CONTACT MONOIDS AND STEIN COBORDISMS

JOHN A. BALDWIN

Abstract. Suppose S is a compact surface with boundary, and let ϕ be a diffeomorphism of S which fixes the boundary pointwise. We denote by $(M_{S,\phi}, \xi_{S,\phi})$ the contact 3-manifold compatible with the open book (S, ϕ). In this note, we construct a Stein cobordism from the contact connected sum $(M_{S,h}, \xi_{S,h})\#(M_{S,g}, \xi_{S,g})$ to $(M_{S,hg}, \xi_{S,hg})$, for any two boundary-fixing diffeomorphisms h and g. This cobordism accounts for the comultiplication map on Heegaard Floer homology discovered in [4], and it illuminates several geometrically interesting monoids in the mapping class group of $(S, \partial S)$.

1. Introduction

Let M be a closed, oriented 3-manifold. In [15], Giroux proves that there is a one-to-one correspondence between isotopy classes of contact structures on M and open book decompositions of M up to an equivalence called positive stabilization. Giroux's work places contact geometry on a more topological footing, and allows us to translate questions about tightness and fillability of contact structures into questions about diffeomorphisms of compact surfaces with boundary. In some cases, this translation is rather well-understood. For instance, Akbulut and Ozbagci [1] and, independently, Giroux [14], have shown that a contact manifold (M, ξ) is Stein fillable if and only if (M, ξ) is supported by an open book (S, ϕ) for which ϕ is a product of right-handed Dehn twists around curves in S. In a similar vein, Honda, Kazez and Matić, generalizing a result of Goodman [16], prove that (M, ξ) is tight if and only if every open book (S, ϕ) supporting (M, ξ) has *right-veering* monodromy ϕ [20].

Let $Mod^+(S, \partial S)$ denote the set of isotopy classes of orientation-preserving diffeomorphisms of S which restrict to the identity on ∂S (where isotopies are also required to fix the boundary pointwise). The subset of $Mod^+(S, \partial S)$ whose elements are represented by compositions of right-handed Dehn twists is obviously closed under composition; in other words, it's a monoid (once we include the isotopy class of the identity). So, too, is the subset consisting of isotopy classes of right-veering diffeomorphisms of S. As discussed above, these monoids have special significance in contact geometry, and strengthen our understanding of the link between contact structures and open books. The main goal of this article is to point out other geometrically interesting "contact" monoids in $Mod^+(S, \partial S)$. Let $(M_{S,\phi}, \xi_{S,\phi})$ denote the contact manifold supported by the open book (S, ϕ). Our primary result is the following.

Theorem 1.1. *$(M_{S,hg}, \xi_{S,hg})$ is the result of contact (-1)-surgery on a Legendrian link $\mathbb{L} \subset (M_{S,h}, \xi_{S,h})\#(M_{S,g}, \xi_{S,g})$.*

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In order to prove Theorem 1.1, we show that $(M_{S,h},\xi_{S,h})\#(M_{S,g},\xi_{S,g})$ is the result of contact $(+1)$-surgery on a Legendrian link in $(M_{S,hg},\xi_{S,hg})$. Once we've established this, Theorem 1.1 follows immediately from work of Ding and Geiges [8]. According to Eliashberg, the cobordism from $(M_{S,h},\xi_{S,h})\#(M_{S,g},\xi_{S,g})$ to $(M_{S,hg},\xi_{S,hg})$ obtained by attaching contact (-1)-framed 2-handles to the components of the link $\mathbb{L}$ in Theorem 1.1 carries a natural Stein structure which is compatible with the contact structures on either end [9]. So, we may alternatively think of Theorem 1.1 as the statement that there exists a Stein 2-handle cobordism from the contact connected sum $(M_{S,h},\xi_{S,h})\#(M_{S,g},\xi_{S,g})$ to $(M_{S,hg},\xi_{S,hg})$.

Theorem 1.1 illuminates three new monoids in the mapping class group $Mod^+(S,\partial S)$. To see this, recall that if the contact structures $\xi_{S,h}$ and $\xi_{S,g}$ are Stein fillable, then so is their connected sum $\xi_{S,h}\#\xi_{S,g}$ (there is a Stein 1-handle cobordism from the disjoint union $(M_{S,h},\xi_{S,h})\sqcup(M_{S,g},\xi_{S,g})$ to $(M_{S,h},\xi_{S,h})\#(M_{S,g},\xi_{S,g})$). The same is true if "Stein" is replaced by "strongly symplectically" or "weakly symplectically." Since contact (-1)-surgery on a Legendrian knot preserves Stein fillability as well as strong and weak symplectic fillability [9, 29, 12], Theorem 1.1 implies the following.

Corollary 1.2. *The set of $[\phi]\in Mod^+(S,\partial S)$ for which $\xi_{S,\phi}$ is Stein fillable forms a monoid. The same is true if "Stein" is replaced by "strongly symplectically" or "weakly symplectically."*

The contact invariant in Heegaard Floer homology is well-behaved with respect to the maps induced by Stein cobordisms (see [23]). Specifically, if (M',ξ') is obtained from (M,ξ) by performing contact (-1)-surgery on a Legendrian knot, and W is the corresponding 2-handle cobordism from M to M', then the map

$$F_{-W}:\widehat{HF}(-M')\to\widehat{HF}(-M)$$

sends $c(\xi')$ to $c(\xi)$. In addition, for two contact manifolds (M_1,ξ_1) and (M_2,ξ_2), the contact invariant $c(\xi_1\#\xi_2)$ is identified with $c(\xi_1)\otimes c(\xi_2)$ via the isomorphism

$$\widehat{HF}(-(M_1\#M_2))\cong\widehat{HF}(-M_1)\otimes_{\mathbb{Z}_2}\widehat{HF}(-M_2).$$

Coupled with these facts, Theorem 1.1 immediately implies the following result from [4].

Corollary 1.3 ([4, Theorem 1.4])**.** *There exists a "comultiplication" map*

$$\widehat{HF}(-M_{S,hg})\to\widehat{HF}(-M_{S,h})\otimes_{\mathbb{Z}_2}\widehat{HF}(-M_{S,g})$$

which sends $c(\xi_{S,hg})$ to $c(\xi_{S,h})\otimes c(\xi_{S,g})$.

In particular, the set of $[\phi]\in Mod^+(S,\partial S)$ for which $c(\xi_{S,\phi})\neq 0$ forms a monoid. This prompts the question below.

Question 1.4. *Does the set of $[\phi]\in Mod^+(S,\partial S)$ for which $\xi_{S,\phi}$ is tight form a monoid?*

Corollary 1.3 does not provide an answer to this question, as there are tight contact structures with vanishing contact invariant (namely those with positive *Giroux torsion*) [21, 13]. In fact, the seemingly more basic question below is open as well.

Question 1.5. *Let t_γ denote the right-handed Dehn twist around a curve $\gamma\subset S$. Is it the case that ξ_{S,ht_γ} is tight whenever $\xi_{S,h}$ is tight? Or, equivalently, does contact (-1)-surgery on a Legendrian knot in a closed contact 3-manifold preserve tightness?*

Clearly, an affirmative answer to Question 1.4 would provide an affirmative answer to Question 1.5. According to Theorem 1.1, the converse is also true. To see this, suppose that the contact structures $\xi_{S,h}$ and $\xi_{S,g}$ are tight; then so is their connected sum $\xi_{S,h}\#\xi_{S,g}$ [7]. If the answer to Question 1.5 is "yes" – that is, if contact (-1)-surgery on a Legendrian knot in a closed contact 3-manifold preserves tightness – then $\xi_{S,hg}$ is tight as well, by Theorem 1.1.

It bears mentioning that Honda has found a tight contact structure on a genus 4 handlebody which becomes overtwisted after performing contact (-1) surgery on a Legendrian knot [19]; so, the assumption in Question 1.5 that the manifold is closed is necessary.

The three new "contact monoids" in Corollary 1.2 have been discovered independently by Baker, Etnyre and Van Horn-Morris [2], who construct their own Stein cobordism from $(M_{S,h}, \xi_{S,h}) \sqcup (M_{S,g}, \xi_{S,g})$ to $(M_{S,hg}, \xi_{S,hg})$. In fact, it was not until hearing of their result that I realized that the cobordism from $(M_{S,h}, \xi_{S,h})\#(M_{S,g}, \xi_{S,g})$ to $(M_{S,hg}, \xi_{S,hg})$ defined implicitly in the last section of my paper with Plamenevskaya [5] carries a very natural Stein structure.

The proof of Theorem 1.1 in this article makes use of standard tools in convex surface theory together with some small input from Heegaard Floer homology. In contrast, the approach of Baker, Etnyre and Van Horn-Morris involves an understanding of the contact structures associated to various cables of the binding of an open book. It would be interesting to determine whether our different approaches yield what are more or less the same Stein cobordisms in the end.

Van Horn-Morris notes that the essential technique used in this paper can be applied in other settings. We describe such an example below. But first, we provide a bit of context.

Suppose that K is a transverse knot in the standard tight contact structure ξ_{std} on S^3. A well-known result of Bennequin asserts that

$$sl(K) \leq -\chi(\Sigma),$$

where $sl(K)$ denotes the self-linking number of K and Σ is any Seifert surface for K [6]. Eliashberg has since shown that this bound holds for transverse knots in any tight contact structure [10]. In [17], Hedden shows that if K realizes its Bennequin bound (which is to say that $sl(K) = -\chi(\Sigma)$ for some Seifert surface Σ) and is fibered, then the open book corresponding to the fibration associated with K supports the contact manifold (S^3, ξ_{std}).[1] If (S, ϕ) denotes this open book, then (S, ϕ^n) is an open book for the contact manifold obtained by taking the n-fold cyclic cover of (S^3, ξ_{std}) branched along K. Since (S, ϕ) supports ξ_{std}, which is Stein fillable, Corollary 1.2 implies the following.

Corollary 1.6. *If K is a fibered transverse knot in (S^3, ξ_{std}) which realizes its Bennequin bound, then the contact manifold obtained by taking the n-fold cyclic cover of (S^3, ξ_{std}) branched along K is Stein fillable.*

Using a slight variation of the main technique in this paper as suggested by Van Horn-Morris, combined with the ideas in [3], we can prove the same result without the condition

[1] Baker, Etnyre and Van Horn-Morris have since proved a generalization of this result which applies to transverse knots in any tight contact manifold [3]. The corollary below generalizes accordingly.

that K is fibered. In fact, we can prove the stronger statement below. We defer the proof of this theorem to a future paper.

Theorem 1.7. *If K is a transverse knot in a Stein fillable contact manifold (M, ξ) which realizes its Bennequin bound, then the contact manifold obtained by taking the n-fold cyclic cover of (M, ξ) branched along K is Stein fillable. The same is true if "Stein" is replaced by "strongly symplectically" or "weakly symplectically."*

It is worth noting that Theorem 1.7 overlaps with a similar result in [5, Corollary 1.3]. There, we show, using entirely different methods, that if K is a transverse knot in (S^3, ξ_{std}) which belongs to a *quasi-alternating* knot type (see [24]) and satisfies the equality

$$sl(K) = \sigma(K) - 1, \tag{1}$$

then the contact manifold obtained by taking the double cover of (S^3, ξ_{std}) branched along K is tight (here, $\sigma(K)$ denotes the signature of K).

In general, transverse knots in (S^3, ξ_{std}) satisfy the Bennequin-like bound [25, 27],

$$sl(K) \leq s(K) - 1, \tag{2}$$

where $s(K)$ is the numerical concordance invariant defined by Rasmussen using Khovanov homology in [26]. In his paper, Rasmussen shows that

$$|s(K)| \leq 2g_4(K),$$

and uses this fact to give a combinatorial proof of the Milnor conjecture (here, $g_4(K)$ denotes the 4-ball genus of K). In particular, the inequality above implies that

$$s(K) - 1 \leq -\chi(\Sigma) \tag{3}$$

for any Seifert surface Σ which bounds K. If K is quasi-alternating, then $sl(K) = \sigma(K)$ [26, 22]. If, in addition, K realizes its Bennequin bound, then Equations (2) and (3) force the equality in Equation (1), and both the result in [5, Corollary 1.3] described above and Theorem 1.7 imply that the contact manifold obtained by taking the double cover of (S^3, ξ_{std}) branched along K is tight.

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2. Proof of Theorem 1.1

First, we describe the contact 3-manifold, $(M_{S,\phi}, \xi_{S,\phi})$, which is compatible with the open book (S, ϕ). Let U be the handlebody defined by $U = S \times [-1, 1]/\sim$, where $(x, t) \sim (x, 0)$ for all $x \in \partial S$ (see Figure 1). The oriented curve $\Gamma = \partial S \times \{0\}$ divides $\Sigma = \partial U$ into two pieces, $\Sigma^+ = S \times \{1\}$ and $\Sigma^- = -S \times \{-1\}$. We may therefore view ϕ as a boundary-fixing diffeomorphism of Σ^+. Note that $\partial\Sigma^+ = \Gamma = -\partial\Sigma^-$, and let $r : \Sigma \to \Sigma$ be the orientation-reversing involution defined by reflection across Γ.

It is not hard to prove that there exists a unique (up to isotopy) tight contact structure ξ_0 on U for which Σ is convex with dividing set Γ (see [11], for example). Let (U_1, ξ_1) and (U_2, ξ_2) be identical copies of (U, ξ_0), with $\partial U_1 = \Sigma = \partial U_2$. According to Torisu [28], $(M_{S,\phi}, \xi_{S,\phi})$ is

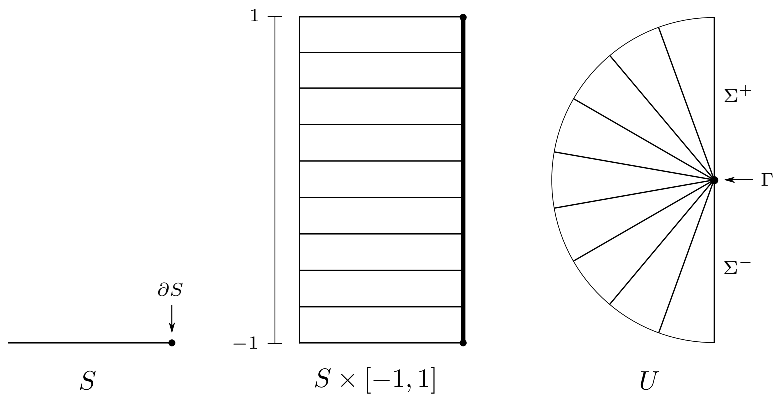

FIGURE 1. The diagram on the left represents the surface S. The diagram in the middle represents $S\times[-1,1]$; we have drawn some of the $S\times\{t\}$ fibers. The diagram on the right represents the handlebody U obtained from $S\times[-1,1]$ by collapsing $\partial S\times[-1,1]$ to $\Gamma=\partial S\times\{0\}$.

the contact 3-manifold obtained by gluing (U_2,ξ_2) to (U_1,ξ_1) via the orientation-reversing diffeomorphism $A_\phi:\partial U_2\to\partial U_1$ defined by

$$A_\phi(x)=\begin{cases} r(\phi(x)), & x\in\Sigma^+,\\ r(x), & x\in\Sigma^-.\end{cases}$$

(The orientation on $M_{S,\phi}$ is specified by $M_{S,\phi}=U_1-U_2$.) The fact that A_ϕ sends $\Gamma\subset\partial U_1$ to $\Gamma\subset\partial U_2$ is what makes it possible to glue these two contact structures together, by Giroux's Flexibility Theorem [14].

Now suppose that ϕ is the composition hg. Let I be the interval $[-\epsilon,\epsilon]$, and let ξ_I be the I-invariant contact contact structure on $\Sigma\times I$ for which each $\Sigma\times\{t\}$ is convex with dividing set $\Gamma\times\{t\}$. Then $(M_{S,hg},\xi_{S,hg})$ may also be obtained by first gluing (U_2,ξ_2) to $(\Sigma\times I,\xi_I)$ by the diffeomorphism from ∂U_2 to $\Sigma\times\{\epsilon\}$ which sends x to $(A_g(x),\epsilon)$, and then gluing the resulting contact manifold to (U_1,ξ_1) by the diffeomorphism from $\Sigma\times\{-\epsilon\}$ to ∂U_1 which sends $(x,-\epsilon)$ to $A_h(r(x))$. See Figure 2 for reference.

If S has genus g and r boundary components, then Σ has genus $n=2g+r-1$. Let $b_1,\dots,b_n$ be disjoint, properly embedded arcs in S for which $S-\cup_i b_i$ is a disk. For $i=1,\dots,n$, we define the curve $\beta_i\subset\Sigma$ by

$$\beta_i=b_i\times\{-1\}\cup b_i\times\{1\}.$$

(See Figure 3 for an example.) Note that β_i bounds the attaching disk $b_i\times[-1,1]\subset U$. In particular, U may be recovered from Σ by thickening the surface, attaching 2-handles to one side along the curves β_i, and then gluing a 3-ball to the S^2 boundary component of the resulting manifold.

Let $\mathbb{L}_\beta$ be the link, contained in the $\Sigma\times I$ portion of $M_{S,hg}$, whose components are the curves $\beta_i\times\{0\}\subset\Sigma\times\{0\}$. The link $\mathbb{L}_\beta$ is *nonisolating* in the convex surface $\Sigma\times\{0\}$; that is, $\mathbb{L}_\beta$ is transverse to $\Gamma\times\{0\}$, and the closure of every component of $\Sigma\times\{0\}-(\Gamma\times\{0\}\cup\mathbb{L}_\beta)$ intersects $\Gamma\times\{0\}$. Therefore, by the Legendrian Realization Principle, we may assume that

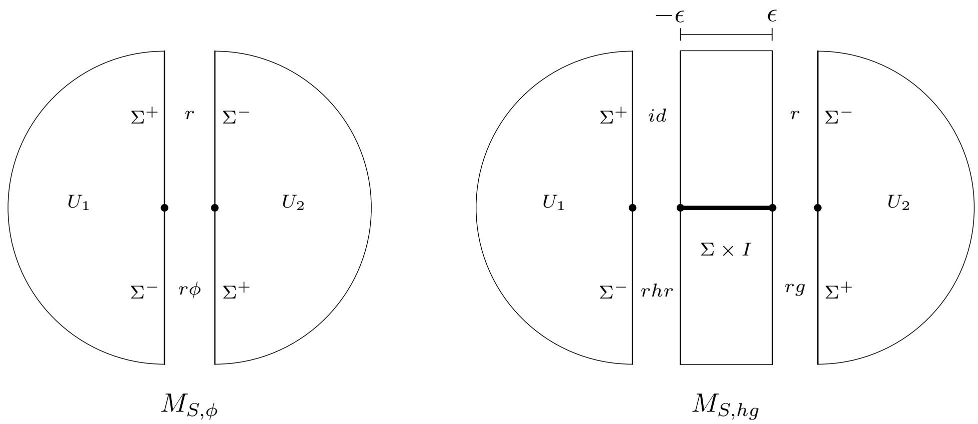

FIGURE 2. The diagram on the left illustrates the process of gluing U_2 to U_1 to form $M_{S,\phi}$. Alternatively, $M_{S,hg}$ can be formed by gluing U_2 to $\Sigma \times I$, and then gluing the result to U_1, as shown in the diagram on the right.

$\mathbb{L}_\beta$ is Legendrian [18]. Moreover, each $\beta_i \times \{0\}$ intersects the dividing set $\Gamma \times \{0\}$ in exactly two places. It follows that $tw(\beta_i \times \{0\}, \Sigma \times \{0\})$, which measures the contact framing of $\beta_i \times \{0\}$ relative to the framing induced by the surface $\Sigma \times \{0\}$, is

$$-\frac{1}{2}\#(\beta_i \times \{0\} \cap \Sigma \times \{0\}) = -1.$$

Therefore, contact (+1)-surgery on $\mathbb{L}_\beta$ is the same as 0-surgery on $\mathbb{L}_\beta$ with respect to the framing induced by $\Sigma \times \{0\}$.

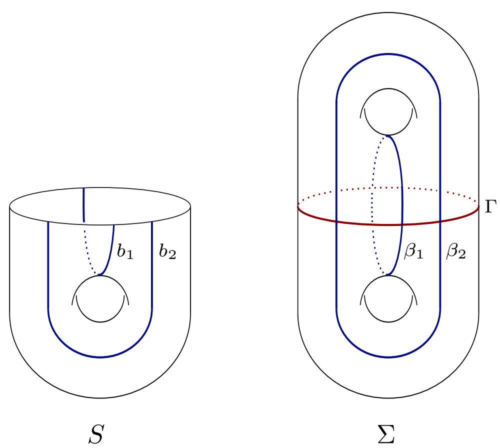

FIGURE 3. In this example, S is a genus one surface with one boundary component. The diagram on the right shows the curves β_1 and β_2 in blue, and the dividing set Γ in red.

For any contact manifold (M, ξ), and any Legendrian link $\mathbb{L} \subset M$, let us denote by $(M_\mathbb{L}, \xi_\mathbb{L})$ the contact manifold obtained from M via contact (+1)-surgery on $\mathbb{L}$.

Proposition 2.1. *The contact manifold* $((\Sigma\times I)_{\mathbb{L}_\beta},(\xi_I)_{\mathbb{L}_\beta})$ *is tight.*

Proof. By construction, $((\Sigma\times I)_{\mathbb{L}_\beta},(\xi_I)_{\mathbb{L}_\beta})$ embeds into $((M_{S,hg})_{\mathbb{L}_\beta},(\xi_{S,hg})_{\mathbb{L}_\beta})$ for any h and g. So, it is enough to find an h and g for which the latter is tight. Let h and g each be the identity. In this case, $M_{S,hg}=M_{S,id}\cong\#^n(S^1\times S^2)$, and each component $\beta_i\times\{0\}$ of $\mathbb{L}_\beta$ bounds the disk

$$b_i\times[-1,1]\cup\beta_i\times[-\epsilon,0]\,\subset\,U_1\cup\Sigma\times[-\epsilon,0].$$

Moreover, the framings induced by these disks agree with the framings induced by $\Sigma\times\{0\}$. So, topologically, $(M_{S,id})_{\mathbb{L}_\beta}$ is the result of 0-surgery on an n component unlink in $\#^n(S^1\times S^2)$; that is, $(M_{S,id})_{\mathbb{L}_\beta}\cong\#^{2n}(S^1\times S^2)$. If W is the cobordism from $M_{S,id}$ to $(M_{S,id})_{\mathbb{L}_\beta}$ obtained by attaching 0-framed 2-handles to the unknots $\beta_i\times\{0\}$, then it follows that the map

$$F_{-W}:\widehat{HF}(-M_{S,id})\to\widehat{HF}(-(M_{S,id})_{\mathbb{L}_\beta})$$

is injective [24, Proposition 6.1]. By [23], this map sends $c(\xi_{S,id})$ to $c((\xi_{S,id})_{\mathbb{L}_\beta})$. The contact invariant $c(\xi_{S,id})$ is non-zero since $\xi_{S,id}$ is Stein fillable; hence, $c((\xi_{S,id})_{\mathbb{L}_\beta})$ is non-zero as well, by the injectivity of F_{-W}. Thus, $(\xi_{S,id})_{\mathbb{L}_\beta}$ is tight, and so is $(\xi_I)_{\mathbb{L}_\beta}$.

□

Proposition 2.2. *The contact manifold* $((M_{S,hg})_{\mathbb{L}_\beta},(\xi_{S,hg})_{\mathbb{L}_\beta})$ *is the contact connected sum* $(M_{S,h},\xi_{S,h})\#(M_{S,g},\xi_{S,g})$.

As mentioned in the introduction, Proposition 2.2 implies Theorem 1.1.

Proof of Proposition 2.2. Let $N_i\subset\Sigma\times I$ be a tubular neighborhood of $\beta_i\times\{0\}$ such that ∂N_i is the union of two annuli, $A_i^1\subset\Sigma\times[-\epsilon,0]$ and $A_i^2\subset\Sigma\times[0,\epsilon]$. And let us think of S^1 as the union of two intervals, $S^1=I_1\cup I_2$. Topologically, $(\Sigma\times I)_{\mathbb{L}_\beta}$ is obtained from $\Sigma\times I$ by performing 0-surgery on $\mathbb{L}_\beta$ with respect to the framing induced by $\Sigma\times\{0\}$, as discussed above. $(\Sigma\times I)_{\mathbb{L}_\beta}$ is therefore the result of gluing solid tori $D_i^2\times S^1$ to $\Sigma\times I-\cup_i\operatorname{int}N_i$ so that $\partial D_i^2\times I_1$ is glued to A_i^1, and $\partial D_i^2\times I_2$ is glued to A_i^2 (see Figure 4). So, $(\Sigma\times I)_{\mathbb{L}_\beta}$ is the union

$$(\Sigma\times[-\epsilon,0]-\cup_i\operatorname{int}N_i)\cup_i(D_i^2\times I_1)\bigcup(\Sigma\times[0,\epsilon]-\cup_i\operatorname{int}N_i)\cup_i(D_i^2\times I_2).\tag{4}$$

Each of these two pieces is homeomorphic to the manifold obtained by thickening Σ and attaching 2-handles to one side of this thickened surface along the curves β_i; in other words, each piece is the complement of a 3-ball in a genus n handlebody, and these pieces are attached along their common S^2 boundary component.

Let us denote the left and right pieces in (4) by U_3-B^3 and U_4-B^3, respectively, where U_3 and U_4 are genus n handlebodies with $\partial U_3=-\Sigma\times\{-\epsilon\}$ and $\partial U_4=\Sigma\times\{\epsilon\}$. According to [14], their common S^2 boundary component can be made convex in $((\Sigma\times I)_{\mathbb{L}_\beta},(\xi_I)_{\mathbb{L}_\beta})$ after a small isotopy. By Proposition 2.1, the restriction of $(\xi_I)_{\mathbb{L}_\beta}$ to U_i-B^3 is tight, for $i=3,4$. Therefore, by Honda's Gluing Theorem [19, Theorem 2.5], the restriction $(\xi_I)_{\mathbb{L}_\beta}|_{U_i-B^3}$ is isotopic to the contact structure on the complement of a Darboux ball in (U_i,ξ_i), where ξ_i is the unique tight contact structure on U_i for which ∂U_i is convex with dividing set $\Gamma\times\{-\epsilon\}$ when $i=3$, and

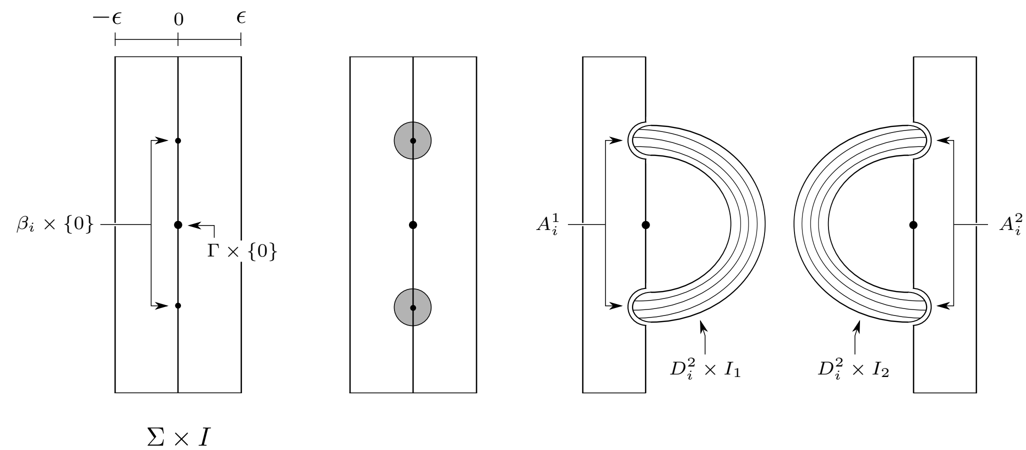

FIGURE 4. The diagram on the left shows the knot $\beta_i \times \{0\} \subset \Sigma \times \{0\}$. The shaded disks in the middle diagram represent the tubular neighborhood N_i. The diagram on the right illustrates the process of performing 0-surgery on $\beta_i \times \{0\}$ by removing N_i and gluing 2-handles along the annuli A_i^1 and A_i^2. We have drawn some of the $D^2 \times \{t\}$ fibers in these 2-handles.

$\Gamma \times \{\epsilon\}$ when $i = 4$. Said differently, $((\Sigma \times I)_{\mathbb{L}_\beta}, (\xi_I)_{\mathbb{L}_\beta})$ is the contact connected sum of identical copies, (U_3, ξ_3) and (U_4, ξ_4), of the contact handlebody (U, ξ_0).

As a result, $((M_{S,hg})_{\mathbb{L}_\beta}, (\xi_{S,hg})_{\mathbb{L}_\beta})$ may be pieced together as follows. First, glue (U_2, ξ_2) to (U_4, ξ_4) by the diffeomorphism from ∂U_2 to $\partial U_4 = \Sigma \times \{\epsilon\}$ which sends x to $(A_g(x), \epsilon)$; this forms $(M_{S,g}, \xi_{S,g})$. Next, glue (U_3, ξ_3) to (U_1, ξ_1) by the diffeomorphism from $-\partial U_3 = \Sigma \times \{-\epsilon\}$ to ∂U_1 which sends $(x, -\epsilon)$ to $A_h(r(x))$; this forms $(M_{S,h}, \xi_{S,h})$. Finally, remove Darboux balls from the U_3 and U_4 portions of $M_{S,h}$ and $M_{S,g}$, and glue the resulting contact manifolds together by a diffeomorphism which identifies the dividing curves on their S^2 boundary components. This process realizes $((M_{S,hg})_{\mathbb{L}_\beta}, (\xi_{S,hg})_{\mathbb{L}_\beta})$ as the contact connected sum of $(M_{S,h}, \xi_{S,h})$ with $(M_{S,g}, \xi_{S,g})$.

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Department of Mathematics, Princeton University, Princeton, NJ 08544-1000

E-mail address: `baldwinj@math.princeton.edu`