ON A FOURTH ORDER PDE IN CONFORMAL GEOMETRY

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Dedicated to Professor A. P. Calderón

§0. Introduction

In this paper, we will survey some of the recent study of a fourth order partial differential operator - namely the Paneitz operator. The study of this operator arises naturally from the consideration of problems in conformal geometry. However the natural partial differential equations associated with the operator, which when restricted to domains in $\mathbb{R}^n$ becomes the bi-Laplacian operator, also are interesting in themselves. The content of this paper is an expanded version of a talk the author gave in the conference to honor Professor A.P. Calderón on the occasion of his seventieth birthday.

On a Riemannian manifold (M^n, g) of dimension n, a most well-studied differential operator is the Laplace Beltrami operator $\Delta = \Delta_g$ which in local coordinate is defined as

$$\Delta = \frac{1}{\sqrt{|g|}} \sum \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^i} \right)$$

where $g = (g_{ij})$, $g^{ij} = (g_{ij})^{-1}$, $|g| = \det(g_{ij})$. On the same manifold, if we change the metric g to a new metric h, we say h is conformal to g if there exists some positive function ρ such that $h = \rho g$. Denote $\rho = e^{2\omega}$, then $g_\omega = e^{2w} g$ is a metric conformal to g. When the dimension of the manifold M^n is two, Δ_{g_w} is related to Δ_g by the simple formula:

$$\Delta_{g_w}(\varphi) = e^{-2\omega} \Delta_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^2)\,. \tag{0.1}$$

When dimension of M^n is greater than two, an operator which enjoys property similar to (0.1) is the conformal Laplacian operator $L \equiv -c_n \Delta + R$ where $c_n = \frac{4(n-1)}{n-2}$ and R is the scalar curvature of the metric. We have

$$L_{g_w}(\varphi) = e^{-\frac{n+2}{2}\omega} L_g \left(e^{\frac{n-2}{2}\omega} \varphi \right) \tag{0.2}$$

for all $\varphi \in C^\infty(M)$.

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In general, we call a metrically defined operator A conformally covariant of bidegree (a,b), if under the conformal change of metric $g_\omega = e^{2\omega} g$, the pair of corresponding operators A_ω and A are related by

$$A_\omega(\varphi) = e^{-b\omega} A(e^{a\omega}\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^n)\,. \tag{0.3}$$

It turns out that there are many operators besides the Laplacian Δ on compact surfaces and the conformal Laplacian L on general compact manifold of dimension greater than two which have the conformal covariant property. A particularly interesting one is a fourth order operator on 4-manifolds discovered by Paneitz [Pa] in 1983:

$$P\varphi \equiv \Delta^2\varphi + \delta\left(\frac{2}{3}RI - 2\mathrm{Ric}\right) d\varphi \tag{0.4}$$

where δ denotes the divergence, d the deRham differential and Ric the Ricci tensor of the metric. The *Paneitz* operator P (which we will later denote by P_4) is conformal covariant of bidegree $(0,4)$ on 4-manifolds, i.e.

$$P_{g_\omega}(\varphi) = e^{-4\omega} P_g(\varphi) \quad \text{for all} \quad \varphi \in C^\infty(M^4)\,. \tag{0.5}$$

For manifold of general dimension n, when n is even, the existence of a n-th order operator P_n conformal covariant of bidegree $(0,n)$ was verified in [GJMS]. However it is only explicitly known on the standard Euclidean space $\mathbb{R}^n$ and hence on the standard sphere S^n. The explicit formula for P_n on standard sphere S^n has appeared in Branson [Br-1] and independently in Beckner [B] and will be discussed in section 2 below.

This paper is organized as follows. In section one, we will list some properties of the Laplacian operator and compared it, from the point of view of conformal geometry, to some analogous properties of the Paneitz operator P_4. In section 2, we will discuss some natural PDE associated with the Paneitz operators P_n on S^n. We will discuss extremal properties of the variational functional associated with the PDE. We will also discuss some uniqueness property of the solutions of the PDE. The main point here is that although P_n is an n-th order elliptic operator (which in the case when n is odd, is a pseudo-differential operator), moving plane method of Alexandrov [Al] and Gidas-Ni-Nirenberg [GNN] can still be applied iteratively to establish the spherical symmetry of the solutions of the PDE. In section 3, we discuss some general existence and uniqueness results of the corresponding functional for the Paneitz operator P_4 on general compact 4-manifolds. In section 4, we will discuss another natural geometric functional-namely the zeta functional determinant for the conformal Laplacian operator- where Paneitz operator plays an important role. We will survey some existence and regularity results of the extremal metrics of the zeta functional determinant, and indicate some recent geometric applications by M. Gursky of the extremal metrics to characterize some compact 4-manifolds. Finally in section 5, we will discuss some existence result for

P_3 operator, which is conformally covariant of bidegree $(0,3)$, operating on functions defined on the boundary of compact 4-manifolds. The existence of P_3 would allow us to study boundary value problems associated with the P_4 operator. Our point of view here is that, the relation of P_3 to P_4 is parallel to the relation of the Neumann operator to the Laplacian operator. Thus, for examples, for domains in $\mathbb{R}^4$, P_3 is a higher-dimensional analogue with respect to the bi-harmonic functions to that of the Dirichlet-Neumann operator with respect to the harmonic functions. Some understanding of the P_3 operator also leads to the study of zeta functional determinant on 4-manifolds with boundary.

In this paper we are dealing with an area of research which can be approached from many different directions. Thus many important research work should be cited. But due to the limited space and the very limited knowledge of the author, we will survey here mainly some recent work of the author with colleagues: T. Branson, M. Gursky, J. Qing, L. Wang and P. Yang; we will mainly only cite research papers which has strongly influenced our work.

§1. Properties of the Paneitz operator

On a compact Riemannian manifold (M^n, g) without boundary, when the dimension of the manifold $n = 2$, we denote $P_2 \equiv -\Delta = -\Delta_g$, the Laplacian operator. When the dimension $n = 4$, we denote $P = P_4$ the Paneitz operator as defined on (0.4). Thus both operators satisfy conformal covariant property $(P_n)_\omega = e^{-n\omega} P_n$, where $(P_n)_\omega$ denote the operator with respect to (M^n, g_ω), $g_\omega = e^{2\omega} g$. Other considerations in conformal geometry and partial differential equation also identifies P_4 as a natural analogue of $-\Delta$. Here we will list several such properties for comparison.

(i) On a compact surface, a natural curvature invariant associated with the Laplace operator is the Gaussian curvature K. Under the conformal change of metric $g_\omega = e^{2\omega} g$, we have

$$\Delta\omega + K_\omega e^{2\omega} = K \quad \text{on} \quad M^2 \tag{1.1$_a$}$$

where K_ω denotes the Gaussian curvature of (M^2, g_ω). While on 4-manifold, we have

$$-P_4\omega + 2Q_w e^{4\omega} = 2Q \quad \text{on} \quad M^4 \tag{1.1$_b$}$$

where Q is the curvature invariant

$$12Q = -\Delta R + R^2 - 3|\mathrm{Ric}|^2 \tag{1.2}$$

(ii) The analogy between K and Q becomes more apparent if one considers the Gauss-Bonnet formulae:

$$2\pi\chi(M) = \int K dv \qquad \text{on} \quad M = M^2 \tag{1.3$_a$}$$

$$(1.3)_b \qquad 4\pi^2\chi(M) = \int \left(Q + \frac{|C|^2}{8}\right) dv \quad \text{on} \quad M = M^4$$

where $\chi(M)$ denotes the Euler-characteristic of the manifold M, and $|C|^2$= norm squared of the Weyl tensor. Since $|C|^2dv$ is a pointwise invariant under conformal change of metric, $\int Qdv$ is the term which measures the conformal change in formula $(1,3)_b$.

(iii) When $n \geq 3$, another natural analogue of $-\Delta$ on M^2 is the conformal Laplacian operator L as defined on (0.2). In this case, if we denote the conformal change of metric as $g_u = u^{\frac{4}{n-2}}g$ for some positive function u, then we may rewrite the conformal covariant property (0.2) for L as

$$(1.4)_a \qquad L_u(\varphi) = u^{-\frac{n+2}{n-2}}L(u\varphi) \quad \text{on} \quad M^n, n \geq 3$$

for all $\varphi \in C^\infty(M^n)$.

A differential equation which is associated with the operator L is the Yamabe equation:

$$(1.5)_a \qquad Lu = R_u u^{\frac{n+2}{n-2}} \qquad \text{on} \quad M^n, \quad n \geq 3\,.$$

Equation $(1.5)_a$ has been intensively studied in the recent decade. For example the famous Yamabe problem in differential geometry is the study of the equation $(1.5)_a$ for solutions $R_u \equiv$ constant; the problem has been completely solved by Yamabe [Y], Trudinger [T-1], Aubin [Au] and Schoen [Sc-2].

(iv) It turns out there is also a natural fourth order Paneitz operator P_4^n in all dimension $n \geq 5$, which enjoys the conformal covariance property with respect to conformal changes in metrics also. The relation of this operator to the Paneitz operator in dimension four is completely analogous to the relation of the conformal Laplacian to the Laplacian in dimension two. On (M^n, g) when $n \geq 4$, define

$$P_4^n = (-\Delta)^2 - \delta(a_n R + b_n R_{i,j})d + \frac{n-4}{2}Q_4^n;$$

where

$$Q_4^n = c_n|\rho|^2 + d_n R^2 - \frac{1}{2(n-1)}\Delta R,$$

and $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$, $b_n = -\frac{4}{n-2}$, $c_n = -\frac{2}{(n-2)^2}$, $d_n = \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}$ are dimensional constants. Thus $P_4^4 = P_4$, $Q_4^4 = Q$. Then (Branson [Br-1]), we have for $g_u = u^{\frac{4}{n-4}}g$, $n \geq 5$

$$(1.4)_b \qquad \left(P_4^n\right)_u(\varphi) = u^{-\frac{n+4}{n-4}}\left(P_4^n\right)(u\varphi)$$

for all $\varphi \in C^\infty(M^n)$. We also have the analogue for the Yamabe equation:

$$(1.5)_b \qquad P_4^n u = Q_4^n u^{\frac{n+4}{n-4}} \qquad \text{on} \quad M^n, \quad n \geq 5\,.$$

We would like to remark on $\mathbb{R}^n$ with Euclidean metric, $P_4^n = (-\Delta)^2$ the bi-Laplacian operator. Equation $(1.5)_b$ takes the form $(-\Delta)^2 u = c_n u^{\frac{n+4}{n-4}}$, an equation which has been studied in literature e.g. [PuS].

§2. Uniqueness result on S^n

In this section we will consider the behavior of the Paneitz operator on the standard spheres (S^n, g). First we recall the situation when $n = 2$. On (S^2, g), when one makes a conformal change of metric $g_\omega = e^{2\omega} g$, the Gaussian curvature $K_\omega = K(g_\omega)$ satisfies the differential equation

$$\Delta\omega + K_\omega e^{2\omega} = 1 \tag{2.1}$$

on S^2, where Δ denotes the Laplacian operator with respect to the metric g on S^2.

When $K_w \equiv 1$ on (2.1), the Cartan-Hadamard theorem asserts that $e^{2w} g$ is isometric to the standard metric g by a diffeomorphism φ; and the conformality requirements says φ is a conformal transformation of S^2. In particular, $w = \frac{1}{2}\log|J_\varphi|$, where J_φ denotes the Jacobian of the transformation φ.

In [ChL], Chen and Li studied the corresponding equation of (2.1) on $\mathbb{R}^2$ with $K_w \equiv 1$, and they proved, using the method of moving plane, the stronger result that when u is a smooth function defined on $\mathbb{R}^2$ satisfying

$$-\Delta u = e^{2u} \qquad \text{on} \quad \mathbb{R}^2 \tag{2.2}$$

with $\int_{\mathbb{R}^2} e^{2u} dx < \infty$, then $u(x)$ is symmetric with respect to some point $x_0 \in \mathbb{R}^2$ and there exists some $\lambda > 0$, so that $u(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2}$ on $\mathbb{R}^2$. There is an alternative argument by Chanillo-Kiessling ([ChK]) for this uniqueness result using the isoperimetric inequality.

As we have mentioned in the previous section, when $n \geq 3$, a natural generalization of the Gaussian curvature equation (2.1) above is the Yamabe equation under conformal change of metric. On (S^n, g), denote $g_u = u^{\frac{4}{n-2}} g$ the conformal change of metric of g, where u is a positive function, then the scalar curvature $R_u = R(g_u)$ of the metric is determined by the following differential equation

$$c_n \Delta u + R_u u^{\frac{n+2}{n-2}} = Ru \tag{2.3}$$

where $c_n = \frac{4(n-1)}{n-2}$, $R = n(n-1)$. When $R_u = R$, a uniqueness result established by Obata [Ob] again states that this happens if the metric g_u is isometric to g or equivalently $u = |J_\varphi|^{\frac{n-2}{2n}}$ for some conformal transformation φ of S^n. In [CGS] Caffarelli-Gidas-Spruck studied the corresponding equation of (2.3) on $\mathbb{R}^n$:

$$-\Delta u = n(n-2) u^{\frac{n+2}{n-2}}, \qquad u > 0 \quad \text{on} \quad \mathbb{R}^n\,. \tag{2.4}$$

They classified all solutions of (2.4), via the method of moving plane, as $u(x) = \frac{2\lambda}{\lambda^2 + |x - x_0|^2}^{\frac{2}{n-2}}$ for some $x_0 \in \mathbb{R}^n$, $\lambda > 0$.

For all n, on (S^n, g), there also exists a n-th order (pseudo) differential operator $\mathbb{P}_n$ which is the pull back via sterographic projection of the operator $(-\Delta)^{n/2}$ from $\mathbb{R}^n$ with Euclidean metric to (S^n, g). $\mathbb{P}_n$ is conformal covariant of bi-degree (0, n), i.e. $(\mathbb{P}_n)_w = e^{-nw}\mathbb{P}_n$. The explicit formulas for $\mathbb{P}_n$ on S^n has been computed in Branson [Br-1] and Beckner [B]:

$$(2.5)\quad \begin{cases} \text{For} \quad n \quad \text{even} \quad \mathbb{P}_n &= \prod_{k=0}^{\frac{n-2}{2}}(-\Delta + k(n-k-1)), \\ \text{For} \quad n \quad \text{odd} \quad \mathbb{P}_n &= \left(-\Delta + \left(\frac{n-1}{2}\right)^2\right)^{1/2} \prod_{k=0}^{\frac{n-3}{2}}(-\Delta + k(n-k-1)). \end{cases}$$

On general compact manifolds in the cases when the dimension of the manifold is two or four, there exist natural curvature invariants $\tilde{Q}_n$ of order n which, under conformal change of metric $g_w = e^{2w}g$, is related to $P_n w$ through the following differential equation:

$$(2.6)\qquad -P_n w + (\tilde{Q}_n)_w e^{nw} = \tilde{Q}_n \quad \text{on} \quad M\,.$$

In the case when $n = 2$, P_2 is the negative of the Laplacian operator, $\tilde{Q}_2 = K$, the Gaussian curvature. When $n = 4$, P_4 is the Paneitz operator, $\tilde{Q}_4 = 2Q_4$ as defined in (1.2). In the special case of (S^2, g), $P_2 = \mathbb{P}_2$, similarly on (S^4, g), $P_4 = \mathbb{P}_4$. In section 5 below, we will also discuss the existence of P_1, P_3 operators and corresponding curvature invariants Q_1 and Q_3 defined on boundaries of general compact manifolds of dimension 2 and 4 respectively.

On (S^n, g), when the metric g_w is isometric to the standard metric, then $(\tilde{Q}_n)_w = \tilde{Q}_n = (n-1)!$. In this case, equation (2.6) becomes

$$(2.7)\qquad -P_n w + (n-1)! e^{nw} = (n-1)! \quad \text{on} \quad S^n$$

One can establish the following uniqueness result for solutions of equation (1.7).

Theorem 2.1. *[CY-4] On (S^n, g), all smooth solutions of the equation (2.7) are of the form $e^{2w}g = \varphi^*(g)$ for some conformal transformation φ of S^n; i.e. $w = \frac{1}{n}\log|J_\varphi|$ for the transformation φ.*

We now re-formulate the equation (2.7) on $\mathbb{R}^n$. For each point $\xi \in S^n$, denote x its corresponding point under the sterographic projection π from S^n to $\mathbb{R}^n$, sending the north pole on S^n to ∞; i.e. Suppose $\xi = (\xi_1, \xi_2, \dots, \xi_{n+1})$ is a point $\in S^n \subset \mathbb{R}^{n+1}$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, then $\xi_i = \frac{2x_i}{1+|x|^2}$ for $1 \le i \le n$; $\xi_{n+1} = \frac{1-|x|^2}{1+|x|^2}$. Suppose w is a smooth function on S^n, denote $\varphi(x) = \log\frac{2}{1+|x|^2} = \log|J_{\pi^{-1}}|$, $u(x) = \varphi(x) + w(\xi)$. Since the Paneitz operator P_n is the pull back under π of the operator $(-\Delta)^{n/2}$ on $\mathbb{R}^n$ ([c.f. Br-2, Theorem 3.3]), w satisfies the equation (1.6) on S^n if and only if u satisfies the corresponding equation

$$(2.8)\qquad (-\Delta)^{n/2}u = (n-1)! e^{nu} \quad \text{on} \quad \mathbb{R}^n\,.$$

Thus Theorem 2.1 above is equivalent to the following result:

Theorem 2.2. *On $\mathbb{R}^n$, suppose u is a smooth function satisfying the equation (2.8). Suppose in addition that*

$$u(x) = \log \frac{2}{1+|x|^2} + w(\xi(x))$$

for some smooth function w defined on S^n, then $u(x)$ is symmetric w.r.t. some point $x_0 \in \mathbb{R}^n$, and these exists some $\lambda > 0$ so that

$$u(x) = \log \frac{2\lambda}{\lambda^2 + |x - x_0|^2} \quad \text{for all} \quad x \in \mathbb{R}^n . \tag{2.9}$$

We remark that, in the case when w is a minimal solution of the functional with Euler-Lagrange equation (1.7), the result in Theorem 2.2 is a consequence of some sharp Sobolev type inequalities of Milin-Lebedev when $n = 1$, Moser [Mo-1] and Onofri [On] when $n = 2$ and Beckner [B] for general n. Sharp inequalities of this type played an important role in a number of geometric PDE problems, see for example the article by Beckner [B] and lecture notes by the author [C-1].

We would like also to mention that during the course of preparation of the paper, above theorem was independently proved by C.S. Lin [L] and X. Xu [X] when $n = 4$ for functions satisfying equation (2.8) under some less restrictive growth conditions at infinity. In general, it remains open whether there exists some natural geometric conditions under which functions satisfying equation (2.8) are necessarily of the form (2.9).

We now describe briefly the method of moving plane. First we recall a fundamental result of Gidas-Ni-Nirenberg.

Theorem 2.3. *[GNN] Suppose u is a positive $\mathcal{C}^2$ function satisfying*

$$\begin{cases} -\Delta u &= f(u) \quad \text{on} \quad B \\ \quad u &= 0 \quad\ \ \text{on} \quad \partial B \end{cases} \tag{2.10}$$

in the unit ball B in $\mathbb{R}^n$ and f is a Lipschitz function. Then $u(x) = u(|x|)$ is a radial symmetric decreasing function in $r = |x|$ for all $x \in B$.

To set up the proof in [GNN] of above theorem we introduce the following notations. For each point $x \in \mathbb{R}^n$, denote $x = (x_1, x')$ where $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{n-1}$. For each real number λ, denote

$$\begin{aligned} &\Sigma_\lambda = \{x = (x_1, x') \mid x_1 < \lambda\} , \\ &T_\lambda = \{x = (x_1, x') \mid x_1 = \lambda\} , \\ &x_\lambda = (2\lambda - x_1, x') \quad \text{the reflection point of} \quad x \quad w.r.t. \quad T_\lambda . \end{aligned}$$

Define

$$w_\lambda(x) = u(x) - u(x_\lambda) \equiv u(x) - u_\lambda(x) .$$

Suppose u satisfies equation (2.10), the idea of moving plane is to prove that $w_\lambda(x) \geq 0$ on Σ_λ for all $0 \leq \lambda \leq 1$ and actually $w_{\lambda=0} \equiv 0$. This is achieved by an application of the maximum principle and Hopf's boundary lemma to the function u. Thus $u(x_1, x') = u(-x_1, x')$ for $x \in B$. Since one can repeat this argument for any hyperplane passing through the origin of the ball B, one establishes that u is radially symmetric w.r.t. to the origin.

If one attempts to generalize above argument to higher order elliptic equation such as $(-\Delta)^2 u = f(u)$ with suitable boundary conditions, one quickly realizes that, due to a lack of maximum principle for higher order elliptic equation, such result in general cannot be expected to hold. Nevertheless, it turns out that for a special class of Lipschitz functions f; namely for functions f satisfying $f(0) \geq 0$ with f monotonically increasing, e.g. $f(u) = e^u$, one can modify the argument in [GNN]. The key observation is that for each f, if $(-\Delta)^2 u = f(u)$ then

$$(-\Delta)^2 w_\lambda(x) = f(u) - f(u_\lambda) = c(x) w_\lambda(x) \tag{2.11}$$

where $c(x)$ is some positive function whose value at x lies between $f(u(x))$ and $f(u_\lambda(x))$. From (2.11) one then concludes that $w_\lambda(x) \geq 0$ on Σ_λ if and only if $(-\Delta)^2 w_\lambda \geq 0$ on $\sum_\lambda$. Since $w_\lambda(x) = (-\Delta) w_\lambda(x) = 0$ for $x \in T_\lambda$, $w_\lambda(x) \geq 0$ on $\sum_\lambda$ also happens if and if $(-\Delta) w_\lambda(x) \geq 0$ on Σ_λ. This suggests that one should apply the maximum principle and Hopf's lemma to the function $(-\Delta) w_\lambda$ to generalize result in [GNN] to higher order elliptic operators like $(-\Delta)^2$.

In [CY-4], Theorem 2.2 was proved by applying above argument to the function $(-\Delta)^{m-1} w_\lambda(x)$ where $m = [\frac{n+1}{2}]$. In the case when n is even $(n = 2m)$, one can then apply directly some technical lemmas about "harmonic asymptotic" behavior of w at infinity in [CGS] to the method of moving planes to finish the proof of Theorem 2.2. In the case when n is odd $(n = 2m - 1)$, a form of Hopf's lemma for the pseudo-differential operators $(-\Delta)^{1/2}$ was established, and the technical lemmas in [CGS] was modified to a version adapted to the operator $(-\Delta)^{1/2}$, one can then apply the method of moving planes to finish the proof of Theorem 2.2.

§3. Existence and regularity result on general 4-manifold

On (M^2, g) with Gaussian curvature $K = K_g$, consider the functional

$$J[w] = \int |\nabla w|^2 dv + 2 \int K w dv - (\int K dv) \log ⨍ e^{2w} dv \tag{3.1}$$

where the gradient, the volume form are taken with respect to the metric g, and $⨍ \varphi dv = \int \varphi dv/\text{volume}$ for all φ.

The Euler-Lagrange equation for J is:

$$\Delta \omega + c e^{2\omega} = K \qquad \text{on} \quad M^2 \tag{3.2}$$

where c is a constant. Notice that (3.2) is a special case of equation $(1.1)_a$ with $K_\omega \equiv c$. For the special manifold (S^2, g), $K \equiv 1$, (3.2) is a special case of equation (2.1).

In the special case of (S^2, g), the functional $J[\omega]$ has been extensively studied in [Mo-2], [On], [CY-1], [CY-2], [ChD], [H], [ChL] in connection with the "Nirenberg problem"; that is, the problem to characterize the set of functions K_ω defined on S^2 which are the Gaussian curvature function of a metric $g_\omega = e^{2\omega} g$ conformal to g. For a general compact surface (M^2, g), there is an intrinsic analytic meaning for the functional $J[\omega]$. Namely it is the logarithmic quotient of determinant of the Laplacian operator with respect to g_ω and g respectively. This is generally known as the Ray-Singer-Polyakov formula [RS], [Po] which we shall briefly describe in section 4.

A key analytic fact which has been used in the study of the functional $J[\omega]$ is a sharp Sobolev inequality established by Trudinger [T-2], Moser [Mo-1]: Given a bounded smooth domain Ω in $\mathbb{R}^2$, denote by $W_0^{1,2}(\Omega)$ the closure of the Sobolev space of functions with first derivative in L^2 and with compact support contained in Ω, then $W_0^{1,2}(\Omega) \subseteq \exp L^2$, and there exists a best constant $\beta(1,2) = 4\pi$ such that for all $u \in W_0^{1,2}(\Omega)$ with $\int |\nabla u|^2 dx \leq 1$, there exists some constant c (independent of u) so that $\int_\Omega e^{\beta |u|^2} dx \leq c|\Omega|$ for all $\beta \leq \beta(1,2)$. Moser's inequality has been generalized to the cases of functions satisfying Neumann boundary condition, and to domains with corners in [CY-2], to general domains in $\mathbb{R}^n$ by Adams [A] and to general compact manifolds [F].

We now state some results which generalize the study of the functional $J[\omega]$ to 4-manifolds. On a compact 4-manifold (M^4, g), denote by $k_p = \int Q dv$, and define

$$II[\omega] = \int (P_4 \omega)\omega + 4 \int Q w dv - (\int Q dv) \log(⨍ e^{4\omega} dv) \tag{3.3}$$

Theorem 3.1. *([CY-3]) Suppose $k_p < 8\pi^2$, and suppose P_4 is a positive operator with* ker $P = \{\text{constants}\}$*; then* $\inf II_{w \in W^{2,2}}[w]$ *is attained. Denote the infimum by w_p, then the metric $g_p = e^{2w_p} g$ satisfies $Q_p \equiv \text{constant} = k_p / \int dv$.*

Remarks. (1) In general, the positivity of P_4 is a necessary condition for the functional II to be bounded from below. But some recent work of M. Gursky [Gu-3] indicates that under the additional assumption that $k_p > 0$ and that g is of positive scalar class, (i.e. under some conformal change of metric, g admits a metric of positive scalar curvature; or equivalently the conformal Laplacian operator $L = Lg$ admits only positive eigenvalues) P_4 is always positive. Furthermore, under the same assumption, $k_p < 8\pi^2$ is always satisfied unless (M^4, g) is conformally equivalent to (S^4, g); in the latter case then $k_p = 8\pi^2$ and the extremal metric for $II[w]$ has been studied in [BCY].

(2) Notice that the extremal function w_p in $W^{2,2}$ for II satisfies the equation

$$-P_4 w_p + 2Q_p e^{4\omega_p} = 2Q \tag{3.4}$$

with $Q_p \equiv$ constant. Thus standard elliptic theory can be applied to establish the smoothness of w_p. This is in contrast with the smoothness property of the extremal function w_d of the log-determinant functional $F[w]$, in which $II[w]$ is one of the term. We will discuss regularity property of w_d in section 4.

(3) A key analytic fact used in establishing Theorem 3.1 above is the generalized Moser inequality established by Adams [A], which in the special case of domains Ω in $\mathbb{R}^4$ states that $W_0^{2,2}(\Omega) \hookrightarrow \exp L^2$ with $\beta(2,4) = 32\pi^2$.

We now briefly mention some partial results in another direction which also indicate P_4 is a natural analogue of $-\Delta$ on 4-manifolds.

On two general compact Riemannian manifolds (M^n, g) and (N^k, h), consider the energy functional

$$E[w] = -\int (\Delta\omega)\omega dv = \int |\nabla\omega|^2 dv\,. \tag{3.5}$$

Defined for all system of functions $\omega \in W^{1,2} : (M^n, g) \to (N^k, h)$. Critical points of $E[w]$ are defined as harmonic maps. Regularity of harmonic maps has been and still is a subject under intensive study in geometric analysis. (see e.g. the articles [Sc-1], [Si]). We will here mention some results on compact surfaces which are relevant to the statement of Theorem 3.2 below. In the case when the dimension of M^n is 2, a classical result of Morrey [M] states that all minimal solutions of $E[\omega]$ are smooth. Morrey's result has been extended to all solutions of $E[\omega]$ by the recent beautiful work of Helein [He-1] [He-2]. In the case when $n \geq 3$, harmonic maps in general are not smooth. The Hausdroff dimension of the singularity sets of the stationary solutions of $E[\omega]$ has also been studied for example in [ScU], [E], [Be].

In some recent joint work [CWY] with L. Wang and P. Yang, we studied the functional

$$E_4[\omega] = \int (P_4\omega)\omega dv\,. \tag{3.6}$$

which is a natural analogue of the functional $E[\omega]$.

A preliminary result we have obtained so far is:

Theorem 3.2. (i) *For general target manifold (N^k, h), weak solutions of the Euler equations of $E_4[\omega]$ which minimizes $E_4[\omega]$ are smooth.*

(ii) *When the target manifold is (S^k, g), then all critical points of $E_4[\omega]$ are smooth.*

In the case when the target manifold is (S^k, g), Euler equation for the functional E_4 takes the following form :

$$\Delta\Delta w^\alpha = -w^\alpha\Big(\sum_\beta (\Delta w^\beta)^2 + 2(\nabla^2 w^\beta)^2 + 4\nabla w^\beta \cdot \nabla(\Delta w^\beta)\Big) + \text{lower order terms} \tag{3.7}$$

where $w = (w^1, \cdots w^k)$ for all $1 \leq \alpha \leq k$.

A key step in the proof of (ii) in Theorem 3.2 above is to establish that the function f^α which denotes the function in the right hand side of equation (3.7) is in fact a function in the Hardy space H^1. Thus duality result of H^1-BMO ([FeS]) may be applied to establish the continuity of the weak solution ω^α. This is completely parallel to the proof by Helein [He-1] in establishing the smoothness of harmonic maps for compact surfaces. But the reason for the function f^α to be in H^1 is not quite the same as the case in [He-1]; in particular compensated compactness results of [CLMS] can not be applied directly to establish that f^α be in H^1.

In view of result of Helein [He-2], it is most plausible that for all general target manifold (N^k, h), all critical solutions of E_4 are smooth. We have also been informed that recently R. Hardt and L. Mou have some regularity results for critical points of E_4 on general manifold M^n of dimension $n \geq 5$.

§4. Zeta functional determinant

There is an interesting connection of the functional $J[\omega]$ in (3.1) to a geometric variation problem. On compact surface (M^2, g), let $\{0 < \lambda_1 \leq \lambda_2 \leq \cdots\}$ be the spectrum of the (negative of) Laplacian -Δ_g. Let $\zeta(s) = \sum \lambda_i^{-s}$ defined for $Res > \frac{1}{2}$, then ζ has a meromorphic continuation to the whole plane and is regular at the origin using the heat kernel expansion of Δ_g. $-\zeta'(0)$ is well-defined, and one may define $\log\det\Delta_g$ to be $-\zeta'(0)$ (as in Ray-Singer [RS]). In [Po], Polyakov further computed the logarithm of the ratio of determinant of two conformally related metrics $g_w = e^{2w}g$ on a compact surface without boundary.

$$F[w] = \log\frac{det\Delta_w}{det\Delta} = \frac{1}{3}\int_M (|\nabla w|^2 + 2Kw)dv_g \tag{4.1}$$

under the normalization that $\mathrm{vol}(g_w) = \mathrm{vol}(g)$. Notice that $F[w]$ is essentially the same as the functional $J[w]$ in (3.1). In a series of papers, Osgood-Phillips-Sarnak ([OPS-1], [OPS-2]) have further studied the functional $F[w]$, and they have shown among other things that $F[w]$ enjoys a certain compactness property on account of the Moser-Trudinger inequality, and proved that in each conformal class, the functional $F[w]$ attains its extrema at the constant curvature metrics.

When the dimension of a closed manifold is odd, it was shown in Branson [Br-2] that $\log\det L_g$ is a conformal invariant. Thus the next natural dimension to study the generalized Polyakov formula (4.1) is four.

Suppose (M, g) is a compact, closed 4-manifold, and suppose A is a conformally covariant operator satisfying (0.3) with $b - a = 2$. In [BO] Branson-Orsted gave an explicit computation of the normalized form of $\log\frac{\det A_w}{\det A}$ which may be expressed as:

$$F[w] = \gamma_1 I[w] + \gamma_2 II[w] + \gamma_3 III[w] \tag{4.2}$$

where $\gamma_1, \gamma_2, \gamma_3$ are constants depending only on A and

$$I[w] = 4\int |C|^2 w dv - \left(\int |C|^2 dv\right) \log \fint e^{4w} dv$$

$$II[w] = \langle Pw, w\rangle + 4\int Qw dv - \left(\int Q dv\right) \log \fint e^{4w} dv,$$

$$III[w] = 12\left(Y(w) - \frac{1}{3}\int (\triangle R)\ w dv\right),$$

where C is the Weyl tensor, and $Y(w) = \int \left(\frac{\Delta(e^w)}{e^w}\right)^2 - \frac{1}{3}\int R\,|\nabla w|^2$. We also remark that the functional $III[w]$ [BCY] may be written as

$$III[w] = \frac{1}{3}\left[\int R_w^2\, dv_w - \int R^2\, dv\right]$$

so that when the background metric is assumed to be the Yamabe metric in a positive conformal class, the functional III is non-negative.

In [BCY] we made two observations. The first is that on the standard 4-sphere (S^4, g), the functional $F[w]$ for the conformal Laplacian L (and Dirac square $\nabla\!\!\!/^2$) is extremized in a strong way, that each term $II[w]$ and $III[w]$ are extremized by the standard metric $g_w = g_0$. For $III[w]$ this is a consequence of Obata's result ([Ob]) that the constant scalar curvature conformal metrics on S^4 are standard. For the functional $II[w]$, this is a consequence of Beckner's [B] inequality which is valid for all dimensions, (see the discussion in section 2). The second observation is that the functional $F[w]$ enjoys certain compactness properties for the operators L and $\nabla\!\!\!/^2$ for most compact locally symmetric Einstein 4-manifolds. The basic analytic inequality required is Adam's inequality [A].

In [CY-3], we continue the study of the log-determinant formula (4.2) on general 4-manifolds. We then extend the compactness criteria to a more general class of 4-manifolds. We define the conformal invariant:

$$\begin{aligned} k_d &= -\gamma_1 \int |C|^2 dv - \gamma_2 \int Q dv \\ &= (-\gamma_2)\, 4\pi^2 \chi(M) + \left(\frac{\gamma_2}{8} - \gamma_1\right) \int |C|^2 dv \end{aligned} \tag{4.3}$$

Theorem 4.1. *If the functional F satisfies $\gamma_2 < 0$, $\gamma_3 < 0$, and $k_d < (-\gamma_2) 8\pi^2$, then $\sup_{w \in W^{2,2}} F[w]$ is attained by some function w_d and the metric $g_d = e^{2w_d} g_0$ satisfies the equation*

$$\gamma_1 |C_d|^2 + \gamma_2\, Q_d - \gamma_3 \triangle_d R_d = -k_d \cdot Vol(g_d)^{-1}. \tag{4.4}$$

Further, all functions $\varphi \in W^{2,2}$ satisfy the inequality:

$$k_d \log \fint e^{4(\varphi - \tilde{\varphi})} dv_d \le (-\gamma_2)\, \langle P\varphi, \varphi \rangle - 12\gamma_3 Y_d(\varphi). \tag{4.5}$$

where $\tilde{\varphi}$ denotes the mean value of φ with respect to the metric g_d, and $\fint$ denotes $\frac{1}{vol(M, g_d)} \int_M dv_d$.

In particular for the operator L and $\nabla\!\!\!/^2$, we obtain existence results for extremal metrics of the corresponding log-determinant functional. Thus for a large class of conformal 4-manifolds, we have the existence of several extremal metrics in addition to the Yamabe metric. It is an interesting problem to study the relation among these metrics. For example, we found in [BCY] that on S^4 all these extremal metrics coincide; while on $S^3 \times S^1$ with the standard metric, we found in [CY-3], [CGY-1] depending on the parameter t of S^1, g_d and the Yamabe metric may not agree. In order to identify these extremal metrics in special circumstances, we provide some uniqueness result:

Theorem 4.2. *If $k_d \leq 0$, the extremal metric g_d for the functional F corresponding to the conformal Laplacian operator L is unique.*

This uniqueness assertion is obtained as consequence of the convexity of the corresponding functionals. Applying the uniqueness result, we were able to identify some of the extremal metrics with known metric in special circumstances.

We remark that, the extremal function ω_d in Theorem 4.1, 4.2, when first established in [CY-3], are functions in $W^{2,2}$ and satisfying the equation (4.4) weakly in $W^{2,2}$. If we re-write the equation (4.4) expressing $|C_d|^2$, Q_d, $\Delta_d R_d$ all in terms of the background metric g (with $g_d = e^{2\omega d} g$), then $\omega = \omega_d$ satisfies the following equation

$$\Delta\Delta\omega = c_1|\nabla\omega|^4 + c_2(\Delta\omega)^2 + c_3\Delta\omega|\nabla\omega|^2 + \text{lower order terms} \tag{4.6}$$

for some constants c_1, c_2, c_3 depending only on γ_1, γ_2, γ_3. Equation equation (4.6) should be compared to the equation (3.7) in section 3.

In a recent joint work [CGY-2] with M. Gursky and P. Yang, we established a general regularity result for weak $W^{2,2}$ solution of equation (4.6) on general compact 4-manifolds. A special case of our result is the following theorem:

Theorem 4.3. *Let $F[\omega]$ be as in Theorem 4.1, then $\sup_{\omega \in \omega^{2,2}} F[\omega]$, when attained, is a smooth function.*

Main idea in the proof of Theorem 4.3 follows the same line as the regularity result for harmonic maps in Schoen-Uhlenbeck [ScU]. The property of the maximal solution of $F[\omega]$ is used in some crucial way which enable us to compare $F[\omega]$ with $F[h]$; where h is a bi-harmonic extension of the boundary value of ω when restricted to a geodesic ball B_r. Some very basic properties for bi-harmonic functions are established to estimate the growth of $E_4[\omega]$ on B_r. We will mention below a few such properties.

For simplicity, we denote B_r as a ball of radius r in $\mathbb{R}^4$. For a given function $\omega \in W^{2,2}(B_{2r})$ let h be the solution of the linear equation:

$$\begin{cases} \Delta\Delta h & = 0 \quad \text{on} \quad B_r \\ h & = \omega \quad \text{on} \quad \partial B_r \\ \frac{\partial h}{\partial n} & = \frac{\partial \omega}{\partial n} \quad \text{on} \quad \partial B_r \end{cases}$$

Lemma 4.4. (i) $\int_{B_r} |h|^2 \leq Cr \int_{\partial B_r} |\nabla^2\omega|^2$, *for some constant C.*

(ii) *Suppose $\omega|_{\partial B_r}$ is Holder of order α, and $\phi = \frac{\partial\omega}{\partial n}$ is in $L^p(\partial B_r)$ for some $p > 3$, than h is Holder of order $\beta \leq \min(\gamma, 1 - 3/p)$ on B_r.*

Apply Lemma 4.4, one can prove that the extremal function ω of the functional $F[\omega]$ is Holder continuous. From there, one can apply some iterative arguments to show that all week $W^{2,2}$ solutions of (4.6) which are Holder continuous are in fact C^∞ smooth. It remains open whether all critical solution of the functional $F[\omega]$ are smooth. It is also interesting to study the equation (4.6) on domains of dimension ≥ 5.

In another direction, recently M. Gursky [Gu-1] gave some beautiful applications of the extremal metric of the log-determinant functional $F[\omega]$ in Theorem 4.1 and Theorem 4.3 above to characterize certain class of compact 4-manifolds. To state his results, we will first make some definitions. On a compact manifold (M^n, g), define the Yamabe invariant of g as

$$
Y(g) = \inf_{g_\omega = e^{2\omega} g} \mathrm{vol}(g_\omega)^{-\frac{n-2}{n}} \int R_{g_\omega} dv_{g_\omega} \,. \tag{4.7}
$$

By the work of Yamabe, Trudinger, Aubin and Schoen mentioned in section 1, every compact manifold M^n admits a metric g_ω conformal to g which achieves $Y(g)$, hence g_ω has constant scalar curvature. We say (M^n, g) is of positive scalar class if $Y(g) > 0$.

On compact 4-manifolds, both $Y(g)$ and $\int Q_g dv_g$ are conformal invariants. The following result of Gursky [Gu-1] indicates that these two conformal invariants constrain the topological type of M^4.

Theorem 4.4. *Suppose (M^4, g) is a compact manifold with $Y(g) > 0$,*

(i) *If $\int Q_g dv_g > 0$, then M admits no non-zero harmonic 1-forms. In particular, the first Betti number of M vanishes.*

(ii) *If $\int Q_g dv_g = 0$, and if M admits a non-zero harmonic 1-form, then (M, g) is conformal equivalent to a quotient of the product space $S^3 \times \mathbb{R}$. In particular (M, g) is locally conformally flat.*

As a corollary of part (ii) of Theorem 4.4, one can characterize quotient of the product space $S^3 \times \mathbb{R}$ as compact, locally conformally flat 4-manifold with $Y(g) > 0$ and $\chi(M) = 0$.

A crucial step in the proof of theorem above is to prove that for suitable choice of γ_1, γ_2, γ_3, the extremal metric g_d for the log-determinant functional $F[\omega]$ exists and is unique. Furthermore under the assumption $Y(g) > 0$, one has $R_{g_d} > 0$; if $Y(g) = 0$ then $R_{g_d} \equiv 0$. In the case $\int Q_g dv_g = 0$, existence of non-zero harmonic 1-form actually indicates that $R_{g_d} \equiv$ positive constant. Gursky's proof is highly ingenious.

Using similar ideas, Gursky [Gu-2] has also applied above line of reasoning to the study of Kahler-Einstein surfaces. Suppose M^4 is compact, 4-manifold in the positive sclar class which also admits non-zero self-dual harmonic 2-form, he established a lower bound for the Weyl functional $\int_M |C|^2 dV$ over all non-negative conformal classes on M^4 and proved that the bound is attained precisely at the conformal classes of Kahler-Einstein metrics.

§5. P_3 – a boundary operator

In the previous sections, we have discussed the behavior of the Laplacian and the Paneitz operator P_4 on functions defined on compact manifolds without boundary. It turns out, associated with these operators, there also exist some natural boundary operators for functions defined on the boundary of compact manifolds. We will now briefly describe such operators on boundary of M^n for $n = 2$ and $n = 4$. Most of the material described in this section is contained in the joint work of Jie Qing with the author [CQ-1], [CQ-2] and [CQ-3]. The reader is also referred to the lecture notes [C-2] for a more detailed description of such operators derived in conjunction with the generalized formula [BO] [BCY] [CY] of Polyakov-Alvarez [Po], [Al], [Ok] of zeta functional determinant for 4-manifolds with boundary. We start with terminology. On compact manifold (M^n, g) with boundary, we say a pair of operators (A, B) satisfy the *conformal assumptions* if:

Conformal Assumptions. *Both A and B are conformally covariant of bidegree (a_1, a_2) and (b_1, b_2) in the following sense*

$$A_w(f) = e^{-a_1 \omega} A(e^{a_2 \omega} f)$$
$$B_w(g) = e^{-b_1 \omega} B(e^{b_2 \omega} g),$$

for any $f \in C^\infty(M), g \in C^\infty(\partial M)$. Assume also that

$$B(e^{a_2 \omega} g) = 0 \text{ if and only if } B_w(g) = 0,$$

for any $\omega \in C^\infty(\bar{M})$, where A_w, B_w denote the operator A, B respectively with respect to the conformal metric $g_w = e^{2w} g$.

Examples: The typical examples of pairs (A, B) which satisfy all three assumptions above are:

(i) when $n = 2$, $A = -\Delta, B = \frac{\partial}{\partial n}$ (negative of) the Laplacian operator and the Neumann operator respectively.

(ii) when $n \geq 3$, $A = \mathcal{L} = -\frac{4(n-1)}{n-2}\Delta + R$ the conformal Laplacian of bidegree $(\frac{n+2}{2}, \frac{n-2}{2})$, and R is the scalar curvature, and B is either the Dirichlet boundary condition or $B = \mathcal{R} = \frac{2(n-1)}{n-2}\frac{\partial}{\partial n} + H$ the Robin operator of bidegree $(\frac{n}{2}, \frac{n-2}{2})$, where H is the trace of the second fundamental form (the mean curvature) of the boundary ∂M.

(iii) when $n = 4$, in [CQ-1] we have discovered a boundary operator P_3 conformal of bidegree (0,3) on the boundary of a compact 4-manifold. On 4-manifolds, (P_4, P_3) is a pair of operators satisfying the conformal covariant assumptions, which in the sense we shall describe below, is a natural analogue of the pair of operators $(-\Delta, \frac{\partial}{\partial n})$ defined on compact surfaces.

As we have mentioned before, on compact surfaces, from the point of view of conformal geometry, a natural curvature invariant associated with the Laplacian operator is the Gaussian curvature K. K enters the Gauss-Bonnet formula $(1.3)_a$. The Laplacian operator and K are related by the differential equation $(1.1)_a$ through the conformal change of metrics $g_w = e^{2w} g$.

On compact surface M with boundary, the Gauss-Bonnet formula takes the form

$$2\pi\chi(M) = \int_M K dv + \oint_{\partial M} k d\sigma, \tag{5.1}$$

where k denotes the geodesic curvature of ∂M and $d\sigma$ the arc length measure on ∂M. Through conformal change of metric $g_w = e^{2w}g$ for w defined on $\bar{M}$, the Neumann operator $\frac{\partial}{\partial n}$ is related to the geodesic curvature k via the differential equation

$$-\frac{\partial w}{\partial n} + k_w e^w = k \ \ on \quad \partial M. \tag{5.2}$$

Equation $(1.1)_a$ and (5.2) suggest that we search for the right pair of curvature functions and their corresponding differential operators through the Gauss-Bonnet formula. Recall that on 4-manifolds, Paneitz operator P_4 and the 4-th order curvature operator Q are related via the equation $(1.1)_b$. It turns out that on 4-manifolds there also exists a boundary local invariant of order 3 and a conformal covariant operator P_3 of bidegree $(0,3)$, the relation of (Q,T) to (P_4,P_3) on 4-manifolds is parallel to that of (K,k) to $(\Delta, \frac{\partial}{\partial n})$ on compact surfaces. The expression of P_3 and T on general compact 4-manifolds , like that of P_4, are quite complicated but can be explicitly written down in terms of geometric intrinsic quantities as in [CQ-1]. [CQ-2]. In particular, via the conformal change of metrics $g_w = e^{2w}g$, P_3 and T satisfies the equation:

$$-P_3 w + T_w e^{3w} = T \quad \text{on} \quad \partial M, \tag{5.3}$$

and

$$(P_3)_w = e^{-3w} P_3 \ \ on \ \ \partial M. \tag{5.4}$$

Perhaps the best way to understand how T and P_3 were discovered in [CQ-2] is via the Chern-Gauss-Bonnet formula for 4-manifolds with boundary:

$$\chi(M) = (32\pi^2)^{-1} \int_M (|C|^2 + 4Q)dx + (4\pi^2)^{-1} \oint_{\partial M} (T - \mathcal{L}_4 - \mathcal{L}_5)dy, \tag{5.5}$$

where $\mathcal{L}_4$ and $\mathcal{L}_5$ are boundary invariant of order 3 which are invariant under conformal change of metrics. Hence for a fixed conformal class of metrics,

$$\frac{1}{2}\int_M Q dv + \oint_{\partial M} T ds$$

is a fixed constant. We would like to remark that in the original Chern-Gauss-Bonnet formula T is not exactly the term as we have defined in [CQ-2] , actually it differs from T by $\frac{1}{3}\tilde{\Delta}H$, which does not affect the integration formula (5.5).

Thus on 4-manifolds with boundary it is natural to study the energy functional

$$E[w] = \frac{1}{4}\int wP_4 w + \frac{1}{2}\int wQ + \frac{1}{2}\oint_{\partial M} wP_3 w + \oint_{\partial M} wT. \tag{5.6}$$

In view of the complicated expressions of the operators P_4, P_3, Q and T, at this moment it is difficulty to study the functional $E[w]$ defined as above on general compact manifolds. But in the special case of (B^4, S^3) with the standard metrics, we have

$$P_4 = (\Delta)^2, P_3 = \frac{1}{2}N\Delta + \tilde{\Delta}N + \tilde{\Delta} \quad \text{and} \quad Q = 0, \quad \text{and} \quad T = 3, \tag{5.7}$$

where $\tilde{\Delta}$ denotes the the Laplacian operator Δ on (S^3, g). Thus the expression in $E[w]$ becomes relatively simple. In this special case, we are able to study the functional $E[w]$. The main analytic tool is the following sharp inequality of Lebedev-Milin type on (B^4, S^3).

Theorem 5.1. *Suppose $w \in C^\infty(\bar{B}^4)$. Then*

$$\log\left\{\frac{1}{2\pi^2}\oint_{S^3} e^{3(w-\bar{w})}dy\right\} \le \frac{3}{4\pi^2}\left\{\frac{1}{4}\int_{B^4} w\Delta^2 w + \oint_{S^3} \frac{1}{2}wP_3 w - \frac{1}{4}\frac{\partial w}{\partial n} + \frac{1}{4}\frac{\partial^2 w}{\partial n^2}\right\}, \tag{5.8}$$

under the boundary assumptions $\int_{S^3} \tau[w]ds[w] = 0$ where τ is the scalar curvature of S^3. Moreover the equality holds if and only if $e^{2w}g$ on B^4 is isometric to the canonical metric g.

The key step in the proof of theorem is the following analytic lemma.

Lemma 5.2. *Suppose w solves*

$$\begin{cases} \Delta^2 w = 0 & \text{in } R^4 \\ w \mid_{S^3} = u \\ \frac{\partial w}{\partial n} \mid_{S^3} = \phi, \end{cases}$$

then

$$\Delta w\Big|_{\partial B^4} = 2\tilde{\Delta}u + 2\left\{(-\tilde{\Delta}+1)^{\frac{1}{2}} + 1\right\}\phi \tag{5.9}$$

$$-\frac{\partial}{\partial n}\Delta w\Big|_{\partial B^4} = 2\mathbb{P}_3 u + 2\tilde{\Delta}u - 2\tilde{\Delta}\phi \tag{5.10}$$

where $\mathbb{P}_3 = (-\tilde{\Delta}+1)^{\frac{1}{2}}(-\tilde{\Delta})$ is the same as the $\mathbb{P}_3$ operator defined on S^3 as in section 2.

We would also like to remark that the second term in (5.9) above, i.e. the term $\left\{(-\tilde{\Delta}+1)^{\frac{1}{2}} + 1\right\}$ is the Dirichlet to Neumann operator on (B^4, S^3).

We can get the following result as a direct corollary of Lemma 5.2.

Corollary. *On B^4,*

$$P_3(w) = \mathbb{P}_3(w) \text{ on } \partial B^4,$$

provided that $\Delta^2 w = 0$ *in* B^4*.*

That is, on (B^4, S^3), P^3 is an extension of the Dirichlet- Neumann with respect to bi-harmonic functions. Thus the study of the P^3 operator can be viewed as an extension of the study of Dirichlet-Neumann operator.

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