \tag{2.9}$$

We also define the isoperimetric ratios:

$$
\begin{aligned}
C_{3,4}(t) &= \frac{V_3^{\frac{4}{3}}(t)}{(\frac{1}{2}\pi^2)^{\frac{1}{3}} V_4(t)}, \\
C_{2,3}(t) &= \frac{V_2(t)}{(\frac{1}{2}\pi^2)^{\frac{1}{3}} V_3^{\frac{2}{3}}(t)}, \\
C_{1,2}(t) &= \frac{V_1^{\frac{2}{3}}(t)}{(\frac{1}{2}\pi^2)^{\frac{1}{3}} V_2^{\frac{1}{3}}(t)},
\end{aligned}
\tag{2.10}
$$

and

$$
\begin{aligned}
C_{2,4}(t) =& C_{3,4}^{\frac{1}{3}}(t) C_{2,3}^{\frac{2}{3}}(t), \\
C_{1,3}(t) =& C_{2,3}^{\frac{1}{4}}(t) C_{1,2}^{\frac{3}{4}}(t), \\
C_{1,4}(t) =& C_{3,4}^{\frac{1}{9}}(t) C_{2,3}^{\frac{2}{9}}(t) C_{1,2}^{\frac{2}{3}}(t).
\end{aligned}
\tag{2.11}
$$

Then we have

$$
C_{2,3}(t) = C_{1,2}(t) = C_{1,3}(t) = v'(t). \tag{2.12}
$$

Furthermore, when both $V_4(t)$ and $V_3(t)$ tend to infinity as t tends to infinity, we also have, via the L'Hospital's rule,

$$
\lim_{t\to\infty} C_{3,4}(t) = \lim_{t\to\infty} C_{1,4}(t) = \lim_{t\to\infty} C_{2,4}(t) = \lim_{t\to\infty} v'(t). \tag{2.13}
$$

In the proof of Theorem 2.1 below, we have also established (2.13) for the general cases.

On the other hand, by the Chern-Gauss-Bonnet formula we have,

$$
4\pi^2 - \int_{s\le t} Qe^{4v} = \int_{s=t} Te^{3v}, \tag{2.14}
$$

where as defined in [CQ, Remark 3.1], in the special case of R^4 with cylindrical coordinate, we have

$$
Te^{3v} = P_3 v = -\frac{1}{2}v''' + 2v'. \tag{2.15}
$$

Therefore

$$
4\pi^2 - \int_{s\le t} Qe^{4v} = \frac{1}{2}|S^3|(-v'''(t) + 4v'(t)). \tag{2.16}
$$

Now to relate the total Paneitz curvature with any of the isoperimetric constants $C_{l,m}$ as defined above, we would like to show that, under suitable conditions,

$$\lim_{t\to\infty} v'''(t) = \lim_{t\to\infty} v''(t) = 0. \tag{2.17}$$

Thus we turn to study the behavior of v. For convenience, we denote $2Qe^{4v}$ by F. Equation (2.4) is equivalent to the following ODE

$$f'''' - 4f'' = F, \quad -\infty < t < \infty \tag{2.18}$$

where $\int_{-\infty}^{\infty} |F| dt < \infty$. We will first find a special solution to (2.18). Denote $f'' = C(t)e^{-2t}$, then $C(t)$ satisfies

$$C''(t) - 4C'(t) = F(t)e^{2t},$$

or equivalently

$$(C'(t)e^{4t})' = F(t)e^{-2t}.$$

Thus we can solve for $C(t)$ as:

$$\begin{aligned} C(t) &= -\int_{-\infty}^{t} e^{4x} \int_{x}^{\infty} F(y)e^{-2y} dy dx \\ &= -\frac{1}{4} e^{4t} \int_{t}^{\infty} F(x)e^{-2x} dx - \frac{1}{4} \int_{-\infty}^{t} F(x)e^{2x} dx. \end{aligned} \tag{2.19}$$

Therefore

$$f''(t) = -\frac{1}{4} e^{2t} \int_{t}^{\infty} F(x)e^{-2x} dx - \frac{1}{4} e^{-2t} \int_{-\infty}^{t} F(x)e^{2x} dx, \tag{2.20}$$

and

$$\begin{aligned} f'(t) = \frac{1}{8} \{ & e^{-2t} \int_{-\infty}^{t} F(x)e^{2x} dx - e^{2t} \int_{t}^{\infty} F(x)e^{-2x} dx \\ & + (K_1 - K_2) + \int_{t}^{\infty} F(x) dx - \int_{-\infty}^{t} F(x) dx \} \end{aligned} \tag{2.21}$$

where

$$K_1 = \lim_{t\to-\infty} e^{2t} \int_{t}^{\infty} F(x)e^{-2x} dx, \quad K_2 = \lim_{t\to\infty} e^{-2t} \int_{-\infty}^{t} F(x)e^{2x} dx. \tag{2.22}$$

Thus we obtained

$$v(t) = c_0 + c_1 t + c_2 e^{-2t} + c_3 e^{2t} + f(t) \tag{2.23}$$

for some constants c_0, c_1, c_2, c_3.

Lemma 2.1. $K_1 = K_2 = 0$.

Proof. We prove, for instance, $K_2 = 0$ first.

$$
\begin{aligned}
&e^{-2t}\int_{-\infty}^{t} F(x)e^{2x}dx \\
&=e^{-2t}\{\int_{-\infty}^{T} F(x)e^{2x}dx + \int_{T}^{t} F(x)e^{2x}dx\} \\
&\le e^{-2(t-T)}\int_{-\infty}^{\infty} |F(x)|dx + \int_{T}^{\infty} |F(x)|dx,
\end{aligned}
\tag{2.24}
$$

thus if we take $T < t$, say take $T = \frac{1}{2}t$, and let t tends to ∞, we obtain $K_2 = 0$. Similarly we have

$$
\begin{aligned}
&e^{2t}\int_{t}^{\infty} F(x)e^{-2x}dx \\
&=e^{2t}\{\int_{t}^{T} F(x)e^{-2x}dx + \int_{T}^{\infty} F(x)e^{-2x}dx\} \\
&\le \int_{-\infty}^{T} |F(x)|dx + e^{2(t-T)}\int_{-\infty}^{\infty} |F(x)|dx,
\end{aligned}
\tag{2.25}
$$

thus $K_1 = 0$ by a similar argument.

As direct consequences of Lemma 2.1, we have

Corollary 2.2.

$$\lim_{t\to\infty} f'(t) = -\frac{1}{8}\int_{-\infty}^{\infty} F(x)dx, \tag{2.26}$$

$$\lim_{t\to-\infty} f'(t) = \frac{1}{8}\int_{-\infty}^{\infty} F(x)dx \tag{2.27}$$

Lemma 2.3. $c_2 = 0$.

Proof. From (2.20) and Lemma 2.1, we have

$$\lim_{t\to-\infty} f''(t) = -\frac{1}{4}K_1,$$

which is zero. Also we have

$$v''(t) = w''_{rr}r^2 + w'_r r \tag{2.28}$$

with $r = e^t$ tends to zero as t tends to negative infinity. Thus from (2.23) we conclude that $c_2 = 0$.

Lemma 2.4. $c_1 = 1 - \frac{1}{8}\int_{-\infty}^{\infty} F(x)dx$.

Proof. Since $v'(t) \to 1$ as $t \to -\infty$, this lemma follows from (2.23) and (2.27).

We now try to eliminate the c_3 coefficient of the e^{2t} term in $v(t)$. We notice that without any further assumption in addition to Q being absolutely integrable this is not always possible. For example, we may consider $v(t) = e^{2t} + t$. Then $Q = 0$, and the metric $e^{2(v-t)}|dx|^2 = e^{2r^2}|dx|^2$ is complete on R^4.

lemma 2.5. *Suppose we assume that the scalar curvature is nonnegative at infinity, then* $c_3 = 0$.

Proof. This is a consequence of the transformation formula for the scalar curvature. Denote R the scalar curvature of the metric. Then with the cylindrical coordinate, we have

$$-v'' - (v')^2 + 1 = \frac{1}{6} Re^{2v}. \tag{2.29}$$

Thus if $R \geq 0$ at infinity, it follows from (2.23), (2.26), (2.27), (2.29), and Lemma 2.4 that $c_3 = 0$.

Remark 2.6 It is even easier to see that if $v''(t) = O(1)$ as $t \to \infty$, then c_3 has to be zero, in light of (2.23) and (2.26). We will use this observation in the proof of Theorem 3.8 in the next section.

Theorem 2.7. *Suppose that w is a radial function on R^4 and $e^{2w}|dx|^2$ is a complete metric with $\int |Q|e^{4w}dx < \infty$ and the scalar curvature nonnegative at infinity. Then*

$$\lim_{t\to\infty} v'(t) = 1 - \frac{1}{4\pi^2}\int_{R^4} Qe^{4w}dx \geq 0. \tag{2.30}$$

Moreover, we have

$$\lim_{t\to\infty} C_{l,m}(t) = 1 - \frac{1}{4\pi^2}\int_{R^4} Qe^{4w}dx, \tag{2.31}$$

for all $1 \leq l \leq m \leq 4$, where $C_{l,m}$ are as defined in (2.10) and (2.11).

Proof. Under the assumptions, we have by Lemma 2.3 and 2.4 that

$$v(t) = c_0 + c_1 t + f(t). \tag{2.32}$$

Thus we have $v''(t) = f''(t)$ which tends to zero as t tends to infinity. We also have

$$\lim_{t\to\infty} v'(t) = c_1 + \lim_{t\to\infty} f'(t) = 1 - \frac{1}{4}\int_{-\infty}^{\infty} F(x)dx = 1 - \frac{1}{4\pi^2}\int_{R^4} Qe^{4w}.$$

From the completeness of $e^{2w}|dx|^2$, we then conclude that $\lim_{t\to\infty} v'(t) \geq 0$. Thus if this limit is strictly positive, then both $V_4(t)$ and $V_3(t)$ tend to infinity as t tends to infinity; (2.31) then follows from L'Hospital's rule. On the other hand if $v'(t)$ tends to zero and $\lim_{t\to\infty} V_4(t)$ is bounded, then $\lim_{t\to\infty} e^{4v(t)} = 0$, hence $\lim_{t\to\infty} e^{kv(t)} = 0$ for $1 \leq k \leq 3$. We again have $\lim_{t\to\infty} C_{3,4}(t) = 0 = \lim_{t\to\infty} v'(t)$. Thus we have established (2.31).

3. Normal Metrics

In this section we will first define normal metrics as a generalization of the definition of normal metrics in two dimension given by Robert Finn in [F]. Then we will verify that, for a complete normal metric on R^4, the generalized Chern-Gauss-Bonnet formula (1.9) holds.

Suppose that $e^{2w}|dx|^2$ is a metric on R^4 with its Paneitz curvature Q absolutely integrable. i.e.

$$\int_{R^4} |Q|e^{4w} dx < \infty. \tag{3.1}$$

Definition 3.1. *A conformal metric $e^{2w}g_0$ satisfying (3.1) is defined to be normal if*

$$w(x) = \frac{1}{4\pi^2}\int_{R^4} \log\frac{|y|}{|x-y|} Q(y)e^{4w(y)}dy + C. \tag{3.2}$$

A natural question is: what are the normal metrics on R^4? It turns out that this is not an easy question. But we will show that a fairly large class of metrics on R^4 are normal in the following section. More precisely, we will prove that, if the scalar curvature of the metric $e^{2w}|dx|^2$ is nonnegative at infinity, then the metric $e^{2w}|dx|^2$ is normal.

As in the previous section, one defines the mixed volumes on the manifold $(R^4, e^{2w}|dx|^2)$ as follows:

$$V_4(r) = \int_{B_r} e^{4w} dx \tag{3.3}$$

$$V_3(r) = \frac{1}{4}\int_{\partial B_r} e^{3w} d\sigma(x) \tag{3.4}$$

$$V_2(r) = \frac{1}{12}\int_{\partial B_r} H_1 e^{3w} d\sigma(x) = \frac{1}{4}\int_{\partial B_r} (\frac{1}{r} + \frac{\partial w}{\partial r})e^{2w} d\sigma(x) \tag{3.5}$$

$$V_1(r) = \frac{1}{12}\int_{\partial B_r} H_2 e^{3u} d\sigma(x) = \frac{1}{4}\int_{\partial B_r} (\frac{1}{r} + \frac{\partial w}{\partial r})^2 e^{w} d\sigma(x) \tag{3.6}$$

We also define $C_{l,m}(r)$ as in the previous section.

In the following, we will adopt some techniques used in Finn's paper [F] to compare $V_m(r)$ defined in the above with $\bar{V}_m(r)$: the mixed volume for the metric $e^{2\bar{w}}|dx|^2$ which may be considered to be the average of the metric $e^{2w}|dx|^2$, where $\bar{w}$ is defined as:

$$\bar{w}(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} w(x) d\sigma(x).$$

Lemma 3.2. *Suppose that the metric $e^{2w}|dx|^2$ on R^4 is a normal metric. Then, for any number $k > 0$,*

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} e^{kw}d\sigma(x) = e^{k\bar{w}(r)}e^{o(1)}, \tag{3.7}$$

where $o(1) \to 0$ as $r \to \infty$.

Proof. Suppose $e^{2w}|dx|^2$ is a normal metric, we re-write w as

$$\begin{aligned} w(x) =& \frac{1}{4\pi^2}\int_{B_{\frac{|x|}{2}}(0)} \log\frac{|y|}{|x-y|}Q(y)e^{4w(y)}dy + C \\ &+ \frac{1}{4\pi^2}\int_{R^4\setminus B_{\frac{|x|}{2}}(0)} \log\frac{|y|}{|x-y|}Q(y)e^{4w(y)}dy \\ =& w_1(x) + w_2(x). \end{aligned} \tag{3.7}$$

Then we write

$$\begin{aligned} w_1(x) =& \frac{1}{4\pi^2}\int_{|y|\le\frac{1}{2}|x|} \log\frac{|y|}{|x|}Q(y)e^{4w(y)}dy + C \\ &+ \frac{1}{4\pi^2}\int_{|y|\le\frac{1}{2}|x|} \log\frac{|x|}{|x-y|}Q(y)e^{4w(y)}dy \\ =& f(|x|) + w_1^0(x), \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} |w_1^0(x)| \le& C\{\int_{|y|\le\eta|x|} + \int_{\eta|x|\le|y|\le\frac{1}{2}|x|}\}\log\frac{|x|}{|x-y|}|Q(y)|e^{4w(y)}dy \\ \le& C\log\frac{1}{1-\eta} + C\int_{\eta|x|\le|y|\le\frac{1}{2}|x|}|Q(y)|e^{4w(y)}dy \\ =& o(1). \end{aligned} \tag{3.9}$$

Since

$$\int_{R^4}|Q(y)|e^{4w(y)}dx < \infty, \qquad |\log\frac{|x|}{|x-y|}| \le \log 3 \tag{3.10}$$

and we may take $\eta \to 0$ while $\eta|x| \to \infty$, as $|x| \to \infty$, then $|w_1^0(x)| = o(1)$ as $|x| \to \infty$ and we have

$$w_1(x) = f(|x|) + o(1), \quad \text{for some function } f, \tag{3.11}$$

which in turn implies

$$\begin{aligned} &k\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)}(w(y) - w_2(y))d\sigma(y) \\ =& \log\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} e^{k(w(y)-w_2(y))}d\sigma(y) + o(1). \end{aligned} \tag{3.12}$$

Next we will study the term

$$\begin{aligned}
(3.13)\qquad &\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} w_2(x)d\sigma(x)\\
=&\frac{1}{8\pi^2}\int_{|y|\geq\frac{1}{2}r}\{\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)}\log\frac{|y|}{|x-y|}d\sigma(x)\}Q(y)e^{4w(y)}dy.
\end{aligned}$$

And we claim that the term

$$\begin{aligned}
(3.14)\qquad L=&|\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)}\log\frac{|y|}{|x-y|}d\sigma(x)|\\
\leq&\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)}|\log\frac{|y|}{|x-y|}|d\sigma(x)
\end{aligned}$$

is bounded for $|y|\geq\frac{1}{2}r$. To prove the claim we first rewrite L as

$$\begin{aligned}
(3.15)\qquad L\leq&\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)\setminus\{x:|x-y|\leq\theta|y|\}}|\log\frac{|y|}{|x-y|}|d\sigma(x)\\
&+\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)\bigcap\{x:|x-y|\leq\theta|y|\}}|\log\frac{|y|}{|x-y|}|d\sigma(x)\\
=&L_1+L_2,
\end{aligned}$$

where θ is chosen to be smaller than $\frac{1}{2}$, for example. Then it is easily seen that

$$(3.16)\qquad L_1\leq\log\frac{1}{\theta}$$

and

$$(3.17)\qquad L_2\leq\frac{1}{|\partial B_{\frac{r}{|y|}}(0)|}\int_{\partial B_{\frac{r}{|y|}}(0)\bigcap\{x:|x-\frac{y}{|y|}|\leq\theta\}}|\log\frac{1}{|x-\frac{y}{|y|}|}|d\sigma(x),$$

which is bounded because

$$(3.18)\qquad \frac{1}{1-\theta}\leq\frac{r}{|y|}\leq 2$$

and $\log\frac{1}{|x-\frac{y}{|y|}|}$ is certainly integrable. This proved the claim (3.14). Thus from (3.13) we conclude that

$$(3.19)\qquad \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} w_2(x)d\sigma(x)=o(1)\quad\text{as } r\to\infty$$

when Q is absolutely integrable on $(R^4, e^{2w}|dx|^2)$. Finally, we consider the term

$$(3.20)\qquad \frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)}(e^{w_2(x)}-1)d\sigma(x)=\frac{1}{|\partial B_1(0)|}\int_{\partial B_1(0)}(e^{w_2(r\sigma)}-1)d\sigma.$$

Following [F], we will estimate the term $E_M = \{\sigma \in S^3 : |w_2(r\sigma)| > M\}$. We have

$$
\begin{aligned}
M \cdot |E_M| \leq & \int_{E_M} |w_2| d\sigma \\
\leq & \frac{1}{8\pi^2} \int_{R^4 \backslash B_{\frac{r}{2}}(0)} \{ \int_{E_M} |\log \frac{|y|}{|r\sigma - y|} | d\sigma \} |Q(y)| e^{4w(y)} dy \\
= & \frac{1}{8\pi^2} \int_{|y| \geq \frac{r}{2}} H(y) |Q(y)| e^{4w(y)} dy
\end{aligned} \tag{3.21}
$$

We will estimate the function $H(y)$ for $|y| \geq \frac{r}{2}$. Again we decompose H as:

$$
\begin{aligned}
H(y) = & \int_{E_M \backslash \{\sigma : |r\sigma - y| \leq \frac{1}{3}|y|\}} |\log \frac{|y|}{|r\sigma - y|}| d\sigma \\
& + \int_{E_M \bigcap \{\sigma : |r\sigma - y| \leq \frac{1}{3}|y|\}} |\log \frac{|y|}{|r\sigma - y|}| d\sigma \\
= & H_1 + H_2
\end{aligned} \tag{3.22}
$$

Clearly

$$
H_1 \leq \log 3 \cdot |E_M|. \tag{3.23}
$$

It is more difficult to estimate the term H_2. To do this, we observe that for $|r\sigma - y| \leq \frac{1}{3}|y|$ we have

$$
\begin{aligned}
|\log \frac{|y|}{|r\sigma - y|}| \leq & |\log \frac{|y|}{r}| + |\log |\sigma - \frac{y}{r}|| \\
\leq & \log \frac{3}{2} + |\log |\sigma - \frac{y}{r}||.
\end{aligned} \tag{3.24}
$$

Thus we may estimate H_2 when E_M is a three dimensional disk centered at $\frac{y}{r}$, perpendicular to y, and of the size $|E_M|$ as:

$$
H_2 \leq C \cdot |E_M| + C|E_M| \cdot \log \frac{1}{|E_M|} \leq C(1 + \log \frac{1}{|E_M|})|E_M|. \tag{3.25}
$$

Combining (3.21), (3.22), (3.23) and (3.25), we obtain

$$
M \leq o(1)(1 + \log \frac{1}{|E_M|}), \tag{3.26}
$$

where $o(1) \to 0$ as $r \to \infty$. (3.26) then implies

$$
|E_M| \leq Ce^{-\frac{M}{o(1)}}. \tag{3.27}
$$

Thus

$$
\begin{aligned}
&|\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)}(e^{kw_2(x)}-1)d\sigma(x)| \\
(3.28)\qquad &= \frac{k}{|\partial B_1(0)|}\int_{-\infty}^{+\infty}(e^{kM}-1)|E_M|dM \\
&= o(1). \quad \text{as } r\to\infty
\end{aligned}
$$

Combining (3.12), (3.19) and (3.28) we arrive at

$$\frac{k}{|\partial B_r(0)|}\int_{\partial B_r(0)} w(x)d\sigma(x) = \log\frac{1}{|\partial B_r(0)|}\int_{\partial B_r(0)} e^{kw(x)}d\sigma(x)+o(1),$$

or equivalently,

$$\int_{\partial B_r(0)} e^{kw(x)}d\sigma(x) = e^{o(1)}|\partial B_r(0)|e^{k\bar{w}(r)}.$$

Thus we have established (3.7).

As a direct consequence we have

Corollary 3.3. *Suppose that the metric $e^{2w}|dx|^2$ on R^4 is a normal metric. Then*

$$V_3(r)=\bar{V}_3(r)(1+o(1)), \tag{3.29}$$

and

$$\frac{d}{dr}V_4(r)=(\frac{d}{dr}\bar{V}_4(r))(1+o(1)) \tag{3.30}$$

To deal with V_1, V_2, we will first establish the following technical lemma:

Lemma 3.4. *Suppose that $e^{2w}|dx|^2$ is a normal metric on R^4. Then*

$$\frac{1}{|\partial B_r|}\int_{\partial B_r}(\frac{\partial w}{\partial r})^k d\sigma(x) = O(\frac{1}{r^k}), \quad \textit{for } k=1,2,3, \tag{3.31}$$

and

$$\frac{1}{|\partial B_r|}\int_{\partial B_r}(\frac{\partial w}{\partial r})^2 d\sigma(x) = (\frac{\partial \bar{w}}{\partial r})^2(r)+o(\frac{1}{r^2}) \tag{3.32}$$

Proof. We observe that

$$\frac{\partial \bar{w}}{\partial r}(r)=\frac{1}{|\partial B_r|}\int_{\partial B_r}\frac{\partial w}{\partial r}d\sigma(x).$$

To prove (3.31), we have for a normal metric $e^{2w}|dx|^2$,

$$\frac{\partial w}{\partial r}(x) = \int_{R^4} K(x,y)F(y)dy, \tag{3.33}$$

where $K(x,y) = \frac{1}{8\pi^2|x|}\frac{|x|^2 - x\cdot y}{|x-y|^2}$ and $F(y) = 2Q(y)e^{4w(y)}$ which is integrable over R^4. Then

$$\begin{aligned}
&\frac{1}{|\partial B_r|}\int_{\partial B_r}(\frac{\partial w}{\partial r})^k d\sigma(x)\\
&\quad \le \{\int_{R^4}\frac{1}{|\partial B_r|}\int_{\partial B_r}|K(x,y)|^k d\sigma(x)|F(y)|dy\}\cdot\{\int_{R^4}|F(y)|dy\}^{k-1}
\end{aligned}$$

Therefore to prove (3.31) it suffices to prove

$$\frac{1}{|\partial B_r|}\int_{\partial B_r}|K(x,y)|^k d\sigma(x) = O(\frac{1}{r^k}) \ \text{ for all } y \in R^4. \tag{3.34}$$

To verify (3.34), we write

$$2(|x|^2 - x\cdot y) = |x-y|^2 + |x|^2 - |y|^2.$$

Therefore it leads to verify that

$$\frac{1}{|\partial B_r|}\int_{\partial B_r}\frac{||x|^2 - |y|^2|^3}{|x-y|^6} \le C, \tag{3.35}$$

for some constant independent of $y \in R^4$. By the homogeneity of the integrand, we only need to consider, for any $y \in R^4$,

$$\int_{\partial B_1}\frac{||x|^2 - |y|^2|^3}{|x-y|^6}d\sigma(x) \le C. \tag{3.36}$$

It is easily seen that (3.36) holds if $|y| \le 1-\delta$ or $|y| \ge 1-\delta$, for any given $\delta > 0$. But for $|y| \in (1-\delta, 1)\cup(1, 1+\delta)$, we have the following calculus inequality for any $N \ge 3$:

$$\int_{\partial B_1}\frac{d\sigma(x)}{|x-y|^N} \le \frac{C}{|1-|y||^{N-3}}. \tag{3.37}$$

We have thus established (3.36), hence (3.34) and (3.31). To prove (3.32), we note that, since $\frac{1}{|x-y|^2}$ is the Green's function for the Laplacian on R^4:

$$\frac{1}{|S_r^3|}\int_{S_r^3}\frac{1}{|x-y|^2}d\sigma(x) = \begin{cases}\dfrac{1}{r^2} & \text{when } |y| < r\\ \dfrac{1}{|y|^2} & \text{when } |y| > r.\end{cases} \tag{3.38}$$

It follows that we can write

$$\frac{\partial \bar{w}}{\partial r}(r) = \int_{R^4} \bar{K}(x,y)F(y)dy,$$

where

$$\bar{K}(x,y) = \begin{Bmatrix} \dfrac{r}{16\pi^2|y|^2} & \text{when } |y| > r \\ \dfrac{1}{16\pi^2 r}(2 - \dfrac{|y|^2}{r^2}) & \text{when } |y| < r. \end{Bmatrix} \tag{3.39}$$

Thus

$$\begin{aligned} &\frac{1}{|\partial B_r|}\int_{\partial B_r} |(\frac{\partial w}{\partial r})^2 - (\frac{\partial \bar{w}}{\partial r})^2| d\sigma(x) \\ &\leq \int_{R^4} |F|(y)dy \cdot \int_{R^4} (\frac{1}{|\partial B_r|}\int_{\partial B_r} |K(x,y) - \bar{K}(x,y)|^2 d\sigma(x))|F|(y)dy. \end{aligned}$$

We again compute

$$|K(x,y) - \bar{K}(x,y)| = \begin{cases} \dfrac{1}{16\pi^2|x|}|\dfrac{(|y|^2 - |x|^2)(|x-y|^2 - |y|^2)}{|x-y|^2|y|^2}|, & \text{for } |y| > |x|, \\ \dfrac{1}{16\pi^2|x|}|\dfrac{(|y|^2 - |x|^2)(|x-y|^2 - |x|^2)}{|x-y|^2|x|^2}| & \text{for } |y| < |x|. \end{cases}$$

Therefore

$$\frac{1}{|\partial B_r|}\int_{\partial B_r} |K(x,y) - \bar{K}(x,y)|^2 d\sigma(x) = O(\frac{1}{|x|^2}) \text{ as } |x| \to \infty \text{ and } |y| > |x|^{\frac{1}{2}}, \tag{3.40}$$

and

$$\begin{aligned} &|K(x,y) - \bar{K}(x,y)| \leq C|x||\frac{1}{|x-y|^2} - \frac{1}{|x|^2}| \\ &= o(\frac{1}{|x|}) \text{ as } |x| \to \infty \text{ and } |y| \leq |x|^{\frac{1}{2}}. \end{aligned} \tag{3.41}$$

Combining (3.39), (3.40), and (3.41), we obtain

$$\begin{aligned} &\frac{1}{|\partial B_r|}\int_{\partial B_r} |(\frac{\partial w}{\partial r})^2 - (\frac{\partial \bar{w}}{\partial r})^2| d\sigma(x) \\ &= \int_{|y|>|x|^{\frac{1}{2}}} O(\frac{1}{|x|^2})|F|(y)dy + \int_{|y|\leq|x|^{\frac{1}{2}}} o(\frac{1}{|x|^2})|F|(y)dy = o(\frac{1}{|x|^2}). \end{aligned}$$

This establishes (3.32).

Now we are ready to estimate $V_1(r)$ and $V_2(r)$.

Lemma 3.5. *Suppose $e^{2w}|dx|^2$ is a normal metric on R^4. And suppose that*

$$\lim_{r\to\infty}(1+r\frac{\partial\bar{w}}{\partial r})$$

exists and is positive. Then

$$V_2(r)=\bar{V}_2(r)(1+o(1)) \quad \textit{as } r\to\infty \tag{3.42}$$

and

$$V_1(r)=\bar{V}_1(r)(1+o(1)) \quad \textit{as } r\to\infty. \tag{3.43}$$

Proof. Let us denote by $a=\frac{\partial\bar{w}}{\partial r}(r)$, and $b=e^{\bar{w}}(r)$. Then

$$\bar{V}_2(r)=\frac{1}{4}|\partial B_r|(\frac{1}{r}+a)b^2,$$

and

$$V_2(r)-\bar{V}_2(r)=\frac{1}{4}(\frac{1}{r}+a)\int_{\partial B_r}(e^{2w}-b^2)+\frac{1}{4}\int_{\partial B_r}(\frac{\partial w}{\partial r}-a)(e^{2w}-b^2).$$

Applying Lemma 3.2 and Lemma 3.4, we have

$$\begin{aligned} V_2(r)-\bar{V}_2(r)=&|\partial B_r|(\frac{1}{r}+a)b^2o(1)\\ &+(\int_{\partial B_r}(\frac{\partial w}{\partial r}-a)^2)^{\frac{1}{2}}\cdot(\int_{\partial B_r}(e^{2w}-b^2)^2)^{\frac{1}{2}}\\ =&\bar{V}_2(r)o(1)+|\partial B_r|\frac{1}{r}b^2o(1)\end{aligned} \tag{3.44}$$

which implies (3.42) under the assumption that $\lim_{r\to\infty}(1+ra)>0$. Similarly,

$$\bar{V}_1(r)=\frac{1}{4}|\partial B_r|(\frac{1}{r}+a)^2b,$$

and

$$\begin{aligned} V_1(r)-\bar{V}_1(r)=&\frac{1}{4}\int_{\partial B_r}(\frac{1}{r}+a)^2(e^w-b)\\ &+\frac{1}{2r}\int_{\partial B_r}(\frac{\partial w}{\partial r}-a)(e^w-b)+\frac{1}{4}\int_{\partial B_r}((\frac{\partial w}{\partial r})^2-a^2)e^w.\end{aligned}$$

Applying Hölder inequality, Lemma 3.2, and Lemma 3.4, we again obtain

$$V_1(r)-\bar{V}_1(r)=\bar{V}_1(r)o(1)+|\partial B_r|\frac{b}{r^2}o(1),$$

which again implies (3.43) under the assumption that $\lim_{r\to\infty}(1+ra) > 0$. So the Lemma is proved.

We now discuss the metric which is an average of the metric $e^{2w}|dx|^2$ on R^4 over the spheres $\partial B_r(0)$. For convenience, we would use the cylindrical coordinates again. Then $\bar{w}(r) = \bar{w}(e^t)$, and $v(t) = \bar{w}(r) + t$. As we have verified in Section 2, v satisfies:

$$v'''' - 4v'' = \frac{2}{|\partial B_r(0)|}\int_{\partial B_r(0)} Qe^{4v}\,d\sigma(x) = F(t), \quad -\infty < t < \infty \tag{3.45}$$

with

$$\int_{-\infty}^{\infty} F(t)dt = \frac{2}{|S^3|}\int_{R^4} Qe^{4w}dx = \frac{1}{\pi^2}\int_{R^4} Qe^{4w}dx < \infty \tag{3.46}$$

and

$$\int_{-\infty}^{\infty} |F(t)|dt \le \frac{1}{\pi^2}\int_{R^4} |Q|e^{4w}dx < \infty. \tag{3.47}$$

Lemma 3.6. *Suppose that $(R^4, e^{2w}|dx|^2)$ is a complete normal metric. Then its averaged metric $(R^4, e^{2\bar{w}(r)}|dx|^2)$ is also a complete metric.*

Proof. This is basically a consequence of Lemma 3.2. By the argument given in the proof of Lemma above we have

$$\frac{1}{|S^3|}\int_{S^3} e^{w(r\sigma)}d\sigma = e^{\bar{w}(r)}\cdot e^{o(1)}, \tag{3.48}$$

where $o(1) \to 0$ as $r \to \infty$. So

$$\frac{1}{|S^3|}\int_{S^3}\int_{r_0}^{r_1} e^{w(r\sigma)}dr\,d\sigma = \int_{r_0}^{r_1}\frac{1}{|S^3|}\int_{S^3} e^{w(r\sigma)}d\sigma\,dr = \int_{r_0}^{r_1} e^{\bar{w}(r)}\cdot e^{o(1)}\,dr, \tag{3.49}$$

which proves the Lemma.

Recall that in the statement of Theorem 2.7 we have assumed that the sign of the scalar curvature to be positive in order to establish the fact that the coefficient of the e^{2t} term in the expression (2.23) of the conformal factor v vanishes. We have also in Remark 2.6 pointed out that, to achieve the same purpose, it suffices to prove that $v''(t) = O(1)$ as t tends to infinity. We will now establish this later fact for the average metric of a normal metric.

Lemma 3.7. *Suppose that $(R^4, e^{2w}|dx|^2)$ is a normal metric. Then*

$$\Delta\bar{w}(r) \le \frac{C}{r^2}. \tag{3.50}$$

Proof. We compute

$$\Delta \bar{w}(r) = \Delta \frac{1}{|S^3|} \int_{S^3} w(r\sigma) d\sigma = \frac{1}{|S^3_r|} \int_{S^3_r} \Delta w(x)\, d\sigma(x). \tag{3.51}$$

Then, by the assumption that the metric is normal, we have

$$\Delta \bar{w}(r) = \frac{1}{4\pi^2} \int_{R^4} \{ \frac{2}{|S^3_r|} \int_{S^3_r} \frac{1}{|x-y|^2}\, d\sigma(x) \} Q(y) e^{4w(y)} dy. \tag{3.52}$$

Thus it follows from (3.39) and (3.52) that

$$\Delta \bar{w}(r) \leq \frac{1}{2\pi^2 r^2} \int_{R^4} |Q| e^{4w} dy. \tag{3.53}$$

We have thus established the lemma.

Theorem 3.8. *Suppose that* $(R^4, e^{2w}|dx|^2)$ *is a complete normal metric. Then*

$$\int_{R^4} Q e^{4w} dx \leq 4\pi^2,$$

and

$$\lim_{r\to\infty} \frac{(\int_{\partial B_r(0)} e^{3w} d\sigma(x))^{\frac{4}{3}}}{\int_{B_r(0)} e^{4w} dx} = 1 - \frac{1}{4\pi^2} \int_{R^4} Q e^{4w} dx \geq 0. \tag{3.54}$$

Furthermore, if

$$(1 - \frac{1}{4\pi^2} \int_{R^4} Q e^{4w} dx) > 0, \tag{3.55}$$

then

$$\lim_{r\to\infty} C_{l,m}(r) = 1 - \frac{1}{4\pi^2} \int_{R^4} Q e^{4w} dx, \tag{3.56}$$

for all $1 \leq l, m \leq 4$.

Proof. We will follow the outline of the proof as indicated at the beginning of this section. We first look at the averaged metric $e^{2\bar{w}(r)}|dx|^2$. Apply Lemma 3.6, Lemma 3.7, and Remark 2.6 in last section we have

$$\lim_{r\to\infty} r \frac{\partial \bar{w}}{\partial r}(r) = -\frac{1}{2|S^3|} \int_{R^4} Q e^{4w} dx \geq -1. \tag{3.57}$$

We then apply Corollary 3.3 and get

$$\lim_{r\to\infty} \frac{(\int_{\partial B_r(0)} e^{3w} d\sigma(x))^{\frac{4}{3}}}{\int_{B_r(0)} e^{4w} dx} = \lim_{r\to\infty} \frac{|S^3|^{\frac{4}{3}} e^{4\bar{w}} r^4}{V(r)}. \tag{3.58}$$

Thus it follows from (3.29) and (3.30), with a proof similar to the proof of Theorem 2.7, that (3.54) holds. By the same reasoning, (3.56) follows from Lemma 3.5 under the condition of (3.55).

Remark 3.9

In the statements of Theorem 3.8 above, we have established (3.54) without the additional assumption that (3.55) holds. But we have established the further isoperimetric equalities (3.56) for higher order mixed volumes only under the additional assumption (3.55). In view of the fact that the classical isoperimetric inequalities (2.5) hold in general only for convex bodies for domains in R^n, it remains an interesting problem to see if (3.56) hold even in the degenerate case when the constant in (3.55) is zero.

4. Proof of Main Theorems

In this section we will show that a metric $(R^4, e^{2w}|dx|^2)$ is a normal metric if its scalar curvature is nonnegative at infinity. It follows from the result in the last section that for a fairly large class of conformal metrics on R^4 we have the generalized Chern-Gauss-Bonnet formula (3.54) and (3.56) in Theorem 3.8. Before we state and prove our main result, we first remark that in general a complete conformal metric with integrable Paneitz curvature on R^4 may not be a normal metric. To see this, we can use the same example as having been discussed at the beginning of section 2. Recall that, in cylindrical coordinate, denote $v = e^{2t} + t$; then $(R^4, e^{4v}|dx|^2)$ is a complete metric with Paneitz curvature $Q = 0$. It is also clearly not a normal metric. This justifies the additional assumption in the statement of Theorem 4.1 below.

Theorem 4.1. *Suppose the metric $(R^4, e^{2w}|dx|^2)$ is a complete metric with the Paneitz curvature Q integrable, and the scalar curvature nonnegative at infinity. Then it is a normal metric.*

Proof. First denote

$$v(x) = \frac{3}{2\pi^2} \int_{R^4} \log \frac{|y|}{|x-y|} Q e^{4w} dy \tag{4.1}$$

and let $h = w - v$. We will now show that the biharmonic function h on R^4 is a constant. Recall the transformation formula for the scalar curvature

$$\Delta w + |\nabla w|^2 = -\frac{R}{6} e^{2w} \tag{4.2}$$

where R is the scalar curvature for the metric $(R^4, e^{2w}|dx|^2)$. Notice that Δh is a harmonic function. Thus

$$\begin{aligned} \Delta h(x_0) &= \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} \Delta h d\sigma \\ &= -\frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} (|\nabla w|^2 + \frac{R}{6}) d\sigma - \frac{1}{|\partial B_r(x_0)|} \int_{\partial B_r(x_0)} \Delta v d\sigma. \end{aligned} \tag{4.3}$$

The first term on the right of (4.3) is nonpositive when r is large enough by our assumption that R is nonnegative. We now observe that, by an argument similar to the proof of Lemma 3.7 in Section 3, we have

$$\begin{aligned}\int_{\partial B_r(x_0)} \Delta v d\sigma =& \frac{3}{\pi^2}\int_{R^4}\{\frac{1}{|S^3|}\int_{S^3}\frac{1}{|r\sigma + x_0 - y|^2}d\sigma\}Q(y)e^{4w(y)}dy.\\ \le& \frac{3}{\pi^2 r^2}\int_{R^4}|Q|e^{4w}dy.\end{aligned} \tag{4.4}$$

Therefore, by taking $r\to\infty$, we have, for each $x_0 \in R^4$,

$$\Delta h(x_0) \le 0. \tag{4.5}$$

Thus $\Delta h = C_0$ for some non-positive constant by Liouville Theorem for harmonic functions. Thus, any partial derivative of h is harmonic, i.e.

$$\Delta h_{x_i} = 0. \tag{4.6}$$

Apply the mean value theorem again, we have

$$|h_{x_i}(x_0)|^2 = |\frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)} h_{x_i} d\sigma|^2 \le \frac{1}{|\partial B_r(x_0)|}\int_{\partial B_r(x_0)}|\nabla h|^2 d\sigma. \tag{4.7}$$

But

$$|\nabla h|^2 \le 2|\nabla w|^2 + 2|\nabla v|^2 = -2C_0 - \frac{R}{3}e^{2w} + 2|\nabla v|^2 \tag{4.8}$$

and

$$|\nabla v|^2 \le C(\int_{R^4}\frac{1}{|x-y|^2}|Q|e^{4w}dy)(\int_{R^4}|Q|e^{4w}dy) \le C\int_{R^4}\frac{1}{|x-y|^2}|Q|e^{4w}dy. \tag{4.9}$$

Similarly, we conclude that, for each $x_0 \in R^4$,

$$|h_{x_i}(x_0)|^2 \le -2C_0, \tag{4.10}$$

which implies that all partial derivatives of h are constants. Then $\Delta h = C_0 = 0$, which finally implies that all partial derivatives of h vanish by (4.10). Thus h is a constant.

Main Theorem. *Suppose that $(R^4, e^{2w}|dx|^2)$ is a complete metric with its Paneitz curvature absolutely integrable, and its scalar curvature nonnegative at infinity. Then conclusions in Theorem 3.1 hold. In particular,*

$$\lim_{r\to\infty}\frac{(\int_{\partial B_r(0)} e^{3w}d\sigma(x))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}}\int_{B_r(0)} e^{4w}dx} = 1 - \frac{1}{4\pi^2}\int_{R^4} Qe^{4w}dx \ge 0. \tag{4.11}$$

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Sun-Yung Alice Chang, Department of Mathematics, Princeton University, Princeton, NJ 08544 & Department of Mathematics, UCLA, Los Angeles, CA 90095.
E-mail address: `chang@math.princeton.edu`

Jie Qing, Department of Mathematics, University of California, Santa Cruz, Santa Cruz, CA 95064.
E-mail address: `qing@math.ucsc.edu`

Paul Yang, Department of Mathematics, University of Southern California, Los Angeles, CA 90089.
E-mail address: `pyang@math.usc.edu`