BIGNESS IN COMPATIBLE SYSTEMS

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ABSTRACT. Clozel, Harris and Taylor have recently proved a modularity lifting theorem of the following general form: if ρ is an ℓ-adic representation of the absolute Galois group of a number field for which the residual representation $\overline{\rho}$ comes from a modular form then so does ρ. This theorem has numerous hypotheses; a crucial one is that the image of $\overline{\rho}$ must be "big," a technical condition on subgroups of GL_n. In this paper we investigate this condition in compatible systems. Our main result is that in a sufficiently irreducible compatible system the residual images are big at a density one set of primes. This result should make some of the work of Clozel, Harris and Taylor easier to apply in the setting of compatible systems.

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1. INTRODUCTION

Let k be a field, let V be a finite dimensional k-vector space and let G be a subgroup of $\mathrm{GL}(V)$. For an endomorphism g of V and an element α of k we let $V_{g,\alpha}$ denote the α-generalized eigenspace of g. It is naturally a sub and a quotient of V. We say that G is *big* if the following three conditions hold:

- (B1) The space V is absolutely irreducible as a G-module.
- (B2) We have $H^1(G, \mathrm{ad}^\circ V) = 0$.
- (B3) For every irreducible G-submodule W of $\mathrm{ad}\, V$ we can find $g \in G$, $\alpha \in k$ and $f \in W$ such that $V_{g,\alpha}$ is one dimensional and the composite

$$\begin{tikzcd} V_{g,\alpha} \arrow[r, hook] & V \arrow[r, "f"] & V \arrow[r, two heads] & V_{g,\alpha} \end{tikzcd}$$

is non-zero.

Let k'/k be an extension and put $V' = V \otimes k'$. If G is a big subgroup of $\mathrm{GL}(V)$ then it is big when regarded as a subgroup of $\mathrm{GL}(V')$ (see Proposition 3.2). The converse to this statement is false, since (B3) depends on the existence of rational eigenvalues. We say that G is *potentially big* if there exists an extension k'/k such that G is big when regarded as a subgroup of $\mathrm{GL}(V')$.

The bigness condition is important in the work of Clozel, Harris and Taylor [CHT]. They prove a modularity lifting theorem of the following general form: if ρ is an ℓ-adic representation of the absolute Galois group G_F of a number field F such that $\overline{\rho}$ comes from a modular form then so does ρ. There are several hypotheses

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in these theorems, but one crucial one is that the image of $\overline{\rho}$ must be big. In this paper, we investigate the bigness condition in compatible systems and essentially show that it automatically holds at a set of density one. Thus the theorems of [CHT] should become easier to apply in the setting of compatible systems. Precisely, our main theorem is the following:

Theorem 1.1. *Let F be a global field and let $\{\rho_\ell\}_{\ell\in L}$ be a compatible system of n-dimensional ℓ-adic representations of its Galois group G_F, indexed by a set of primes L of density one. Assume that each ρ_ℓ is absolutely irreducible when restricted to any open subgroup of G_F. Then there is a set of primes $L' \subset L$ of density one such that $\overline{\rho}_\ell(G_F)$ is a potentially big subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$ for all $\ell \in L'$. In fact, one can find a finite Galois extension $E/\mathbb{Q}$ such that $\overline{\rho}_\ell(G_F)$ is a big subgroup of $\mathrm{GL}_n(k_\ell)$ for $\ell \in L'$, where k_ℓ is the residue field of E at a prime over ℓ. In particular, $\overline{\rho}_\ell(G_F)$ is a big subgroup of $\mathrm{GL}_n(\mathbb{F}_\ell)$ for a set of primes of positive density.*

Here $\overline{\rho}_\ell$ denotes the mod ℓ reduction of ρ_ℓ. Of course, to form this one must choose a lattice and different lattices can give different reductions. However, we will see below that for a set of ℓ of density one the result is independent of the choice of lattice (indeed, this is implied by the theorem).

For our definition of "compatible system" see §8. We point out here, however, that by an "ℓ-adic representation" we mean one on a $\mathbb{Q}_\ell$-vector space.

1.1. Outline of proof.

Broadly speaking, the proof of Theorem 1.1 has three steps:

1. We first show that if k is a finite field, G/k a reductive group and $r : G \to \mathrm{GL}_n$ an algebraic representation of G then, subject to some conditions, the group $r(G(k))$ is big.
2. Using this, we show that if $\rho : \Gamma \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ is an ℓ-adic representation of a profinite group whose image is a hyperspecial maximal compact subgroup of its Zariski closure (or even a "nearly hyperspecial" subgroup) and ℓ is sufficiently large compared to n then the residual image of ρ is a potentially big subgroup of $\mathrm{GL}(\mathbb{F}_\ell)$.
3. Finally, we combine the above with a result of Larsen to deduce Theorem 1.1.

1.2. Examples.

We should point out that one can construct compatible systems which satisfy the hypotheses of the theorem. Let F be a totally real number field (resp. imaginary CM field) and let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$ satisfying the following conditions:

- (C1) π is *regular algebraic*. This means that π_∞ has the same infinitesimal character as some irreducible algebraic representation of the restriction of scalars from F to $\mathbb{Q}$ of GL_n.
- (C2) π is *essentially self-dual* (resp. *conjugate self-dual*). When F is totally real this means that $\pi^\vee = \chi\otimes\pi$ for some character χ of the idele group of F for which $\chi_v(-1)$ is independent of v, as v varies over the infinite places of F. When F is imaginary CM, this means that $\pi^\vee = \pi^c$, where c denotes complex conjugation.
- (C3) There is some finite place v_0 of F such that π_{v_0} is a twist of the Steinberg representation.

Under these conditions, we can associate to π a compatible system of semi-simple ℓ-adic representations $\{\rho_\ell\}$ of G_F, indexed by the set of all primes (for more precise statements, see [CHT, Proposition 3.2.1] and [CHT, Proposition 3.3.1]). Now, let ℓ be a prime for which $v_0 \nmid \ell$. Assume that F is imaginary CM. By the main result of [TY] the (Frobenius semi-simplification of the) representation $\rho_\ell|_{G_{F,v_0}}$ corresponds to π_{v_0} under the local Langlands correspondence (we write G_{F,v_0} for the decomposition group at v_0). Thus for an element $g = F^n g'$ of G_{F,v_0}, where F is a fixed Frobenius element and g' belongs to the inertia subgroup I_{F,v_0}, we have

$$\rho_\ell(g) = \nu(g)\cdot\begin{pmatrix} \chi^{n-1}(g) & & & \\ & \chi^{n-2}(g) & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}\cdot \exp(t_\ell(g')N)$$

where ν is some character, χ is the cyclotomic character, t_ℓ is the tame ℓ-adic character of I_{F,v_0} and N is a nilpotent matrix with $N^n = 0$ and $N^k \neq 0$ for $k < n$ (one can take N to have 0's on the diagonal and 1's on the off diagonal). It follows that ρ_ℓ is absolutely indecomposable when restricted to any open subgroup of G_{F,v_0}. Since ρ_ℓ is semi-simple, it therefore follows that it is absolutely irreducible when restricted to any open subgroup of G_F. When F is totally real we can still conclude that ρ_ℓ has this property by making an appropriate abelian base change to an imaginary CM field and appealing to the above argument. Thus $\{\rho_\ell\}_{\ell\in L}$ satisfies the hypotheses of the theorem, where L is the set of all primes except for the single prime lying under v_0.

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2. Notation and conventions

For a scheme X/S and a map $T \to S$ we denote by $X(T)$ the T-points of X and by X_T the base change of X to T. For functorial constructs we use an underline to denote the sheaf version and no underline to denote the usual version. For example, if G/S is an algebraic group then $\mathrm{Aut}(G)$ is its group of automorphisms while $\underline{\mathrm{Aut}}(G)$ is the sheaf which assigns to $T \to S$ the group $\mathrm{Aut}(G_T)$.

For a vector space V over a field k we let $\mathrm{GL}(V)$ denote the group of automorphisms of V and we let $\underline{\mathrm{GL}}(V)$ denote the algebraic group whose k-points are $\mathrm{GL}(V)$. We write $\operatorname{ad} V$ for the vector space of endomorphisms of V and $\operatorname{ad}^\circ V$ for the subspace of trace zero endomorphisms.

Reductive and semi-simple groups (over fields) are connected. For a group G over a field k we let G° denote the connected component of the identity, G^{ad} the adjoint group of the quotient of G° by its radical, which is a semi-simple group, and G^{sc} the simply connected cover of G^{ad}. We also write G^{der} for the derived subgroup of G°, which is semi-simple if G° is reductive.

More notation is defined in the body of the paper.

3. Elementary properties of bigness

In this section we establish some elementary properties of bigness. Throughout this section, k denotes an arbitrary field, V a finite dimensional vector space over k and G a subgroup of $\mathrm{GL}(V)$.

We begin by giving a reformulation of condition (B3). Consider the following condition.

(B3') The natural map

$$\operatorname{ad} V \to \bigoplus_{(g,\alpha)\in S} \mathrm{End}(V_{g,\alpha})$$

is injective, where S is the set of pairs $(g,\alpha) \in G \times k$ for which $V_{g,\alpha}$ is one dimensional. (An element $f \in \operatorname{ad} V$ gives an element of $\mathrm{End}(V_{g,\alpha})$ as in the statement of (B3); the “natural map” above is the sum of these maps.)

Note that the above map is G-equivariant, using the obvious G-structure on the target. We then have the following:

Proposition 3.1. *The conditions (B3) and (B3') are equivalent.*

Proof. First say that G satisfies (B3'). Let A denote the map in the statement of (B3'). Let W be an irreducible submodule of $\operatorname{ad} V$. Given any non-zero $f \in W$ we have that $A(f)$ is non-zero and so one of the coordinates of $A(f)$ is non-zero. Thus f induces a non-zero endomorphism on $V_{g,\alpha}$ for some (g,α) with $V_{g,\alpha}$ one dimensional. This proves that G satisfies (B3).

Now say that G does not satisfy (B3'). Then $\ker A$ is non-zero and thus contains an irreducible submodule W of $\operatorname{ad} V$. Since $A(W) = 0$ we see that for any $f \in W$ and any (g,α) with $V_{g,\alpha}$ one dimensional the map induced by f on $V_{g,\alpha}$ is zero. This shows that G does not satisfy (B3). Thus the two conditions are equivalent. ☐

Proposition 3.2. *Let k'/k be an extension and put $V' = V \otimes k'$. If G is a big subgroup of* $\mathrm{GL}(V)$ *then it is a big subgroup of* $\mathrm{GL}(V')$.

Proof. Conditions (B1) and (B2) are immediate. We prove (B3). Let S (resp. S') be the set of (g, α) in $G \times k$ (resp. $G \times k'$) such that $V_{g,\alpha}$ (resp. $V'_{g,\alpha}$) is one dimensional. Of course S is naturally a subset of S'. Now, we are given that the map

$$\mathrm{ad}\, V \to \bigoplus_{(g,\alpha)\in S} \mathrm{End}(V_{g,\alpha})$$

is injective. Tensoring with k' and replacing S by S', we see that the map

$$\mathrm{ad}\, V' \to \bigoplus_{(g,\alpha)\in S'} \mathrm{End}(V'_{g,\alpha})$$

is injective. We thus conclude that $G \subset \mathrm{GL}(V')$ satisfies (B3). ☐

Proposition 3.3. *If G has a normal subgroup which is big then G is big.*

Proof. Let H be a normal subgroup of G which is big. Then V is absolutely irreducible for H and thus for G as well. Thus G satisfies (B1). Next, we have an exact sequence

$$H^1(G/H, (\mathrm{ad}^\circ V)^H) \to H^1(G, \mathrm{ad}^\circ V) \to H^1(H, \mathrm{ad}^\circ V)^{G/H}.$$

Since H is big we have $H^1(H, \mathrm{ad}^\circ V) = 0$ and so the rightmost term vanishes. Since V is absolutely irreducible for H we have $(\mathrm{ad}^\circ V)^H = 0$ and so the leftmost term vanishes. Thus $H^1(G, \mathrm{ad}^\circ V) = 0$ and G satisfies (B2). Finally let W be a G-irreducible submodule of $\mathrm{ad}\, V$. Let W' be an H-irreducible submodule of W. Since H is big, we can find $g \in H$, $\alpha \in k$ and $f \in W'$ such that $V_{g,\alpha}$ is one dimensional and $f(V_{g,\alpha})$ has non-zero projection to $V_{g,\alpha}$. Of course, g also belongs to G and f also belongs to W. Thus G satisfies (B3) as well and is therefore big. ☐

Proposition 3.4. *The group G is big if and only if $k^\times G$ is, where $k^\times$ denotes the group of scalar matrices in* $\mathrm{GL}(V)$.

Proof. Since G is a normal subgroup of $k^\times G$, the bigness of the former implies that of the latter by Proposition 3.3. Now assume that $k^\times G$ is big. Since V is absolutely irreducible for $k^\times G$ it is for G as well. Thus G satisfies (B1). Next, we have an exact sequence

$$1 \to G \to k^\times G \to H \to 1$$

where H is a quotient of $k^\times$. We thus have an exact sequence

$$H^1(k^\times G, \mathrm{ad}^\circ V) \to H^1(G, \mathrm{ad}^\circ V)^H \to H^2(H, (\mathrm{ad}^\circ V)^G).$$

The group on the left vanishes by hypothesis. The group on the right vanishes since $(\mathrm{ad}^\circ V)^G = 0$. Thus the group in the middle vanishes. Now, the action of H on $H^1(G, \mathrm{ad}^\circ V)$ is trivial. (Proof: Let $f : G \to \mathrm{ad}^\circ V$ be a 1-cocycle representing representing a cohomology class $[f]$ and let h be an element of H. Then $h \cdot [f]$ is represented by the 1-cocycle $g \mapsto \widetilde{h} f(\widetilde{h}^{-1} g \widetilde{h})$ for any lift $\widetilde{h}$ of h. We can pick a lift $\widetilde{h}$ of h which belongs to $k^\times$. Thus $\widetilde{h}$ acts trivially on G by conjugation and acts trivially on $\mathrm{ad}^\circ V$. Therefore $h \cdot [f] = [f]$.) It thus follows that $H^1(G, \mathrm{ad}^\circ V)$ vanishes and so G satisfies (B2).

As for the last condition, let W be an irreducible G-submodule of $\mathrm{ad}\, V$. Then it is also an irreducible $k^\times G$-module. Thus we can find $g \in k^\times G$, $\alpha \in k$ and $f \in W$ such that $V_{g,\alpha}$ is one dimensional and $f(V_{g,\alpha})$ has non-zero projection to $V_{g,\alpha}$. We can write $g = zg'$ where z belongs to $k^\times$ and g' belongs to G. Put $\alpha' = \alpha z^{-1}$. Then $V_{g',\alpha'} = V_{g,\alpha}$. Thus this space is one dimensional and $f(V_{g',\alpha'})$ has non-zero projection to $V_{g',\alpha'}$. We have therefore shown that G satisfies (B3). Thus G is big. ☐

4. Background on representations of algebraic groups

Let S be a scheme. A group scheme G/S is *reductive* (resp. *semi-simple*) if it is smooth, affine and its geometric fibers are reductive (resp. semi-simple). This implies that its geometric fibers are connected, by our conventions. Such a group is a *torus* if it is fppf locally isomorphic to $\mathbb{G}_m^n$. A torus is *split* if it is (globally) isomorphic to $\mathbb{G}_m^n$ (it may be best to allow n to be a function of the connected components of the base S; this will not be an issue for us). By a *maximal torus* in G we mean a subtorus which is maximal in each geometric fiber. Similarly, by a *Borel subgroup* we mean a closed subgroup which is smooth over S and a Borel subgroup in each geometric fiber. A reductive group G/S is *split* if it has a split maximal torus T such that the weight spaces of T on $\underline{\mathrm{Lie}}(G)$ are *free* coherent sheaves on S. See [SGA3] for the general theory.

4.1. **Borel-Weil type representations.**

Let G/S be a split reductive group over a connected locally noetherian base S. We first review a Borel-Weil type construction of representations of G. Let B be a Borel subgroup of G and let $T \subset B$ be a split maximal torus. Let λ be a dominant weight of T and let $\mathcal{L}_S(\lambda)$ be the natural G-equivariant line bundle on G/B associated to λ. Put $V_{S,\lambda} = f_*\mathcal{L}_S(\lambda)$, where $f : G/B \to S$ is the structure map. Thus $V_{S,\lambda}$ is a coherent sheaf on S with a natural action of G. We omit the S from the notation if it is clear from context. Consider a cartesian diagram

$$\begin{tikzcd}
(G/B)_{S'} \arrow[r, "g'"] \arrow[d, "f'"'] & G/B \arrow[d, "f"] \\
S' \arrow[r, "g"] & S.
\end{tikzcd}$$

Note that formation of $\mathcal{L}_S(\lambda)$ commutes with pull-back, that is, $(g')^*\mathcal{L}_S(\lambda) = \mathcal{L}_{S'}(\lambda)$. Kempf's vanishing theorem [Jan, Proposition II.4.5] states that if S' is a geometric point of S then $R^i f'_* \mathcal{L}_{S'}(\lambda) = 0$ for $i > 0$. Thus, using a combination of the formal functions theorem and the proper base change theorem (see also the chapter "Cohomology and base change" in [Mum]), we see that $V_{S,\lambda}$ is a locally free sheaf on S and its formation commutes with base change, that is, for any diagram as above we have $V_{S',\lambda} = g^*V_{S,\lambda}$.

Assume now that $S = \mathrm{Spec}(k)$ with k an algebraically closed field. If k has characteristic zero then V_λ is an irreducible representation of G. Furthermore, every irreducible representation of G is isomorphic to a unique V_λ. This is the classical Borel-Weil theorem. If k does not have characteristic zero then V_λ may not be irreducible. However, it has a unique irreducible submodule $\mathrm{soc}(V_\lambda)$ and any irreducible representation of G is isomorphic to a unique $\mathrm{soc}(V_\lambda)$ (see [Jan, Corollary II.2.7]). The representation $\mathrm{soc}(V_\lambda)$ is the unique irreducible with λ as its highest weight.

4.2. **Representations of restricted norm.**

Let G/S be a split reductive group and let $T \subset B$ be as above. For a weight λ of T we let $|\lambda|_\infty$ be the maximum value of $|\langle \lambda, \alpha^\vee \rangle|$ as α varies over the roots of G. Let V be a representation of G; that is, V is a coherent sheaf on S and there is a given map $G \to \underline{\mathrm{GL}}(V)$. We let $|V|_\infty$ be the maximum value of the $|\lambda|_\infty$ among the weights λ appearing in V. If V is an extension of V' by V'' then $|V|_\infty = \max(|V'|_\infty, |V''|_\infty)$. If $p : G \to G'$ is a surjection of reductive groups and V is a representation of G' then $|p^*V|_\infty = |V|_\infty$ (this statement holds for some other types of maps p, *e.g.*, the inclusion of G^{der} into G). If G is semi-simple then there are only finitely many weights of a given norm.

We let $\mathrm{Rep}(G)$ be the category of representations of G. For an integer n we let $\mathrm{Rep}^{(n)}(G)$ be the full subcategory of $\mathrm{Rep}(G)$ on those representations V which satisfy $|V|_\infty < n$. If

$$0 \to V' \to V \to V'' \to 0$$

is an exact sequence in $\mathrm{Rep}(G)$ then V belongs to $\mathrm{Rep}^{(n)}(G)$ if and only if both V' and V'' do. In other words, $\mathrm{Rep}^{(n)}(G)$ is a Serre subcategory of $\mathrm{Rep}(G)$.

Proposition 4.1. *Let $S = \mathrm{Spec}(k)$ with k an algebraically closed field and let G/S be a reductive group. Assume $\mathrm{char}\, k$ is zero or large compared to n and G. Then the simple objects of $\mathrm{Rep}^{(n)}(G)$ are of the form V_λ for appropriate λ.*

Proof. The statement is well-known for k of characteristic zero; thus assume k has positive characteristic. The simple objects of $\mathrm{Rep}(G)$ are exactly the $\mathrm{soc}(V_\lambda)$. Now, $|\mathrm{soc}(V_\lambda)|_\infty \ge |\lambda|_\infty$ as λ occurs as a weight in $\mathrm{soc}(V_\lambda)$. Thus if $\mathrm{soc}(V_\lambda)$ belongs to $\mathrm{Rep}^{(n)}(G)$ then $|\lambda|_\infty \le n$. On the other hand, it is known that for $\mathrm{char}\, k$ large compared to $\dim G$ and $|\lambda|_\infty$ the representation V_λ is irreducible (see [Spr]). This establishes the proposition. □

Proposition 4.2. *Let $S = \mathrm{Spec}(k)$ with k an algebraically closed field and let G/S be a reductive group. Assume that $\mathrm{char}\, k$ is zero or sufficiently large compared to $\dim G$ and n. Then the category $\mathrm{Rep}^{(n)}(G)$ is semi-simple.*

Proof. First note that the simple objects of $\mathrm{Rep}^{(n)}(G)$ have dimension bounded in terms of n and $\dim G$. Here is why: The group G is the pull-back to k of a unique split reductive group over $\mathbb{Z}$, which we still call G. The simple $V_{k,\lambda}$ is just $V_{\mathbb{Z},\lambda} \otimes k$. Thus the dimension of $V_{k,\lambda}$ is the same as the dimension of $V_{\mathbb{C},\lambda}$. This dimension can then be bounded in terms of $\dim G$ and $|\lambda|_\infty$ using the Weyl dimension formula [FH, Corollary 24.6] and the fact that there are only finitely many root systems of a given rank.

Now, it is known (see [Jan2], [Lar2]) that any representation of G with small dimension compared to $\mathrm{char}\, k$ is semi-simple. It thus follows that, for $\mathrm{char}\, k$ large, if A and B are two simples of $\mathrm{Rep}^{(n)}(G)$ then any extension of A by B is semi-simple, and thus $\mathrm{Ext}^1(A, B) = 0$. This shows that $\mathrm{Rep}^{(n)}(G)$ is semi-simple. □

Proposition 4.3. *Let $S = \mathrm{Spec}\, \mathscr{O}_K$ where $K/\mathbb{Q}_p$ is a finite extension and let G/S be a reductive group. Let V be a free sheaf on S with a representation of G such that the representation of $G_{\overline{K}}$ on $V_{\overline{K}}$ is irreducible. Assume that p is large compared to $\mathrm{rk}\, V$. Then the representation of $G_{\overline{k}}$ on $V_{\overline{k}}$ is irreducible, where k is the residue field of $\mathscr{O}_K$. In any case, $|V_{\overline{k}}|_\infty = |V_{\overline{K}}|_\infty$ and this is bounded in terms of $\mathrm{rk}\, V$.*

Proof. By enlarging K if necessary we can assume that G is split. Let T be a split maximal torus in G. As maps of tori are rigid, the weights of T in $V_{\overline{K}}$ and $V_{\overline{k}}$ are the same. This shows that their norms agree. We now show that $V_{\overline{k}}$ is irreducible, assuming $V_{\overline{K}}$ is. First observe that $|V_{\overline{K}}|_\infty$ can be bounded in terms of $\mathrm{rk}\, V$ (this is a statement about representations of complex Lie groups) and so we may assume that p is large compared to $|V_{\overline{k}}|_\infty$. It follows that $V_{\overline{k}}$ can be written as $\bigoplus V_{\overline{k},\lambda_i}$. From this we see that the representations $V_{\overline{K}}$ and $\bigoplus V_{\overline{K},\lambda_i}$ of $G_{\overline{K}}$ have the same weights. This implies that they are isomorphic. We conclude that there must be only one term in the sum, and so $V_{\overline{k}}$ is irreducible. □

4.3. **Representations of** $G(k)$**.** Let k be a finite field and let G/k be a reductive group. We denote by $\mathrm{Rep}(G(k))$ the category of representations of the finite group $G(k)$ on $\overline{k}$-vector spaces.

Proposition 4.4. *Let k be a finite field and G/k a semi-simple simply connected group. Assume $\mathrm{char}\, k$ is sufficiently large compared to $\dim G$ and n. Then the functor $\mathrm{Rep}^{(n)}(G_{\overline{k}}) \to \mathrm{Rep}(G(k))$ is fully faithful and the essential image is a Serre subcategory of $\mathrm{Rep}(G(k))$.*

Proof. The functor $\mathrm{Rep}^{(n)}(G_{\overline{k}}) \to \mathrm{Rep}(G(k))$ is clearly faithful and exact. The desired properties now follow from the fact that $\mathrm{Rep}^{(n)}(G_{\overline{k}})$ is semi-simple and the fact that if V is an irreducible representation of $G_{\overline{k}}$ with norm small compared to $\mathrm{char}\, k$ then it stays irreducible when restricted to $G(k)$ (see [Lar, §1.13]). □

4.4. **Representations of** $\mathrm{Lie}(G)$**.** We will need the following result:

Proposition 4.5. *Let k be the algebraic closure of a finite field, G/k a semi-simple group and V an irreducible representation of G. Pick a system of positive roots P in $\mathfrak{g} = \mathrm{Lie}(G)$. Assume $\mathrm{char}\, k$ is large compared to $|V|_\infty$ and $\dim G$. Then the subspace of V annihilated by P is one dimensional. (This subspace is the highest weight space of V with respect to P.)*

Proof. Denote by G still the unique split group over $\mathbb{Z}$ giving rise to G/k. By our hypotheses we have $V = V_{k,\lambda}$ for some dominant weight λ. We know that $V_{k,\lambda} = V_{\mathbb{Z},\lambda} \otimes k$ and similarly $V_{\mathbb{C},\lambda} = V_{\mathbb{Z},\lambda} \otimes \mathbb{C}$. Now, since G is split over $\mathbb{Z}$ for each $r \in P$ we can find $X_r \in \mathfrak{g}_{\mathbb{Z}}$ which generates the r root space of $\mathfrak{g}_{\mathbb{Z}}$. Consider the map

$$V_{\mathbb{Z},\lambda} \to \bigoplus_P V_{\mathbb{Z},\lambda}, \qquad v \mapsto (X_r \cdot v)_{r \in P}.$$

This is a linear map of finite free $\mathbb{Z}$-modules. After tensoring with $\mathbb{C}$ the kernel of this map is the subspace of $V_{\mathbb{C},\lambda}$ annihilated by P. This is one dimensional by the usual highest weight theory over $\mathbb{C}$. It thus follows that for p sufficiently large, the reduction of the map modulo p will have one dimensional kernel. This proves the proposition. ☐

5. Highly regular elements of semi-simple groups

Fix a finite field k of cardinality q. The purpose of this section is to demonstrate the following result:

Proposition 5.1. *Let G/k be a semi-simple group and let $T \subset G$ be a maximal torus defined over k. Let n be an integer and assume q is large compared to* $\dim G$ *and n. Then there exists an element $g \in T(k)$ for which the map*

$$\{\lambda \in X(T_{\overline{k}}) \mid |\lambda|_\infty < n\} \to \overline{k}^\times, \qquad \lambda \mapsto \lambda(g)$$

is injective.

Before proving the proposition we give a few lemmas.

Lemma 5.2. *Let T/k a torus of rank r. Then $(q-1)^r \leq \#T(k) \leq (q+1)^r$.*

Proof. We have $\#T(k) = \det((q-F)|X(T_{\overline{k}}))$, where F is the Frobenius in $\mathrm{Gal}(\overline{k}/k)$. (For a proof of this, see [Oes, §1.5].) Since T splits over a finite extension, the action of F on $X(T_{\overline{k}})$ has finite order and so its eigenvalues are roots of unity. We thus have $\#T(k) = \prod_{i=1}^r (q - \zeta_i)$ where each ζ_i is a root of unity, from which the lemma easily follows. ☐

Lemma 5.3. *Let G/k be a semi-simple group and let $T \subset G$ be a maximal torus defined over k. Then $\#\{\lambda \in X(T_{\overline{k}}) \mid |\lambda|_\infty < n\}$ is bounded in terms of* $\dim G$ *and n.*

Proof. The quantity $\#\{\lambda \in X(T_{\overline{k}}) \mid |\lambda|_\infty < n\}$ depends only on n and the root datum associated to $(G_{\overline{k}}, T_{\overline{k}})$. Since there are only finitely many semi-simple root data of a given dimension, the result follows. ☐

Lemma 5.4. *Let G/k be a semi-simple group of rank r, $T \subset G$ a maximal torus defined over k and $\lambda \in X(T_{\overline{k}})$ a non-zero character satisfying $|\lambda|_\infty < n$. Then the kernel of the map $\lambda : T(k) \to \overline{k}^\times$ has order at most $C(q+1)^{r-1}$ for some constant C depending only on n and* $\dim G$.

Proof. For a subset S of $\{\lambda \in X(T_{\overline{k}}) \mid |\lambda|_\infty < n\}$ let $C(S)$ denote the cardinality of the torsion of the quotient of $X(T_{\overline{k}})$ by the subgroup generated by S. Let C be the least common multiple of the $C(S)$ over all S. Since C only depends upon n and the root datum associated to $(G_{\overline{k}}, T_{\overline{k}})$, it can be bounded in terms n and $\dim G$.

Now, say the character λ of $T_{\overline{k}}$ is defined over the extension k'/k. Then λ defines a map $T \to \mathrm{Res}_{k'/k}(\mathbb{G}_m)$, where Res denotes restriction of scalars. The kernel of λ is a diagonalizable group scheme whose character group is the cokernel of the map $f : \mathbb{Z}[\mathrm{Gal}(k'/k)] \to X(T_{\overline{k}})$ given by $\sigma \mapsto \sigma \cdot \lambda$ (note that $\mathbb{Z}[\mathrm{Gal}(k'/k)]$ is the character group of $\mathrm{Res}_{k'/k}(\mathbb{G}_m)$). The image of f is spanned by the $\mathrm{Gal}(\overline{k}/k)$ orbit of λ. Since $|\cdot|_\infty$ is Galois invariant, it follows that the torsion in the cokernel of f has order at most C. Furthermore, since λ is non-zero, we see that the rank of the cokernel of f is at most $r-1$.

We have thus shown that the kernel of $T \to \mathrm{Res}_{k'/k}(\mathbb{G}_m)$ is an extension of a finite group scheme of order at most C by a torus of rank at most $r-1$. It follows from Lemma 5.2 that the set of k-points of the kernel has cardinality at most $C(q+1)^{r-1}$, as was to be shown. ☐

We now prove the proposition.

Proof of Proposition 5.1. Let S be the set of all non-zero $\lambda \in X(T_{\overline{k}})$ such that $|\lambda|_\infty < 2n$ and let N be the cardinality of S. We first claim that

$$T(k) \not\subset \bigcup_{\lambda \in S} \ker \lambda.$$

Of course, this is equivalent to $T(k) \neq \bigcup_{\lambda \in S}(\ker \lambda \cap T(k))$. To see this, we look at the cardinality of each side. The right side is a union of N sets, each of which has cardinality at most $C(q+1)^{r-1}$, while the left side has cardinality at least $(q-1)^r$ by Lemma 5.2. Since N and C are small compared to q (by Lemma 5.3 and Lemma 5.4), the claim follows.

Now, pick an element $g \in T(k)$ such that $g \notin \bigcup_{\lambda \in S} \ker \lambda$. Let λ and λ' be distinct elements of $X(T_{\overline{k}})$ each of which has $|\cdot|_\infty < n$. Then $\lambda - \lambda'$ belongs to S. Thus $(\lambda/\lambda')(g) \neq 1$ and so $\lambda(g) \neq \lambda'(g)$. Therefore, $\lambda \mapsto \lambda(g)$ is injective on those λ with $|\cdot|_\infty < n$. ☐

We prove one more simple result here:

Proposition 5.5. *Let G/k be a semi-simple group which splits over k'/k. Then G has a maximal torus defined over k which splits over k'.*

Proof. Let $T_{k'}$ be a split maximal torus of $G_{k'}$. Let F be a generator of the cyclic group $\mathrm{Gal}(k'/k)$. As two split maximal tori are conjugate by an element over the base field, we can find $g \in G(k')$ such that $FT_{k'} = gT_{k'}g^{-1}$. We may change the action of F on $G_{k'}$ to $x \mapsto g^{-1}Fxg$ without affecting the isomorphism class of descent data. Having done this, we find $FT_{k'} = T_{k'}$, and so $T_{k'}$ descends to a maximal torus T of G. ☐

6. Bigness for algebraic representations

In this section we prove that many algebraic representations have big image. The main result is the following:

Proposition 6.1. *Let k be a finite field, let G/k be a reductive group and let ρ be an absolutely irreducible representation of G on a k-vector space V. Assume $\operatorname{char} k$ is large compared to $\dim V$ and $|V|_\infty$. Then $\rho(G(k))$ is a potentially big subgroup of $\mathrm{GL}(V)$ which becomes big over any extension over which G splits.*

Proof. Since ρ is absolutely irreducible, the center of G acts through a character. Thus the restriction of ρ to G^{der} is still absolutely irreducible and still has small norm (since the norm is the same when computed as a representation of G or G^{der}). Since $\rho(G^{\mathrm{der}}(k))$ is a normal subgroup of $\rho(G(k))$, it suffices to show that the former is big. Thus we may as well assume G to be semi-simple.

Now, let H be the kernel of ρ, let H° be the identity component in H and let $G' = G/H^\circ$. The map ρ factors through G'. By Lang's theorem, the natural map $G(k) \to G'(k)$ is surjective. Thus $\rho(G(k)) = \rho(G'(k))$ and so we must show that $\rho(G'(k))$ is big. As V is still absolutely irreducible and still has small norm when regarded as a representation of G', we may as well replace G by G'. We can thus assume that the kernel of ρ is finite. Note that this bounds the dimension of G in terms of the dimension of V.

Finally, let G^{sc} be the simply connected cover of G. The image of $G^{\mathrm{sc}}(k) \to G(k)$ is a normal subgroup and so $\rho(G^{\mathrm{sc}}(k))$ is a normal subgroup of $\rho(G(k))$. It suffices to show that the former is big. Of course, V is still absolutely irreducible and still has small norm when considered as a representation of G^{sc}. We may thus assume that G is simply connected.

We have thus reduced to the case where G is semi-simple and simply connected and the kernel of ρ is finite. As remarked, the dimension of G is bounded in terms of that of V. We now begin the proof proper. To begin with, Proposition 4.4 shows that V is absolutely irreducible as a representation of $G(k)$ and so $\rho(G(k))$ satisfies (B1).

We now examine $H^1(\rho(G(k)), \mathrm{ad}^\circ V)$. By Proposition 4.2 and Proposition 4.4, we have $H^1(G(k), \mathrm{ad}\, V) = 0$ since this group classifies self-extensions of V and any such extension is semi-simple. Since $\operatorname{char} k$ is large compared to $\dim V$ this implies $H^1(G(k), \mathrm{ad}^\circ V) = 0$, as $\mathrm{ad}^\circ(V)$ is then a summand of $\mathrm{ad}\, V$. Let Z be the kernel of ρ. It is a finite central subgroup of G. We have an exact sequence

$$1 \to Z(k) \to G(k) \to \rho(G(k)) \to 1$$

and thus we get an injection

$$H^1(\rho(G(k)), (\mathrm{ad}^\circ V)^{Z(k)}) \to H^1(G(k), \mathrm{ad}^\circ V).$$

The group on the right vanishes and so the group on the left does too. Since $Z(k)$ is central, it acts trivially on $\mathrm{ad}^\circ V$. Thus $H^1(\rho(G(k)), \mathrm{ad}^\circ V) = 0$ and so $\rho(G(k))$ satisfies (B2).

We now turn to condition (B3). Let k'/k be an extension over which G splits and put $V' = V \otimes k'$. By Proposition 5.1 and Proposition 5.5 we can pick a maximal torus T of G which splits over k' and an element $g \in T(k)$ such that $\lambda(g) \neq \lambda'(g)$ for any two distinct characters λ and λ' of $T_{k'}$ which are weights of V'. For a character λ of $T_{k'}$ we let V'_λ denote the λ-weight space of V'. We have a decomposition $V' = \bigoplus V'_\lambda$. Let λ_0 denote the highest weight and put $V'_0 = V'_{\lambda_0}$. The space V'_0 is one dimensional. By our assumption on the element g, for any weight λ appearing in V' the $\lambda(g)$-eigenspace of g on V' is exactly V'_λ. In particular, the $\lambda_0(g)$-eigenspace of g is V'_0 and thus one dimensional.

Now, we have a decomposition $\mathrm{ad}\, V' = \bigoplus V'_\lambda \otimes (V'_\mu)^*$. Let f be an element of $\mathrm{ad}\, V'$. Then the composite

$$\begin{tikzcd} V'_0 \arrow[r, hook] & V' \arrow[r, "f"] & V' \arrow[r, two heads] & V'_0 \end{tikzcd}$$

is non-zero if and only if the projection of f to $V'_0 \otimes (V'_0)^*$ is non-zero. Thus to prove that the image of $G(k)$ in $\mathrm{GL}(V')$ satisfies (B3), it suffices to show that every irreducible $G(k)$-submodule of $\mathrm{ad}\, V'$ has non-zero projection to $V'_0 \otimes (V'_0)^*$.

Thus let W be an irreducible $G(k)$-submodule of $\mathrm{ad}\, V'$. To show that the image of W in $V'_0 \otimes (V'_0)^*$ is non-zero it suffices to show that the image of $\overline{W}$ in $\overline{V}_0 \otimes \overline{V}_0^*$ is, where the bar denotes $- \otimes \overline{k}$. Let $\overline{U}$ be an irreducible $G(k)$-submodule of $\overline{W}$. It is enough, of course, to show that the image of $\overline{U}$ in $\overline{V}_0 \otimes \overline{V}_0^*$ is non-zero. Now, $\overline{U}$ is an irreducible $G(k)$-submodule of $\mathrm{ad}\, \overline{V}$, and so, by Proposition 4.4, it is an irreducible G-submodule of $\mathrm{ad}\, \overline{V}$. Thus to show that the image of $G(k)$ in $\mathrm{GL}(V')$ satisfies (B3) it is enough to prove the following: *every irreducible G-submodule of* $\mathrm{ad}\, \overline{V}$ *has non-zero image in* $\overline{V}_0 \otimes \overline{V}_0^*$. This is established in the following lemma. ☐

Lemma 6.2. *Let k be the algebraic closure of a finite field, let G/k be a semi-simple group and let (ρ, V) be an irreducible representation of G with* $\dim V$ *and* $|V|_\infty$ *small compared to* $\mathrm{char}\, k$*. Let V_0 be the highest weight space of V. Then every irreducible submodule of* $\mathrm{ad}\, V$ *has non-zero projection to* $V_0 \otimes V_0^*$*.*

Proof. We can assume that ρ is faithful, and so $\dim G$ is bounded in terms of $\dim V$. Pick a maximal torus of G and a system of positive roots. For a weight λ let V_λ denote the λ-weight space of V. Let λ_0 be the highest weight for V and let V_0 be the λ_0-weight space. For a root α we pick an element X_α of $\mathrm{Lie}(G)$ which spans the α root space. Any positive element λ of the root lattice has a unique expression $\lambda = \sum n_i \alpha_i$ where the n_i are non-negative integers and the α_i are the simple roots. We let $\mathrm{len}\, \lambda$ be the sum of the n_i.

By a *simple tuple* we mean an ordered tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ consisting of simple roots. For such a tuple $\underline{\alpha}$ we put $|\underline{\alpha}| = \sum \alpha_i$. Note that $\mathrm{len}\, |\underline{\alpha}| = n$. We let $X_{\underline{\alpha}}$ (resp. $Y_{\underline{\alpha}}$) denote the product $X_{\alpha_1} \cdots X_{\alpha_n}$ (resp. $X_{-\alpha_1} \cdots X_{-\alpha_n}$), regarded as an element of the universal enveloping algebra.

Given a weight λ for which V_λ is non-zero the difference $\lambda_0 - \lambda$ is positive and lies in the root lattice. For a simple tuple $\underline{\alpha}$ with $|\underline{\alpha}| = \lambda_0 - \lambda$ the operator $X_{\underline{\alpha}}$ maps V_λ into V_0. The resulting map

$$V_\lambda \to \bigoplus_{|\underline{\alpha}| = \lambda_0 - \lambda} V_0$$

is injective. (Proof: By Proposition 4.5, the only vector annihilated by all of the X_α is the highest weight vector. Thus if $\lambda \neq \lambda_0$ and v belongs to V_λ then we can find some α such that $X_\alpha v$ is non-zero. We can thus move v closer to the λ_0-weight space. By induction on $\mathrm{len}(\lambda_0 - \lambda)$ we can therefore find $\alpha_1, \dots, \alpha_n$ such that $X_{\alpha_n} \cdots X_{\alpha_1} v$ is non-zero and belongs to V_0.) We can thus pick $m = \dim V_\lambda$ simple tuples $\underline{\alpha}_1, \dots, \underline{\alpha}_m$ for which the resulting map is injective. We can then pick a basis $\{v_i\}$ of V_λ such that v_i belongs to the kernel of $X_{\underline{\alpha}_j}$ whenever $i \neq j$ but does not belong to the kernel of $X_{\underline{\alpha}_i}$. We call such a basis *admissible.* Note that in V^* the space V_0^* is a lowest weight space. The same process as above, but with $X_{\underline{\alpha}}$ replaced by $Y_{\underline{\alpha}}$, yields the notion of an admissible basis for V_{α}^*.

Let W be an irreducible submodule of $\operatorname{ad} V$. Let $p : W \to V_0 \otimes V^*$ be the natural projection. We first show that $p(W)$ is non-zero. Among those weights λ for which the projection $W \to V_\lambda \otimes V^*$ is non-zero, pick one for which $\operatorname{len}(\lambda_0 - \lambda)$ is minimal. Let w be an element of W which has non-zero projection to $V_\lambda \otimes V^*$ and write

$$w = \left(\sum v_i \otimes v_i^* \right) + v'$$

where $\{v_i\}$ is an admissible basis of V_λ, the v_i^* belong to V^* and v' belongs to the complement of $V_\lambda \otimes V^*$. Let $\underline{\alpha}_i$ be the simple tuples yielding the basis v_i. Let 1 denote an index such that v_1^* is non-zero. We then have

$$p(X_{\underline{\alpha}_1} w) = (X_{\underline{\alpha}_1} v_1) \otimes v_1^* \tag{1}$$

(explained below). Since the right side is non-zero, it follows that $p(W)$ is non-zero.

We now explain why (1) holds. Recall that if X is an element of $\operatorname{Lie}(G)$ then the formula for how X acts on a pure tensor is

$$X(v \otimes w) = (Xv) \otimes w + v \otimes (Xw).$$

Thus when we apply $X_{\underline{\alpha}_1}$ to a pure tensor $v \otimes w$ we get a sum of terms and in each term some $X_{\alpha_{1,i}}$ land on v and some land on w. We now examine $X_{\underline{\alpha}_1} v$. First consider the v' part. Write $v' = \sum v_i' \otimes u_i'$ where v_i' has weight μ_i. If $\underline{\alpha}'$ is any sub-sequence of $\underline{\alpha}_1$ then $X_{\underline{\alpha}'} v_i'$ lands in the $\mu_i + |\underline{\alpha}'|$ weight space. If $\underline{\alpha}'$ is not all of $\underline{\alpha}_1$ then this cannot equal λ_0 for length reasons. Even if $\underline{\alpha}'$ is all of $\underline{\alpha}_1$ this is not equal to λ_0 since $\lambda_0 = |\underline{\alpha}_1| + \lambda$ and no μ_i is equal to λ. Thus $p(X_{\underline{\alpha}_1} v') = 0$. We now consider the first term in v. The same length argument shows that the only way to land in $V_0 \otimes V^*$ is to have all of $X_{\underline{\alpha}_1}$ land on the first factor. However, $X_{\underline{\alpha}_1}$ kills v_i for $i \neq 1$. We have thus proved (1).

We now show that the image of the projection $q : W \to V_0 \otimes V_0^*$ is non-zero. Among those weights λ for which the projection $W \to V_0 \otimes V_\lambda^*$ is non-zero, pick one for which $\operatorname{len}(\lambda_0 - \lambda)$ is minimal. (Such a weight exists by the previous paragraphs.) Let w be an element of W which has non-zero projection to $V_0 \otimes V_\lambda^*$. We may as well assume that w has weight $\lambda_0 - \lambda$. We can thus write

$$w = \left(\sum v_i \otimes v_i^* \right) + v'$$

where the v_i belong to V_0, $\{v_i^*\}$ is an admissible basis of V_λ^* and v' belongs to the complement of $V_0 \otimes V^*$. Let $\underline{\alpha}_i$ be the simple tuples yielding the basis v_i^*. Let 1 denote an index such that v_1 is non-zero. We then have

$$q(Y_{\underline{\alpha}_1} w) = v_1 \otimes (Y_{\underline{\alpha}_1} v_1^*) \tag{2}$$

(explained below). Since the right side is non-zero, it follows that $q(W)$ is non-zero, proving the proposition.

We now explain (2). The point is that, since $Y_{\underline{\alpha}_1}$ is a lowering operator, the only way for a term of $Y_{\underline{\alpha}_1} w$ to have its first factor in V_0 is if $Y_{\underline{\alpha}_1}$ lands entirely on the second factor. Of course, none of the terms in v' have their first factor in V_0 to begin with, so they will not after applying $Y_{\underline{\alpha}_1}$. As for the first term, $Y_{\underline{\alpha}_1}$ kills v_i^* for $i \neq 1$. This proves (2). □

7. Bigness for nearly hyperspecial groups

Throughout this section K denotes a finite extension of $\mathbb{Q}_\ell$, $\mathscr{O}_K$ its ring of integers and k its residue field.

We begin by recalling some definitions. Let G/K be a reductive group. The group G is *quasi-split* if it has a Borel subgroup. It is *unramified* if it is quasi-split and it splits over an unramified extension of K. A subgroup $\Gamma \subset G(K)$ is *hyperspecial* if there exists a reductive group $\widetilde{G}/\mathscr{O}_K$ with generic fiber G such that $\Gamma = \widetilde{G}(\mathscr{O}_K)$. Hyperspecial subgroups of $G(K)$ are maximal compact subgroups. The group $G(K)$ has a hyperspecial subgroup if and only if G is unramified. Let G^{ad} be the adjoint group of G and let G^{sc} be the simply connected cover of G^{ad}. We have maps

$$\begin{tikzcd} G \arrow[r, "\sigma"] & G^{\mathrm{ad}} & G^{\mathrm{sc}}. \arrow[l, "\tau"'] \end{tikzcd}$$

We say that a subgroup $\Gamma \subset G(K)$ is *nearly hyperspecial* if $\tau^{-1}(\sigma(\Gamma))$ is a hyperspecial subgroup of $G^{\mathrm{sc}}(K)$. (This is not a standard term.)

Let Γ be a profinite group and let $\rho : \Gamma \to \mathrm{GL}(V)$ be a continuous representation of Γ on a K-vector space V. Let G be the Zariski closure of $\rho(\Gamma)$. This is a closed algebraic subgroup of $\underline{\mathrm{GL}}(V)$. We call the representation ρ *connected* if the group G is connected. Note that, in this case, for any subgroup Γ' of finite index in Γ the Zariski closure of $\rho(\Gamma')$ is also G. In particular, if ρ is connected and irreducible then it remains so when restricted to any finite index subgroup of Γ. More generally, if ρ is connected and semi-simple then the group G is reductive. We call ρ *(nearly) hyperspecial* if it is connected and $\rho(\Gamma)$ is a (nearly) hyperspecial subgroup of $G(K)$.

Let ρ be a representation of a group Γ on a K-vector space V. We say that (ρ, V) is *residually absolutely irreducible* if there is some Γ-stable lattice Λ in V such that the representation of Γ on $\Lambda \otimes k$ is absolutely irreducible. In this case, any two Γ-stable lattices in V differ by an element of $K^{\times}$. It is thus reasonable to define $\overline{V}$ to be the reduction of any one of them and let $\overline{\rho}$ be the representation of Γ on $\overline{V}$.

The purpose of this section is to prove the following proposition:

Proposition 7.1. *Let Γ be a profinite group and let ρ be an absolutely irreducible, nearly hyperspecial representation of Γ on a K-vector space V. Assume that $\operatorname{char} k$ is sufficiently large compared to $\dim V$. Then ρ is residually absolutely irreducible and $\overline{\rho}(\Gamma)$ is a potentially big subgroup of $\mathrm{GL}(\overline{V})$. In fact, if the Zariski closure G of $\rho(\Gamma)$ splits over the unramified extension K'/K then $\overline{\rho}(\Gamma)$ becomes big over the corresponding extension k'/k.*

We need some auxiliary lemmas to prove the proposition. We begin with the following lemma.

Lemma 7.2. *Let $\widetilde{G}/\mathscr{O}_K$ be a simply connected semi-simple group and let σ be an automorphism of the generic fiber $G = \widetilde{G}_K$ such that σ maps $\widetilde{G}(\mathscr{O}_K)$ into itself. Then for any tamely ramified finite extension L/K the automorphism σ maps $\widetilde{G}(\mathscr{O}_L)$ into itself.*

Proof. The group $\widetilde{G}(\mathscr{O}_K)$ fixes a point x on the building $B(G, K)$ by [Tits, §2.3.1] or [BT, §4.6.31] which is known to be unique. Similarly, the group $\widetilde{G}(\mathscr{O}_L)$ fixes a unique point x' on the building $B(G, L)$. Furthermore $\widetilde{G}(\mathscr{O}_K)$ (resp. $\widetilde{G}(\mathscr{O}_L)$) is the full stabilizer of x (resp. x') since $\widetilde{G}(\mathscr{O}_K)$ (resp. $\widetilde{G}(\mathscr{O}_L)$) is maximal compact ([Tits, §3.2]). We now claim that under the natural inclusion $B(G, K) \to B(G, L)$ the point x is identified with x'. To see this, first note that if τ is an element of $\mathrm{Gal}(L/K)$ then $\widetilde{G}(\mathscr{O}_L)$ fixes $\tau x'$ and so $\tau x' = x'$ by the uniqueness of x'. Thus x' is fixed by $\mathrm{Gal}(L/K)$ and therefore belongs to $B(G, K)$ by [Tits, 2.6.1] (this uses the hypothesis that L/K is tamely ramified). Since x' is fixed by $\widetilde{G}(\mathscr{O}_L)$ it is certainly also fixed by the subgroup $\widetilde{G}(\mathscr{O}_K)$. By the uniqueness of x we conclude $x = x'$.

Now, the automorphism σ of G acts on $B(G, K)$ and $B(G, L)$ and respects the inclusion map. As σ carries $\widetilde{G}(\mathscr{O}_K)$ into itself it must fix x. It therefore also fixes x' and so must carry its stabilizer, $\widetilde{G}(\mathscr{O}_L)$, into itself. This proves the lemma. □

We can now prove the following:

Lemma 7.3. *Let Γ be a profinite group and let ρ be an absolutely irreducible, nearly hyperspecial representation of Γ on a K-vector space V. Let G be the Zariski closure of $\rho(\Gamma)$. Then we can find:*

- *a Γ-stable lattice Λ in V;*
- *a semi-simple group $\widetilde{G}/\mathscr{O}_K$ with generic fiber equal to G^{sc}; and*
- *a representation $r : \widetilde{G} \to \underline{\mathrm{GL}}(\Lambda)$ which induces the natural map $G^{\mathrm{sc}} \to G$ on the generic fiber,*

such that $\mathscr{O}_K^\times \cdot r(\widetilde{G}(\mathscr{O}_K))$ *is an open normal subgroup of* $\mathscr{O}_K^\times \cdot \rho(\Gamma)$ *(here* $\mathscr{O}_K^\times$ *denotes the center of* $\mathrm{GL}(\Lambda)$*), the index of which can be bounded in terms of* $\dim V$*. Necessarily, the generic fiber of* r *is an absolutely irreducible representation of* $\widetilde{G}_K$ *on* V*.*

Proof. Let G be the Zariski closure of $\rho(\Gamma)$. It is a reductive (and in particular connected) group, by hypothesis. Since ρ is absolutely irreducible, the center Z of G is contained in the center of $\underline{\mathrm{GL}}(V)$. Thus, either Z is finite and G is semi-simple or else $Z = \mathbb{G}_m$. We have maps

$$\begin{tikzcd}
G \arrow[r, "\sigma"] & G^{\mathrm{ad}} & G^{\mathrm{sc}} \arrow[l, "\tau"'] \arrow[dl] \\
 & G^{\mathrm{der}}. \arrow[ul] \arrow[u] &
\end{tikzcd}$$

By hypothesis, $\tau^{-1}(\sigma(\rho(\Gamma)))$ is a hyperspecial subgroup of G^{sc}. Thus we can find a semi-simple group $\widetilde{G}/\mathscr{O}_K$ with generic fiber G^{sc} such that $\widetilde{G}(\mathscr{O}_K) = \tau^{-1}(\sigma(\rho(\Gamma)))$.

Let $r : G^{\mathrm{sc}} \to G$ be the natural map; this factors through G^{der} in the above diagram. Let U be the image of $G^{\mathrm{sc}}(K)$ under τ. It is an open normal subgroup of $G^{\mathrm{ad}}(K)$, the index of which can be bounded in terms of $\dim G$ and thus $\dim V$. Now, we have

$$\sigma(r(\widetilde{G}(\mathscr{O}_K))) = \tau(\widetilde{G}(\mathscr{O}_K)) = \sigma(\rho(\Gamma)) \cap U.$$

Applying σ^{-1}, we find

$$K^\times \cdot r(\widetilde{G}(\mathscr{O}_K)) = K^\times \cdot (\rho(\Gamma) \cap \sigma^{-1}(U)).$$

Since $r(\widetilde{G}(\mathscr{O}_K))$ and $\rho(\Gamma) \cap \sigma^{-1}(U)$ are both compact, it follows that

$$\mathscr{O}_K^\times \cdot r(\widetilde{G}(\mathscr{O}_K)) = \mathscr{O}_K^\times \cdot (\rho(\Gamma) \cap \sigma^{-1}(U)).$$

Thus $\mathscr{O}_K^\times \cdot r(\widetilde{G}(\mathscr{O}_K))$ is an open normal subgroup of $\mathscr{O}_K^\times \cdot \rho(\Gamma)$, the index of which can be bounded in terms of $\dim V$.

We now claim that for any finite unramified extension L/K the group $\rho(\Gamma)$ normalizes $\mathscr{O}_L^\times \cdot r(\widetilde{G}(\mathscr{O}_L))$. To see this, let γ be an element of $\rho(\Gamma)$. Write $\overline{\gamma}$ for the image of γ in $G^{\mathrm{ad}}(K)$ under σ. Thus $\overline{\gamma}$ gives an automorphism of G^{sc}, which we denote by $x \mapsto \overline{\gamma} x \overline{\gamma}^{-1}$. Now, let x be an element of $G^{\mathrm{sc}}(L)$. We then have

$$\gamma r(x) \gamma^{-1} = z \cdot r(\overline{\gamma} x \overline{\gamma}^{-1}),$$

for some $z \in \mathscr{O}_L^\times$, as is easily seen by applying σ. It thus suffices to show that conjugation by $\overline{\gamma}$ carries $\widetilde{G}(\mathscr{O}_L)$ into itself. By Lemma 7.2 it suffices to show that $\overline{\gamma}$ carries $\widetilde{G}(\mathscr{O}_K)$ into itself. Thus let x be an element of $\widetilde{G}(\mathscr{O}_K)$. Using the above formula and the fact that γ normalizes $\mathscr{O}_K^\times \cdot r(\widetilde{G}(\mathscr{O}_K))$, we can find an element y of $\widetilde{G}(\mathscr{O}_K)$ and an element z of $\mathscr{O}_K^\times$ such that $r(\overline{\gamma} x \overline{\gamma}^{-1}) = z r(y)$. It thus follows that $\overline{\gamma} x \overline{\gamma}^{-1} = z' y$ for some element z' of the K-points of center of G^{sc}. However, z' must be contained in $\widetilde{G}(\mathscr{O}_K)$ since it belongs to a compact central group and $\widetilde{G}(\mathscr{O}_K)$ is maximal compact. Thus $\overline{\gamma} x \overline{\gamma}^{-1}$ belongs to $\widetilde{G}(\mathscr{O}_K)$.

Now, the group $\widetilde{G}(\mathscr{O}_K^{\mathrm{un}})$ is bounded in the sense of [Tits, §2.2.1]. Thus, arguing as in [Lar, §1.12], we can find a lattice $\Lambda' \subset V$ such that $\Lambda' \otimes \mathscr{O}_K^{\mathrm{un}}$ is stable under the action of $\widetilde{G}(\mathscr{O}_K^{\mathrm{un}})$. Now, we have shown that $\mathscr{O}_K^\times \cdot r(\widetilde{G}(\mathscr{O}_K))$ has finite index in $\mathscr{O}_K^\times \cdot \rho(\Gamma)$. Let $\gamma_1, \ldots, \gamma_n$ be coset representatives and put

$$\Lambda = \sum_{i=1}^{n} \gamma_i \cdot \Lambda'.$$

Thus Λ is a lattice in V. It is easy to see that Γ maps Λ into itself and $\widetilde{G}(\mathscr{O}_K^{\mathrm{un}})$ maps $\Lambda \otimes \mathscr{O}_K^{\mathrm{un}}$ into itself. Following the argument in [Lar, §1.12] once again, we see that $r : \widetilde{G}_K \to \underline{\mathrm{GL}}(V)$ lifts to a map $\widetilde{G} \to \underline{\mathrm{GL}}(\Lambda)$, which we still call r. This completes the proof of the proposition. ☐

We can now prove the proposition.

Proof of Proposition 7.1. Let Γ be a profinite group and let ρ be an absolutely irreducible, nearly hyperspecial representation of Γ on a K-vector space V. Let G be the Zariski closure of $\rho(\Gamma)$ and let K'/K be an unramified extension over which G splits. (Note that since G^{sc} has a hyperspecial subgroup G is unramified.) Let $\widetilde{G}/\mathscr{O}_K$, Λ and r be as in Lemma 7.3. Let $\overline{\rho}$ be the representation of Γ on $\overline{V} = \Lambda \otimes k$. The group $\widetilde{G}$ splits over $\mathscr{O}_{K'}$ and so the group $\widetilde{G}_k$ splits over k'. By Proposition 4.3, the representation $r_k : \widetilde{G}_k \to \underline{\mathrm{GL}}(\Lambda \otimes k)$ is absolutely irreducible and its norm is bounded in terms of $\dim V$. It thus follows from Proposition 6.1 that $r(\widetilde{G}(k))$ is potentially big and becomes big over k'. Since $k^\times \overline{\rho}(\Gamma)$ contains $k^\times r(H(k))$ as a normal subgroup, it follows from Proposition 3.3 and Proposition 3.4 that $\overline{\rho}(\Gamma)$ is potentially big and becomes big over k'. Note that the representation $\overline{\rho}$ of Γ on $\Lambda \otimes k$ is thus absolutely irreducible and so ρ is residually absolutely irreducible. □

8. Groups with Frobenii, compatible systems and algebraic monodromy groups

A *group with Frobenii* is a pair $(\Gamma, \{F_{\mathfrak{p}}\}_{\mathfrak{p} \in P})$ consisting of a profinite group Γ and a dense set of elements $\{F_{\mathfrak{p}}\}$ of Γ indexed by a set P. We call the $F_{\mathfrak{p}}$ the "Frobenius elements" of Γ and P the set of "places." We will usually denote a group with Frobenii simply by Γ, leaving the set of Frobenii out of the notation.

The motivating example of a group with Frobenii is the Galois group of a global field. Let F be a global field (that is, a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p((t))$) and let Γ be its absolute Galois group. Let P be the set of places of $\overline{F}$ and for each $\mathfrak{p} \in P$ let $F_{\mathfrak{p}} \in \Gamma$ be a Frobenius element at $\mathfrak{p}$. Then Γ together with $\{F_{\mathfrak{p}}\}_{\mathfrak{p} \in P}$ is a group with Frobenii.

Let Γ be a group with Frobenii, let L be a set of rational prime numbers and for each $\ell \in L$ let ρ_ℓ be a continuous representation of Γ on a $\mathbb{Q}_\ell$-vector space V_ℓ. We say that the ρ_ℓ are *compatible*, or that $\{(\rho_\ell, V_\ell)\}_{\ell \in L}$ forms a *compatible system*, if each V_ℓ has the same dimension and if for each $\mathfrak{p} \in P$ there exists a finite set $L_{\mathfrak{p}}$ of primes (the "bad primes" for $F_{\mathfrak{p}}$) such that the following conditions hold:

- The characteristic polynomial of $F_{\mathfrak{p}}$ has rational coefficients and is independent of ℓ for good primes ℓ. Precisely, given $\mathfrak{p} \in P$ there exists a polynomial p with coefficients in $\mathbb{Q}$ such that for all primes $\ell \in L \setminus L_{\mathfrak{p}}$ the characteristic polynomial of $\rho_\ell(F_{\mathfrak{p}})$ is equal to p.
- For any finite collection of primes $L' \subset L$ the Frobenii for which all primes in L' are good form a dense set in Γ. That is, for any such L' the set $\{F_{\mathfrak{p}} \mid L' \cap L_{\mathfrak{p}} = \emptyset\}$ is dense.

By a "compatible system of semi-simple representations" we simply mean a compatible system in which each ρ_ℓ is semi-simple.

Let Γ be a a group with Frobenii and let Γ' be an open subgroup. Given a Frobenius element $F_{\mathfrak{p}}$ of Γ there is some minimal positive integer $n(\mathfrak{p})$ such that $F_{\mathfrak{p}}^{n(\mathfrak{p})}$ belongs to Γ'. We give Γ' the structure of a group with Frobenii by taking the $F_{\mathfrak{p}}^{n(\mathfrak{p})}$ to be its Frobenii. Note that these are dense: in fact, the $F_{\mathfrak{p}}$ which already belong to Γ' are dense. It is clear that if $\{(\rho_\ell, V_\ell)\}_{\ell \in L}$ is a compatible system of ℓ-adic representations of Γ then $\{(\rho_\ell|_{\Gamma'}, V_\ell)\}_{\ell \in L}$ is one for Γ'.

Let $\{(\rho_\ell, V_\ell)\}_{\ell \in L}$ be a compatible system of representations of a group with Frobenii Γ. We define G_ℓ to be the Zariski closure of $\rho_\ell(\Gamma)$ inside of $\mathrm{GL}(V_\ell)$. Thus G_ℓ is a closed algebraic subgroup of $\underline{\mathrm{GL}}(V_\ell)$. We let G_ℓ° denote the connected component of the identity in G_ℓ. Note that if ρ_ℓ is a semi-simple representation then G_ℓ° is a reductive group.

We will need some results that examine how G_ℓ varies with ℓ. The first result that we quote examines how $\pi_0(G_\ell)$ behaves. It is due to Serre.

Proposition 8.1 ([LP, Proposition 6.14]). *Let Γ be a group with Frobenii and let $\{(\rho_\ell, V_\ell)\}_{\ell \in L}$ be a compatible system of semi-simple representations with L of density one. Then the open subgroup $\rho_\ell^{-1}(G_\ell^\circ(\mathbb{Q}_\ell))$ of Γ is independent of ℓ. In particular, the groups $\pi_0(G_\ell)$ for various ℓ are canonically isomorphic.*

The next result, due to Larsen and Pink, examines the extensions over which the G_ℓ split.

Proposition 8.2 ([LP, Proposition 8.9]). *Let Γ be a group with Frobenii and let $\{(\rho_\ell, V_\ell)\}_{\ell \in L}$ be a compatible system of semi-simple representations with L of density one. Then there exists a finite Galois extension $E/\mathbb{Q}$*

and a set $L' \subset L$ of density one, containing no primes which ramify in E, such that for each $\ell \in L'$ the group G_ℓ° is quasi-split and splits over $E\mathbb{Q}_\ell$ (and thus is unramified).

9. Bigness for compatible systems

We can now prove our main theorem:

Theorem 9.1. *Let Γ be a group with Frobenii and let $\{(\rho_\ell, V_\ell)\}_{\ell \in L}$ be a compatible system of ℓ-adic representations of Γ with L of density one. Assume that each ρ_ℓ is absolutely irreducible when restricted to any open subgroup of Γ. Then there is a set of primes $L' \subset L$ of density one such that for $\ell \in L'$ the representation ρ_ℓ is residually absolutely irreducible and $\overline{\rho}_\ell(\Gamma)$ is a potentially big subgroup of $\mathrm{GL}(\overline{V}_\ell)$. In fact, there is a finite Galois extension $E/\mathbb{Q}$ such that $\overline{\rho}_\ell(\Gamma)$ is a big subgroup of $\mathrm{GL}(\overline{V} \otimes k_\ell)$ for $\ell \in L'$, where k_ℓ is the residue field of E at a prime above ℓ. In particular, $\overline{\rho}_\ell(\Gamma)$ is a big subgroup of $\mathrm{GL}(\overline{V})$ for a set of ℓ of positive density.*

Proof. Let Γ° be the open subgroup of Γ which is equal to $\rho_\ell^{-1}(G_\ell^\circ(\mathbb{Q}_\ell))$ for each $\ell \in L$ (see Proposition 8.1). The main theorem of [Lar] states that we can find a set of primes $L'' \subset L$ of density one such that $\rho_\ell|_{\Gamma^\circ}$ is nearly hyperspecial for each $\ell \in L''$. Throwing out a finite number of primes, if necessary, Proposition 7.1 show that $\rho_\ell|_{\Gamma^\circ}$ is residually absolutely irreducible and that $\overline{\rho}_\ell(\Gamma^\circ)$ is big over the residue field of any extension over which G_ℓ° splits (for all $\ell \in L''$). It thus follows that ρ_ℓ is residually absolutely irreducible and that $\overline{\rho}_\ell(\Gamma)$ is big over the residue field of any extension over which G_ℓ° splits (for all $\ell \in L''$). By Proposition 8.2 we can find a set of primes $L''' \subset L$ of density one and a finite Galois extension $E/\mathbb{Q}$ such that G_ℓ° splits over $E\mathbb{Q}_\ell$, for $\ell \in L'''$. Taking $L' = L'' \cap L'''$ establishes the theorem. ☐

Applying Theorem 9.1 in the case where Γ is the absolute Galois group of a global field gives Theorem 1.1 from the introduction.

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