# Clique minors in claw-free graphs

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**Abstract**

Hadwiger's conjecture states that every graph with chromatic number $\chi$ has a clique minor of size $\chi$. Let $G$ be a graph on $n$ vertices with chromatic number $\chi$ and stability number $\alpha$. Then since $\chi\alpha \geq n$, Hadwiger's conjecture implies that $G$ has a clique minor of size $\frac{n}{\alpha}$. In this paper we prove that this is true for connected claw-free graphs with $\alpha \geq 3$. We also show that this result is tight by providing an infinite family of claw-free graphs with $\alpha \geq 3$ that do not have a clique minor of size larger than $\frac{n}{\alpha}$.

## 1 Introduction

In 1943, Hadwiger [11] conjectured that every graph with chromatic number $\chi$ has a clique minor of size $\chi$ (we postpone the definition of clique minor to later in this section). This vast generalization of the Four Color Theorem [1, 2, 17] is still open for $\chi > 7$ (in fact, the cases $\chi = 5, 6$ were shown to be equivalent to the Four Color Theorem, the case $\chi = 5$ by Wagner [19] and the case $\chi = 6$ by Robertson, Seymour, and Thomas [18]).

A *clique* in a graph is a set of vertices all pairwise adjacent. A *stable set* is a set of vertices all pairwise non-adjacent. A triad is a stable set of size three. For a graph $G$, we denote the set of vertices of $G$ by $V(G)$ and the set of edges by $E(G)$. Further, we denote by $\omega(G)$ the size of a maximum clique in $G$ and by $\alpha(G)$ the size of a maximum stable set in $G$ (we call these the *clique number* and the *stability number* of $G$, respectively). The chromatic number of $G$ is denoted by $\chi(G)$.

Let $G$ be a finite loopless graph. In a $\chi(G)$-coloring of $G$, each color class has size at most $\alpha(G)$, and so $\chi(G)\alpha(G) \geq |V(G)|$ (or equivalently, $\chi(G) \geq \frac{|V(G)|}{\alpha(G)}$). Therefore, as Woodall observed in [21], Hadwiger's conjecture implies that $G$ has a clique minor of size $\frac{|V(G)|}{\alpha(G)}$ (from now on we will refer to this as Woodall's conjecture). In fact, for $alpha(G) = 2$, Plummer, Stiebitz, and Toft [15] showed that Woodall's conjecture is equivalent to Hadwiger's conjecture.

Woodall's conjecture is still open in general, but it has been proved up to a constant factor. More specifically, in 1982, Duchet and Meyniel [9] proved that a graph on $n$ vertices and stability number $\alpha$ has a clique minor of size $\frac{n}{2\alpha-1}$. There have been several results [12, 13, 20, 3] making improvements on the factor $2\alpha - 1$ but none of them improving on the constant factor of $\frac{1}{2}$. Recently, Fox showed that the main result of this paper can be used to make the first improvement on the constant $\frac{1}{2}$.

Another way of making progress towards solving Woodall's conjecture is to prove it for special classes of graphs. A graph is claw-free if it does not contain a *claw*, that is a $K_{1,3}$, as an induced subgraph. The main result of this paper is the following:

**1.1** *Let $G$ be a connected claw-free graph on $n$ vertices and with stability number $\alpha \geq 3$. Then $G$ has a clique minor of size $\frac{n}{\alpha}$.*

Following [7], let us say that a graph $G$ is *tame* if there exists a connected claw-free graph $H$ with stability number $\geq 3$, such that $G$ is an induced subgraph of $H$. Let $\nu(G) = \frac{|V(G)|}{\alpha(G)}$. We prove a slight strengthening of 1.1, the following:

**1.2** *Let $G$ be tame. Then $G$ has a clique minor of size $\nu(G)$.*

Our proof of 1.2 uses a structure theorem for claw-free graphs that appears in [6]. We describe this theorem in the next section. However, before we do that we must set up some more notation.

We say that two subgraphs $S_1, S_2$ of $G$ are *adjacent* if there is an edge between $V(S_1)$ and $V(S_2)$. A graph $H$ is said to be a *minor* of a graph $G$ if a graph isomorphic to $H$ can be obtained from a subgraph of $G$ by contracting edges. Let $H$ be a graph with $V(H) = \{v_1, \ldots, v_n\}$. Then $H$ is a minor of $G$ if and only if there are $|V(H)|$ non-null connected subgraphs $A_1, \ldots, A_n$ of $G$, such that $V(A_i) \cap V(A_j) = \emptyset$, and $A_i$ and $A_j$ are adjacent if $v_i$ is adjacent to $v_j$. We say that a graph $G$ has a clique minor of size $t$ if $K_t$ is a minor of $G$.

For $v \in V(G)$, we denote the set of neighbors of $v$ in $G$ by $N_G(v)$ (so $v \notin N_G(v)$) and for $X \subseteq V(G)$, we denote the set $(\bigcup_{x \in X} N_G(x)) \setminus X$ by $N_G(X)$. For $X, Y \subseteq V(G)$, we say that $X$ *dominates* $Y$ if $Y \subseteq N_G(X) \cup X$. For $X \subset V(G)$, let $G|X$ denote the subgraph of $G$ induced on $X$ and let $G \setminus X$ denote the subgraph of $G$ induced on $V(G) \setminus X$.

A *component* is a maximal connected subgraph of $G$. A set $S \subset V(G)$ is a *cutset* if $G \setminus S$ has more components than $G$ and $S$ is a *clique cutset* if it is both a clique and a cutset.

We say that a tame graph $G$ is a *minimum counterexample* to 1.2 if $G$ does not have a clique minor of size $\nu(G)$ and for every tame graph $H$ with $|V(H)| < |V(G)|$ or $|V(H)| = |V(G)|$ and $|E(H)| < |E(G)|$, $H$ has a clique minor of size $\nu(H)$.

This paper is organized as follows. In the next section we state (a corollary of) the structure theorem for claw-free graphs that appears in [6]. Section 3 contains some lemmas about claw-free graphs that are used in later proofs. Sections 4—6 are devoted to dealing with the different outcomes of the structure theorem of [6]; in each of the sections we prove that a minimum counterexample to 1.2 does not fall into the particular class of graphs that section is concerned with. Finally, in Section 7, we collect all these results to prove 1.2 and show that the result is tight by providing an infinite family of claw-free graphs with $\alpha \geq 3$ that do not have a clique minor of size larger than $\frac{n}{\alpha}$. In Section 8 we make some concluding remarks.

# 2 Structure theorem for claw-free graphs

The goal of this section is to state and describe a corollary of the structure theorem for claw-free graphs appearing in [6]. We begin with some definitions which are modified from [6].

Let $X, Y$ be two subsets of $V(G)$ with $X \cap Y = \emptyset$. We say that $X$ and $Y$ are *complete* to each other if every vertex of $X$ is adjacent to every vertex of $Y$, and we say that they are *anticomplete* to each other if no vertex of $X$ is adjacent to a member of $Y$. Similarly, if $A \subseteq V(G)$ and $v \in V(G) \setminus A$, then $v$ is $A$-*complete* if $v$ is adjacent to every vertex in $A$, and $A$-*anticomplete* if $v$ has no neighbor in $A$.

Let $(A, B)$ be disjoint subsets of $V(G)$. The pair $(A, B)$ is called a $W$-*join* in $G$ if $A, B$ are cliques, $A$ is neither complete nor anticomplete to $B$, and for every vertex $v \in V(G) \setminus (A \cup B)$, $v$ is either $A$-complete or $A$-anticomplete and either $B$-complete or $B$-anticomplete.

Here are some classes of claw-free graphs that come up in the structure theorem.

- **Graphs from the icosahedron.** The *icosahedron* is the unique planar graph with twelve vertices all of degree five. Let it have vertices $v_0, v_1, \ldots, v_{11}$, where for $1 \leq i \leq 10$, $v_i$ is adjacent to $v_{i+1}, v_{i+2}$ (reading subscripts modulo 10), and $v_0$ is adjacent to $v_1, v_3, v_5, v_7, v_9$, and $v_{11}$ is adjacent to $v_2, v_4, v_6, v_8, v_{10}$. Let this graph be $G_0$. Let $G_1$ be obtained from $G_0$ by deleting $v_{11}$ and let $G_2$ be obtained from $G_1$ by deleting $v_{10}$.

  Let $G \in \mathcal{I}$ if $G \in \{G_0, G_1, G_2\}$.

- **Long circular interval graphs.** Let $\Sigma$ be a circle, and let $F_1, \ldots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0, 1]$, such that no two of $F_1, \ldots, F_k$ share an endpoint, and no three of them have union $\Sigma$. Now let $V \subseteq \Sigma$ be finite, and let $G$ be a graph with vertex set $V$ in which distinct $u, v \in V$ are adjacent precisely if $u, v \in F_i$ for some $i$. Then $G$ is a *long circular interval graph.*

- **Antiprismatic graphs.** A graph $G$ is *antiprismatic* if for every $X \subseteq V(G)$ with $|X| = 4$, $X$ is not a claw and there are at least two pairs of vertices in $X$ that are adjacent.

Let $G$ be a graph. Then $H$ is an *inflation* of $G$ if for every $v \in V(G)$ there is a nonempty subset $X_v \subseteq V(H)$, all pairwise disjoint and with union $V(H)$ satisfying the following:

- for each $v \in V(G)$, $X_v$ is a clique of $H$
- if $u, v \in V(G)$ are adjacent in $G$, then $X_u$ is complete to $X_v$ in $H$
- if $u, v \in V(G)$ are nonadjacent in $G$, then $X_u$ is anticomplete to $X_v$ in $H$

Next, we define what it means for a claw-free graph to admit a “strip-structure”.

A *hypergraph* $H$ consists of a finite set $V(H)$, a finite set $E(H)$, and an incidence relation between $V(H)$ and $E(H)$ (that is, a subset of $V(H) \times E(H)$). For the statement of the structure theorem, we only need hypergraphs such that every member of $E(H)$ is incident with either one or two members of $V(H)$ (thus, these hypergraphs are graphs if we allow "graphs" to have loops and parallel edges). For $F \in E(H)$, $\overline{F}$ denotes the set of all $h \in V(H)$ incident with $F$.

Let $G$ be a graph. A *strip-structure* $(H, \eta)$ of $G$ consists of a hypergraph $H$ with $E(H) \neq \emptyset$, and a function $\eta$ mapping each $F \in E(H)$ to a subset $\eta(F)$ of $V(G)$, and mapping each pair $(F, h)$ with $F \in E(H)$ and $h \in \overline{F}$ to a subset $\eta(F, h)$ of $\eta(F)$, satisfying the following conditions.

**(SD1)** The sets $\eta(F)$ $(F \in E(H))$ are nonempty and pairwise disjoint and have union $V(G)$.

**(SD2)** For each $h \in V(H)$, the union of the sets $\eta(F, h)$ for all $F \in E(H)$ with $h \in \overline{F}$ is a clique of $G$.

**(SD3)** For all distinct $F_1, F_2 \in E(H)$, if $v_1 \in \eta(F_1)$ and $v_2 \in \eta(F_2)$ are adjacent in $G$, then there exists $h \in \overline{F_1} \cap \overline{F_2}$ such that $v_1 \in \eta(F_1, h)$ and $v_2 \in \eta(F_2, h)$.

There is also a fourth condition, but it is technical and we will not need it in this paper.

Let $(H, \eta)$ be a strip-structure of a graph $G$, and let $F \in E(H)$, where $\overline{F} = \{h_1, \ldots, h_k\}$. Let $v_1, \ldots, v_k$ be new vertices, and let $J$ be the graph obtained from $G|\eta(F)$ by adding $v_1, \ldots, v_k$, where $v_i$ is complete to $\eta(F, h_i)$ and anticomplete to all other vertices of $J$. Then $(J, \{v_1, \ldots, v_k\})$ is called the *strip of* $(H, \eta)$ *at* $F$. A strip-structure $(H, \eta)$ is *nontrivial* if $|E(H)| \geq 2$.

Next, we list some strips $(J, Z)$ that we will need for the structure theorem.

- $\mathcal{Z}_1$: Let $G$ be a graph with vertex set $\{v_1, \ldots, v_n\}$, such that for $1 \leq i < j < k \leq n$, if $v_i, v_k$ are adjacent then $v_j$ is adjacent to both $v_i, v_k$. Let $n \geq 2$, let $v_1, v_n$ be nonadjacent, and let there be no vertex adjacent to both $v_1$ and $v_n$. Then $G$ is a *linear interval graph*. Let $Z = \{v_1, v_n\}$. (Note that an inflation of a linear interval graph is still a linear interval graph).

- $\mathcal{Z}_2$: Let $n \geq 2$. Construct a graph $H$ as follows. Its vertex set is the disjoint union of three sets $A, B, C$, where $|A| = |B| = n + 1$ and $|C| = n$, say $A = \{a_0, a_1, \ldots, a_n\}, B = \{b_0, b_1, \ldots, b_n\}$, and $C = \{c_1, \ldots, c_n\}$. Adjacency is as follows. $A, B, C$ are cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let $a_i, b_j$ be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$, let $c_i$ be adjacent to $a_j, b_j$ if and only if $i \neq j \neq 0$. All other pairs not specified so far are nonadjacent. Now let $X \subseteq A \cup B \cup C \setminus \{a_0, b_0\}$ with $|C \setminus X| \geq 2$. Let $H' = H \setminus X$ and let $G$ be an inflation of $H'$ with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{a_0} = \{a'_0\}$, $X_{b_0} = \{b'_0\}$, and $Z = \{a'_0, b'_0\}$.

$\mathcal{Z}_3$: Let $H$ be a graph, and let $h_1$-$h_2$-$h_3$-$h_4$-$h_5$ be the vertices of a path of $H$ in order, such that $h_1, h_5$ both have degree one in $H$, and every edge of $H$ is incident with one of $h_2, h_3, h_4$. Let $H'$ be obtained from the line graph of $H$ by making the edges $h_2h_3$ and $h_3h_4$ of $H$ (vertices of $H'$) nonadjacent. Let $G$ be an inflation of $H'$ with $|X_{h_1h_2}| = |X_{h_4h_5}| = 1$. Let $X_{h_1h_2} = \{u\}$, $X_{h_4h_5} = \{v\}$, and $Z = \{u, v\}$.

$\mathcal{Z}_4$: Let $H$ be the graph with vertex set $\{a_0, a_1, a_2, b_0, b_1, b_2, b_3, c_1, c_2\}$ and adjacency as follows: $\{a_0, a_1, a_2\}, \{b_0, b_1, b_2, b_3\}, \{a_2, c_1, c_2\}$, and $\{a_1, b_1, c_2\}$ are cliques; $b_2, c_1$ are adjacent; and all other pairs are nonadjacent. Let $G$ be an inflation of $H$ with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{a_0} = \{a_0'\}$, $X_{b_0} = \{b_0'\}$, and $Z = \{a_0', b_0'\}$.

$\mathcal{Z}_5$: Let $H$ be the graph with vertex set $\{v_1, \dots, v_{12}\}$, and with adjacency as follows. $v_1$-$\cdots$-$v_6$-$v_1$ is an induced cycle in $G$ of length 6. Next, $v_7$ is adjacent to $v_1, v_2$; $v_8$ is adjacent to $v_4, v_5$; $v_9$ is adjacent to $v_6, v_1, v_2, v_3$; $v_{10}$ is adjacent to $v_3, v_4, v_5, v_6, v_9$; $v_{11}$ is adjacent to $v_3, v_4, v_6, v_1, v_9, v_{10}$; and $v_{12}$ is adjacent to $v_2, v_3, v_5, v_6, v_9, v_{10}$. No other pairs are adjacent. Let $H'$ be a graph isomorphic to $H \setminus X$ for some $X \subseteq \{v_{11}, v_{12}\}$. Let $G$ be an inflation of $H'$ with $|X_{a_0}| = |X_{b_0}| = 1$. Let $X_{v_7} = \{v_7'\}$, $X_{v_8} = \{v_8'\}$, and $Z = \{v_7', v_8'\}$.

We are now ready to state a structure theorem for claw-free graphs that is an easy corollary of the main result of [6].

**2.1** *Let $G$ be a connected claw-free graph. Then either*

- *$G$ is a circular interval graph, or*
- *$G$ is an inflation of a member of $\mathcal{I}$ or an antiprismatic graph, or*
- *$V(G)$ is the union of three cliques, or*
- *$G$ admits a $W$-join, or*
- *$G$ admits a nontrivial strip-structure such that for each strip $(J, Z)$, $1 \leq |Z| \leq 2$, and if $|Z| = 2$, then either*
  - *$|V(J)| = 3$ and $Z$ is complete to $V(J) \setminus Z$, or*
  - *$(J, Z)$ is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$.*

# 3 Tools and preliminary results

In this section we prove some preliminary results that are useful in the proof of 1.2. First we state a theorem that appears implicitly in [8].

**3.1** *Let $G$ be a graph with $\alpha(G) = 2$ such that $V(G)$ is the union of three cliques. Then $G$ has a clique minor of size $\frac{|V(G)|}{2} = \nu(G)$.*

**3.2** *Let $G$ be tame with $\alpha(G) \leq 2$. Then $G$ has a clique minor of size $\nu(G)$.*

**Proof.** The result is trivial when $\alpha(G) = 1$, so we may assume $\alpha(G) = 2$. Let $H$ be a connected claw-free graph with $\alpha(H) \geq 3$ such that there exists $v \in V(H)$ with $G$ an induced subgraph of $H \setminus \{v\}$ and $\alpha(H \setminus \{v\}) = 2$, and subject to that $|V(H)|$ minimum (such a graph $H$ exists because $G$ is tame).

(1) Either $G$ has a clique of size $\frac{|V(G)|+3}{4}$ or $V(G)$ is the union of three cliques.

Let $G' = H \setminus \{v\}$. Let $U = N_H(v)$ and $W = V(G') \setminus U$. Since $\alpha(H) = 3$, it follows that there exist two nonadjacent vertices $w_1, w_2 \in W$. Also, since $\alpha(G') = 2$, it follows that $\{w_1, w_2\}$ dominates $U$ (hence $w_1, w_2$ have no common non-neighbor in $U$) and for $i = 1, 2$ the non-neighbors of $w_i$ in $U$ are a clique. Suppose that $w_1, w_2$ have a common neighbor $u \in U$. Then $\{u, v, w_1, w_2\}$ induces a claw in $H$, a contradiction. So $w_1, w_2$ have no common neighbor in $U$. Hence, $U$ is the union of two cliques $K_1, K_2$ where $K_1 = N_U(w_1) = U \setminus N_U(w_2)$ and $K_2 = N_U(w_2) = U \setminus N_U(w_1)$.

Next, we show that $W$ is also the union of two cliques. Let $u \in U$. Then because $H$ is claw-free $u$ does not have two nonadjacent neighbors in $W$ and because $\alpha(G') = 2$, $u$ does not have two nonadjacent non-neighbors in $W$. Hence, $W$ is the union of two cliques $K_3, K_4$ where $K_3 = N_W(u)$ and $K_4 = W \setminus N_W(u)$. Now $K_1 \cup \{w_1\}, K_2 \cup \{w_2\}, K_3 \cup u$, and $K_4$ are four cliques of $G'$ whose sizes add up to $|V(G')| + 3$. Hence, one of these cliques has size at least $\frac{|V(G')|+3}{4}$.

Now if $V(G) \cap W$ is not a clique, then $\alpha(H|(V(G) \cup \{v\})) = 3$ and so by the minimality of $H$, $G' = G$. Then by what we showed above, $G$ has a clique of size at least $\frac{|V(G)|+3}{4}$. Hence we may assume that $V(G) \cap W$ is a clique. But now $V(G)$ is the union of three cliques, namely $V(G) \cap W, V(G) \cap K_1$, and $V(G) \cap K_2$. This proves (1).

In [8] it is shown that every graph with stability number 2 and a clique of size at least $\frac{n+3}{4}$ (where $n$ is the number of vertices of the graph), has a clique minor of size at least $\frac{n}{2}$. It follows from (1) and 3.1 that $G$ has a clique minor of size at least $\frac{|V(G)|}{2} = \nu(G)$. This proves 3.2. ■

**3.3** *Let $G, G_1, G_2, \ldots, G_k$ be graphs such that $|V(G)| = \sum_{i=1}^k |V(G_i)|$ and $\alpha(G) = \sum_{i=1}^k \alpha(G_i)$. Then $\max_i \nu(G_i) \geq \nu(G)$.*

**Proof.** Let $n = |V(G)|, \alpha = \alpha(G)$ and for $i = 1, \ldots, k$ let $n_i = |V(G_i)|$ and $\alpha_i = \alpha(G_i)$. By the pigeonhole principle, there is a $j \in \{1, \ldots, k\}$ for which $n_j \geq \frac{\alpha_j}{\alpha} n$. Then

$$\nu(G_j) = \frac{n_j}{\alpha_j} \geq \frac{\frac{\alpha_j}{\alpha} n}{\alpha_j} = \frac{n}{\alpha} = \nu(G).$$

This proves 3.3. ■

**3.4** *Let $G$ be a graph and let $G_1, \dots, G_k$ be minors of $G$ such that $|V(G)| = \sum_{i=1}^{k} |V(G_i)|$, $\alpha(G) = \sum_{i=1}^{k} \alpha(G_i)$ and for $i = 1 \dots, k$, $G_i$ has a clique minor of size $\nu(G_i)$. Then $G$ has a clique minor of size $\nu(G)$.*

**Proof.** This follows immediately from 3.3. ■

**3.5** *Let $G$ be a graph that admits a clique cutset $S$. Then $V(G)$ can be partitioned into two sets $V_1, V_2$ such that $\alpha(G|V_1) + \alpha(G|V_2) = \alpha(G)$. Moreover, if some maximum independent set of $G$ does not meet $S$ and $(X_1, X_2)$ is a partition of $V(G) \setminus S$ such that there are no edges between $X_1$ and $X_2$, then $V_1, V_2$ can be chosen so that $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$.*

**Proof.** Suppose that every maximum independent set of $G$ meets $S$. Then $\alpha(G \setminus S) = \alpha(G) - 1$ and hence $\alpha(G|S) + \alpha(G \setminus S) = \alpha(G)$. Hence, we may assume that for some maximum independent set $I$, $S \cap I = \emptyset$.

Let $(X_1, X_2)$ be a partition of $V(G) \setminus S$ such that there are no edges between $X_1$ and $X_2$ (such a partition exists because $S$ is a cutset). For $i = 1, 2$, let $\alpha_i = |I \cap X_i|$ (note that $\alpha_1 + \alpha_2 = \alpha(G)$). Let $v \in S$. Then $\alpha(G|(X_1 \cup \{v\}) + \alpha(G|(X_2 \cup \{v\}) \leq \alpha(G) + 1$ and so for some $i \in \{1, 2\}$, it holds that $\alpha(G|X_i \cup \{v\}) = \alpha_i$. Therefore, there exists a partition $(S_1, S_2)$ of $S$ such that for $i = 1, 2$, and for each $s \in S_i$, $\alpha(G|(X_i \cup \{s\}) = \alpha_i$. It follows that $\alpha(G|(X_i \cup S_i)) = \alpha_i$. Letting $V_i = X_i \cup S_i$, this completes the proof of 3.5. ■

The following is an easy corollary of 3.5.

**3.6** *If $G$ is a tame graph that admits a clique cutset then $G$ is not a minimum counterexample to 1.2.*

**3.7** *Let $G$ be a minimum counterexample to 1.2 and let $K, J$ be two cliques in $G$. Suppose there exists a partition $(K_1, K_2)$ of $K$, a partition $(J_1, J_2)$ of $J$ and a partition $(Y_1, Y_2)$ of $V(G) \setminus (K \cup J)$ such that there are no edges of $G$ between $Y_1$ and $Y_2 \cup K_2 \cup J_2$ and no edges between $Y_2$ and $Y_1 \cup K_1 \cup J_1$. Then there exist min$(|K|, |J|)$ vertex disjoint paths between $K$ and $J$ in $G$.*

**Proof.** Suppose not. Let $S$ be a smallest cutset separating $K$ and $J$. Then Menger's Theorem [14] implies that $|S| < \min(|K|, |J|)$. It follows that there exists a partition $(X_1, X_2)$ of $V(G) \setminus V(S)$ such that $K \subset X_1 \cup S$, $J \subseteq X_2 \cup S$, and there are no edges between $X_1$ and $X_2$. Let $G'$ be obtained from $G$ by adding an edge $s_1 s_2$ for every pair of nonadjacent vertices $s_1, s_2 \in S$ and let $G_i = G'|(X_i \cup S)$.

*(1) For all $v \in S$, $v$ has a neighbor in $X_1$ and in $X_2$.*

Without loss of generality, suppose there exists $v \in S$ with no neighbor in $X_1$. Then if $v \notin K$ we can add $v$ to $X_2$ and obtain a smaller cutset, $S \setminus \{v\}$, separating $K$ and $J$, contradicting the minimality of $S$. So $v \in S \cap K$, and since $v$ is anticomplete to $X_1$ and

$K \subseteq X_1 \cup S$, it follows that $K \subseteq S$. But $|S| < \min(|K|, |J|)$, which is a contradiction. This proves (1).

*(2) $G_i$ is a claw-free graph for $i = 1, 2$.*

For $v \in X_i$, $v$ has the same neighbors in $G_i$ as in $G$ and the edges between the neighbors in $G_i$ are a superset of those in $G$. Hence, the neighbors of $v$ in $G_i$ still do not contain a triad. For $v \in S$, we claim that the set of neighbors of $v$ in $X_i$ is a clique. For suppose $v$ has two neighbors $x_1, x_1' \in X_1$ that are nonadjacent to each other. By (1), $v$ has a neighbor $x_2 \in X_2$. But now $x_1, x_1', x_2$ are three pairwise nonadjacent vertices in the neighborhood of $v$ in $G$, contrary to the fact that $G$ is a claw-free graph. This proves the claim. Since $N_{G_i}(v) = (N_G(v) \cap X_i) \cup S$, it follows that $G_i$ is claw-free. This proves (2).

*(3) Some maximum independent set of $G'$ does not meet $S$.*

Suppose that every maximum independent set of $G'$ meets $S$. Then since every independent set of $G'$ is also an independent set of $G$, it follows that every maximum independent set of $G$ meets $S$. Since $K$ is a clique (and hence a clique minor) of $G$, it follows that $|K| < \nu(G)$ and so $|S| < |K| < \nu(G) = \frac{|V(G)|}{\alpha(G)}$. Consequently, $|V(G) \setminus S| > \frac{\alpha(G)-1}{\alpha(G)}|V(G)|$ and $\alpha(G \setminus S) \le \alpha(G) - 1$. Therefore,

$$\nu(G \setminus S) = \frac{|V(G) \setminus S|}{\alpha(G \setminus S)} \ge \frac{\frac{\alpha(G)-1}{\alpha(G)}|V(G)|}{\alpha(G) - 1} = \frac{|V(G)|}{\alpha(G)} = \nu(G).$$

By the minimality of $G$, $G \setminus S$ has a clique minor of size $\nu(G \setminus S) \ge \nu(G)$ and hence $G$ has a clique minor of size $\nu(G)$, a contradiction. This proves (3).

Now since $G'$ admits a clique cutset, it follows by 3.5 that there exists a partition $V(G') = (V_1, V_2)$ such that $\alpha(G|V_1) + \alpha(G|V_2) = \alpha(G')$ and because some maximum independent set of $G'$ does not meet $S$ we can choose $V_1, V_2$ such that $X_1 \subseteq V_1$ and $X_2 \subseteq V_2$. By 3.3, $\max(\nu(G'|V_1), \nu(G'|V_2)) \ge \nu(G') = \nu(G)$. Without loss of generality, suppose that $\nu(G'|V_1) \ge \nu(G)$. Let $H = G'|V_1$.

(4) $H$ has a clique minor of size $\nu(H)$.

If $\alpha(H) \ge 3$, then (4) follows from the minimality of $G$. Therefore, we may assume that $\alpha(H) = 2$ (the case $\alpha(H) = 1$ is trivial). First suppose that $V(H) \cap Y_i$ is non-empty for $i = 1, 2$. Then since there are no edges between $Y_1$ and $K_2 \cup Y_2$ in $G$, it follows that $V(H) \cap (K_2 \cup Y_2)$ is a clique. Similarly, $V(H) \cap K_1 \cup Y_1$ is a clique. Then $H$ is the union of three cliques, namely $S$, $V(H) \cap (K_1 \cup Y_1)$, and $V(H) \cap (K_2 \cup Y_2)$. Hence, (4) follows from 3.1.

So we may assume that $V(H) \cap Y_i$ is empty for some $i \in \{1, 2\}$, and without loss of generality say it is $V(H) \cap Y_2$. Then since there are no edges between $Y_1$ and $K_2$, it follows that $V(H) \cap Y_1$ is a clique (or is empty). But then $H$ is the union of three cliques, namely $S$, $K$, and $V(H) \cap Y_1$, and once again (4) follows from 3.1. This proves (4).

Since $V_1 \subseteq V(G_1)$, it follows that $G_1$ has a clique minor of size $\nu(H) \geq \nu(G)$ and hence there exists a set $\mathbb{S}$ of $\nu(G)$ connected disjoint subgraphs of $G_1$ that are pairwise adjacent in $G_1$.

Let $S = \{s_1, \dots, s_n\}$ and let $\mathbb{P} = \{P_1, \dots, P_n\}$ be $|S|$ vertex disjoint paths between $S$ and $K_2$ in $G_2$ such that $s_i \in P_i$. Such paths exist by Menger's Theorem [14] and the minimality of $S$. Let $\phi : S \to \mathbb{P}$ be a bijection defined by $\phi(s_i) = P_i$.

For $H \in \mathbb{S}$ define $\psi(H)$ by

$$\psi(H) = (H \setminus S) \cup \bigcup_{s \in V(H) \cap S} \phi(s).$$

Then $\psi(H)$ is a subgraph of $G$. Define $\mathbb{Q} = \{\psi(H) : H \in \mathbb{S}\}$. Then $\mathbb{Q}$ is a set of $\nu(G)$ connected disjoint subgraphs of $G$. We claim that the members of $\mathbb{Q}$ are pairwise adjacent. Suppose not. Choose $Q_1, Q_2 \in \mathbb{Q}$ that are not adjacent. For $i = 1, 2$, let $H_i$ be the member of $\mathbb{S}$ such that $Q_i = \psi(H_i)$. Since $J$ is a clique in $G$, it follows that not both $V(Q_1)$ and $V(Q_2)$ contain a vertex of $J$, and therefore, not both $V(H_1)$ and $V(H_2)$ contain a vertex of $S$. Since $H_1$ and $H_2$ are adjacent, we deduce that there exist $h_1 \in V(H_1)$ and $h_2 \in V(H_2)$ such that not both $h_1, h_2$ are in $S$ and $h_1h_2$ is an edge of $G_1$. But now by the definition of $\psi$ and $G_1$, $h_1 \in V(Q_1)$, $h_2 \in V(Q_2)$ and $h_1h_2$ is an edge of $G$, contrary to the fact that $Q_1$ and $Q_2$ are nonadjacent. This proves the claim. Hence $G$ has a clique minor of size $\nu(G)$, a contradiction. This completes the proof of 3.7. ■

We conclude this section by proving that a minimal counterexample to 1.2 does not admit a $W$-join. First, we need a lemma that appears in a modified form in [6] and is proved in its current form in [5].

**3.8** *Let $G$ be a claw-free graph and let $(A, B)$ be a $W$-join. Let $H$ be a graph obtained from $G$ by arbitrarily changing the adjacency between some vertices of $A$ and some vertices of $B$ (all the other adjacencies remain unchanged). Then $H$ is claw-free.*

We are now ready to prove the last result of this section.

**3.9** *Let $G$ be a claw-free graph with $\alpha(G) \geq 3$ that admits a $W$-join. Then $G$ is not a minimum counterexample to 1.2.*

**Proof.** Suppose that $G$ is a minimum counterexample to 1.2. Let $(A, B)$ be a $W$-join of $G$. Let $C$ be the set of vertices of $G$ that are $A$-complete and $B$-complete, $D$ be the set of vertices of $G$ that are $A$-complete and $B$-anticomplete, $E$ the set of vertices of $G$ that are $A$-anticomplete and $B$-complete, and $F$ the set of vertices of $G$ that are $A$-anticomplete and $B$-anticomplete.

By the definition of $W$-join, $A$ is neither complete nor anticomplete to $B$. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$ such that $a_1$ is adjacent to $b_1$ and $a_2$ is nonadjacent to $b_2$ (we allow $a_1 = a_2$ or $b_1 = b_2$ but not both). Let $H$ be the graph obtained from $G$ by deleting the edge $a_1b_1$. By 3.8, $H$ is claw-free.

(1) $\alpha(H) = \alpha(G)$

Clearly $\alpha(H) \geq \alpha(G)$. Let $I$ be a maximum independent set of $H$. We claim that $G$ has an independent set of size $|I|$. If either $a_1 \notin I$ or $b_1 \notin I$ then $I$ is also an independent set of $G$, and so the claim holds. So we may assume that $a_1, b_1 \in I$. It follows that $I$ contains no member of $C \cup D \cup E$. But now $(I \setminus \{a_1, b_1\}) \cup \{a_2, b_2\}$ is an independent set of $G$ of cardinality $|I|$. This proves the claim and completes the proof of (1).

Since $|V(H)| = |V(G)|$, from (1) it follows that $\nu(H) = \nu(G)$. We claim that $H$ has a clique minor of size $\nu(H)$. If $H$ is connected, then this follows from the minimality of $G$. Otherwise, every component $C$ of $H$ is a proper induced subgraph of $G$ and so by the minimality of $G$ has a clique minor of size $\nu(H|C)$. But now it follows from 3.4 that $H$ has a clique minor of size $\nu(H)$. This proves the claim. Since $H$ is a subgraph of $G$, it follows that $G$ has a clique minor of size $\nu(H) = \nu(G)$. Hence, we arrive at a contradiction and this proves 3.9. ■

# 4 The icosahedron

In this section we prove that inflations of graphs obtained from the icosahedron (those that appear in the structure theorem 2.1) are not minimal counterexamples to 1.2.

**4.1** *Let $H \in \mathcal{I}$ and let $G$ be an inflation of $H$. Then $G$ is not a minimum counterexample to 1.2.*

**Proof.** Suppose that $G$ is a minimal counterexample to 1.2. Let $v_0, v_1, \ldots, v_{11}$ be as in the definition of the icosahedron and let $G_0, G_1, G_2$ be as in the definition of $\mathcal{I}$. Then $H \in \{G_0, G_1, G_2\}$.

For $0 \leq i \leq 11$, let $X_{v_i}$ be as in the definition of inflation (where $X_{v_{11}}$ is empty when $G$ is an inflation of $G_1$ or $G_2$, and $X_{v_{10}}$ is empty when $G$ is an inflation of $G_2$). For $i = 0, 2, 4, 6, 8, 9$, let $u_i \in X_{v_i}$. Let $P = G|\{u_0, u_2, u_4, u_6, u_8, u_9\}$. Let $G' = G \setminus V(P)$. Since $|V(P)| = 6$ and $\alpha(G) = 3$, it follows that $\nu(G') = \frac{|V(G')|}{\alpha(G')} \geq \frac{|V(G)|-6}{\alpha(G)} = \frac{|V(G)|}{\alpha(G)} - 2 = \nu(G) - 2$. Since $G'$ is tame, by the minimality of $G$ it has a clique minor of size $\nu(G')$. This means that there exists a set $\mathcal{S}$ of $\nu(G')$ connected, disjoint subgraphs of $G'$ that are pairwise adjacent.

Let $S_1 = \{v_0, v_8, v_9\}$ and $S_2 = \{v_2, v_4, v_6\}$. Suppose that no member of $\mathcal{S}$ is a subgraph of $G|(X_{v_0} \cup X_{v_9})$ or $G|(X_{v_2} \cup X_{v_4})$. Then, since $S_1$ dominates $V(G) \setminus (X_{v_2} \cup X_{v_4})$, $S_2$ dominates $V(G) \setminus (X_{v_0} \cup X_{v_9})$, and $G|S_1$ is adjacent to $G|S_2$, it follows that $\mathcal{S} \cup \{G|S_1, G|S_2\}$ is a set of $\nu(G)$ connected, disjoint subgraphs of $G$ that are pairwise adjacent.

Hence, we may assume that some member of $\mathcal{S}$ is a subgraph of $G|(X_{v_0} \cup X_{v_9})$ or $G|(X_{v_2} \cup X_{v_4})$. From symmetry, we may assume that there exists $T \in \mathcal{S}$ such that $V(T) \subseteq X_{v_0} \cup X_{v_9}$. Note that this implies that no member of $\mathcal{S}$ is a subgraph of $G|(X_{v_2} \cup X_{v_4})$ since $X_{v_0} \cup X_{v_9}$ is anticomplete to $X_{v_2} \cup X_{v_4}$. Suppose that no member of $\mathcal{S}$ is a subgraph of $G|X_{v_0}$. Let $S_1' = \{v_0, v_9\}$ and $S_2' = \{v_2, v_4, v_6, v_8\}$. Then, since $S_2'$ dominates $V(G) \setminus X_{v_0}$, and $N_G(T) \setminus S_1' \subseteq N_G(S_1')$, and $G|S_1'$ is adjacent to $G|S_2'$, it follows that $\mathcal{S} \cup \{G|S_1', G|S_2'\}$ is a set of $\nu(G)$ connected, disjoint subgraphs of $G$ that are pairwise adjacent.

Hence, we may assume that some member of $\mathcal{S}$ is a subgraph of $G|X_{v_0}$. So there exists $T \in \mathcal{S}$ such that $V(T) \subseteq X_{v_0}$. Let $S_1'' = \{v_0\}$ and $S_2'' = \{v_2, v_4, v_6, v_8, v_9\}$. Then, since $S_2''$ dominates $V(G)$ and $N_G(T) \setminus S_1'' \subseteq N_G(S_1'')$, and $G|S_1''$ is adjacent to $G|S_2''$, it follows that $\mathcal{S} \cup \{G|S_1'', G|S_2\ ''\}$ is a set of $\nu(G)$ connected, disjoint subgraphs of $G$ that are pairwise adjacent. Hence, $G$ has a clique minor of size $\nu(G)$, a contradiction. This proves 4.1. ■

# 5 Antiprismatic graphs

We begin this section by stating two basic facts about antiprismatic graphs (these are proved in [5]. Then we prove that inflations of antiprismatic graphs are not minimal counterexamples to 1.2.

**5.1** *Let $G$ be an antiprismatic graph such that $G$ does not have two disjoint triads. Then there exists a vertex $v \in V(G)$ meeting all triads of $G$.*

**5.2** *Let $G$ be an antiprismatic graph and let $P = v_1\text{-}v_2\text{-}v_3$ be an induced two-edge path in $G$. Let $X \subseteq V(G)$ be the set of vertices not dominated by $V(P)$. Then $|X| \leq 1$, and if no triad of $G$ contains both $v_1$ and $v_3$ then $|X| = 0$.*

We now prove the main result of this section.

**5.3** *Let $H$ be an antiprismatic graph with $\alpha(H) \geq 3$ and let $G$ be an inflation of $H$. Then $G$ is not a minimum counterexample to 1.2.*

**Proof.** *(1) If $H$ has two disjoint triads, then 5.3 holds.*

Let $\{v_1, v_2, v_3\}$ and $\{u_1, u_2, u_3\}$ be two disjoint triads of $H$. Then, since $H$ is antiprismatic, $H|\{v_1, v_2, v_3, u_1, u_2, u_3\}$ is an induced cycle of length 6, and without loss of generality we may assume that for $i = 1, 2, 3$, $v_i$ is adjacent to $u_i$ and $u_{i+1}$ (where the subscripts are read modulo 3). For $i = 1, 2, 3$, let $X_{v_i}, X_{u_i}$ be as in the definition of inflation. Choose $v_i' \in X_{v_i}$ and $u_i' \in X_{u_i}$.

Let $S_1 = \{v_1', u_1', u_2'\}$, $S_2 = \{v_2', v_3', u_3'\}$ and let $G' = G \setminus (S_1 \cup S_2)$. Since $|S_1 \cup S_2| = 6$ and $\alpha(G) = 3$, it follows that $\nu(G') = \frac{|V(G')|}{\alpha(G')} \geq \frac{|V(G)|-6}{\alpha(G)} = \frac{|V(G)|}{\alpha(G)} - 2 = \nu(G) - 2$. Since $G'$ is tame, by the minimality of $G$, $G'$ has a clique minor of size $\nu(G')$. This means that there exists a set $\mathcal{S}$ of $\nu(G')$ connected, disjoint subgraphs of $G'$ that are pairwise adjacent.

By 5.2, $\{u_1, u_2, v_1\}$ dominates $V(H) \setminus \{u_3\}$ in $H$. Suppose that no member of $\mathcal{S}$ is a subgraph of $G|X_{v_1}$ or $G|X_{u_3}$. Since $G|S_1$ is adjacent to $G|S_2$, it follows that $\mathcal{S} \cup \{G|S_1, G|S_2\}$ is a set of $\nu(G)$ connected, disjoint subgraphs of $G$ that are pairwise adjacent, a contradiction.

Hence, we may assume that some member $T$ of $\mathcal{S}$ is a subgraph of $G|X_{v_1}$ or $G|X_{u_3}$. From symmetry, we may assume that $V(T) \subseteq X_{v_1}$. Since $X_{v_1}$ is anticomplete to $X_{u_3}$, it follows that no member of $\mathcal{S}$ is a subgraph of $G|X_{u_3}$. Let $S_1' = \{v_1'\}$ and $S_2' = \{v_2', v_3', u_1', u_2', u_3'\}$. Then $S_2'$ dominates $V(G)$ since $X_{u_2}$ is complete to $X_{v_1}$. Since $N_{G'}(T) \subseteq N_G(S_1')$ and $G|S_1'$ is adjacent to $G|S_2'$, it follows that $\mathcal{S} \cup \{G|S_1', G|S_2'\}$ is a set of $\nu(G)$ connected, disjoint subgraphs of $G$ that are pairwise adjacent, a contradiction. This proves (1).

In view of (1), we may assume that there are no two disjoint triads in $H$. Then by 5.1, there is a vertex $v$ meeting all triads of $H$. If $N(v)$ is a clique, then $G$ admits a clique cutset and so 5.3 follows from 3.6. So we may assume that there exist $u, w \in N(v)$ such that $u$ and $w$ are nonadjacent. Since $v$ meets all triads, it follows that there is no triad containing $\{u, w\}$ and hence $\{u, w\}$ dominates $V(G)$. Let $S = \{u, v, w\}$. Let $G' = G \setminus S$. Since $|S| = 3$ and $\alpha(G) = 3$, it follows that

$$\nu(G') = \frac{|V(G')|}{\alpha(G')} = \frac{|V(G)| - 3}{\alpha(G')} \geq \frac{|V(G)| - 3}{3} = \nu(G) - 1.$$

By the minimality of $G$, $G \setminus S$ contains a set $\mathcal{S}$ of $\nu(G')$ connected, disjoint subgraphs of $G'$ that are pairwise adjacent. But now $\mathcal{S} \cup \{S\}$ is a set of $\nu(G)$ connected, disjoint subgraphs of $G$ that are pairwise adjacent, a contradiction. This proves 5.3.

# 6 Nontrivial strip-structures

In this section we prove 1.2 for graphs $G$ that admit non-trivial strip structures and appear in 2.1. We begin by stating two lemmas that appear in [5]

**6.1** *Suppose that $G$ admits a nontrivial strip-structure such that $|Z| = 1$ for some strip $(J, Z)$ of $(H, \eta)$. Then either $G$ is a clique or $G$ admits a clique cutset.*

Let $(J, Z)$ be a strip. We say that $(J, Z)$ is a *line graph strip* if $|V(J)| = 3$, $|Z| = 2$ and $Z$ is complete to $V(J) \setminus Z$.

**6.2** *Let $G$ be a graph that admits a nontrivial strip-structure $(H, \eta)$ such that for every $F \in E(H)$, the strip of $(H, \eta)$ at $F$ is a line graph strip. Then $G$ is a line graph.*

We now prove the main result of the section.

**6.3** *Suppose that $G$ admits a nontrivial strip-structure $(H, \eta)$ such that for each strip $(J, Z)$ of $(H, \eta)$, $1 \leq |Z| \leq 2$, and if $|Z| = 2$ then either $|V(J)| = 3$ and $Z$ is complete to $V(J) \setminus Z$, or $(J, Z)$ is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. Then $G$ is not a minimal counterexample to 1.2.*

**Proof.** If $|Z| = 1$ for some strip $(J, Z)$ then by 6.1 either $G$ is a clique or $G$ admits a clique cutset. In the former case 6.3 obviously holds, and in the latter case 6.3 follows from 3.6. Hence, we may assume that $|Z| = 2$ for all strips $(J, Z)$.

Suppose that every strip is a line graph strip. Then the result follows from [16] and 6.2. So we may assume that some strip $(J_1, Z_1)$ is not a line graph strip. Let $Z_1 = \{a_1, b_1\}$. Let $A_1 = N_{J_1}(a_1)$, $B_1 = N_{J_1}(b_1)$, $A_2 = N_G(A_1) \setminus V(J_1)$, and $B_2 = N_G(B_1) \setminus V(J_1)$. Let $C_1 = V(J_1) \setminus (A_1 \cup B_1)$ and $C_2 = V(G) \setminus (V(J_1) \cup A_2 \cup B_2)$. Then $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2 \cup B_2 \cup C_2$.

*(1) If $C_2 = \emptyset$ and $A_2 = B_2$, then 6.3 holds.*

Note that $V(G) = A_1 \cup B_1 \cup C_1 \cup A_2$. Since $|Z_1| = 2$ and $(J_1, Z_1)$ is not a line graph strip, it follows that $(J_1, Z_1)$ is a member of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 \cup \mathcal{Z}_4 \cup \mathcal{Z}_5$. We consider the cases separately:

1. $(J_1, Z_1)$ is a member of $\mathcal{Z}_1$. In this case $J_1$ is a linear interval graph and so $G$ is a long circular interval graph and 6.3 follows from [4].
2. $(J_1, Z_1)$ is a member of $\mathcal{Z}_2, \mathcal{Z}_3$, or $\mathcal{Z}_4$. In all of these cases, $\alpha(G) = 3$, $A_1, B_1$, and $C_1$ are all cliques and so $V(G)$ is the union of three cliques, namely $A_1 \cup A_2, B_1$, and $C_1$. Hence, $G$ has a clique of size $\frac{|V(G)|}{3} = \nu(G)$.
3. $(J_1, Z_1)$ is a member of $\mathcal{Z}_5$. Let $v_1, \ldots, v_{12}, X, H, H'$ be as in the definition of $\mathcal{Z}_5$ and for $1 \leq i \leq 12$ let $X_{v_i}$ be as in the definition of an inflation. Then $A_2$ is complete to $X_{v_1} \cup X_{v_2} \cup X_{v_4} \cup X_{v_5}$. Let $H''$ be the graph obtained from $H'$ by adding a new vertex $a_2$, adjacent to $v_1, v_2, v_4$ and $v_5$. Then $H''$ is an antiprismatic graph and $G$ is an inflation of $H''$, so 6.3 follows from 5.3.

This proves (1).

Therefore, we may assume that either $C_2 \neq \emptyset$, or $A_2 \neq B_2$. For $i = 1, 2$, let $G_i = G|(A_i \cup B_i \cup C_i)$. Let $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$. Without loss of generality, we may assume that $|A| \leq |B|$. It is easy to check that $G, A, B$ satisfy the hypothesis of 3.7, hence there exist $|A|$ vertex disjoint paths between $A$ and $B$ in $G$. From the definitions of $A_1, A_2, B_1$, and $B_2$ it follows that for $i = 1, 2$, $|A_i| \leq |B_i|$ and that there exist $|A_i|$ vertex disjoint paths from $A_i$ to $B_i$ in $G_i$.

*(2) If some maximum independent set of $G$ does not meet $A \cup B$, then 6.3 holds.*

Let $I$ be a maximum independent set of $G$ that does not meet $A \cup B$. For $i = 1, 2$, let $I_i = I \cap V(G_i)$. Suppose that $\alpha(G_1) > |I_1|$ and let $I_1'$ be an independent set of $G_1$ of size $\alpha(G_1)$. Then $I_1' \cup I_2$ is an independent set of $G$ of size greater than $\alpha(G)$, a contradiction. Hence, $\alpha(G_1) = |I_1|$ and similarly $\alpha(G_2) = |I_2|$. Since $G_i$ is tame for $i = 1, 2$, it follows that $G_i$ has a clique minor of size $\nu(G_i)$. Then 6.3 follows from 3.4. This proves (2).

So we may assume that every maximum independent set of $G$ meets $A \cup B$. Suppose that every maximum independent set of $G$ meets $B$. Then $\alpha(G \setminus B) = \alpha(G) - 1$ and hence $\alpha(G|B) + \alpha(G \setminus B) = \alpha(G)$. Then 6.3 follows from 3.4. So we may assume there exists a maximum independent set $L$ of $G$ that does not meet $B$. Then $L$ meets $A$. Since $A$ is a clique, $L \cap (A) = 1$. From symmetry we may assume that $L \cap A_1 = 1$ and $L \cap A_2 = 0$.

Let $G_1'$ be the graph obtained from $G|(A \cup B_1 \cup C_1)$ by making $A_2$ complete to $B_1$. Then since there exist $|A_2|$ vertex disjoint paths between $A_2$ and $B_2$ in $G_2$, it follows that $G_1'$ is a minor of $G$. We claim that $G_1'$ is a claw-free graph. For $v \in C_1$, $G|(N_G(v)) = G_1'|(N_{G_1'}(v))$, and so there is no triad in $N_{G_1'}(v)$. For $v \in A_1$, $N_G(v) = N_{G_1'}(v)$, and if two vertices of $N_{G_1'}(v)$ are adjacent in $G$, then they are also adjacent in $G_1'$. Therefore there is no triad in $N_{G_1'}(v)$. Next let $v \in B_1$, and suppose that in $G_1'$ there is a triad $\{x, y, z\}$ among the neighbors of $v$. Since $G$ is claw-free, we may assume that $x \in A_2$. Consequently, $y, z \in C_1$. Let $b \in B_2$. Now, $\{b, y, z\}$ is a triad among the neighbors of $v$ in $G$, contrary to the fact that $G$ is claw-free. Finally, for $v \in A_2$, the set of neighbors of $v$ in $G_1'$ is the union of two cliques, namely $A \setminus \{v\}$ and $B_1$. This proves the claim that $G_1'$ is a claw-free graph.

Let $L_1 = L \cap (A \cup B_1 \cup C_1)$ and $L_2 = L \cap (B_2 \cup C_2)$.

*(3)* $\alpha(G_1') = |L_1|$

Since $L \cap (A_2 \cup B_1) = \emptyset$ it follows that $\alpha(G_1') \geq |L_1|$. Let $L'$ be a maximum independent set of $G_1'$. We claim that $|L'| \leq |L_1|$. For suppose that $|L'| > |L_1|$. If $L' \cap A_2 = \emptyset$, then $L' \cap L_2$ is an independent set of $G$ of size greater than $|L|$, a contradiction. So we may assume that there exists a vertex $v \in L' \cap A_2$. It follows that $L' \cap (A_1 \cup B_1) = \emptyset$. Hence, $(L' \setminus \{v\}) \cup L_2$ is an independent set of $G$ of size $\geq |L| = \alpha(G)$ that does not meet $A \cup B$, a contradiction. This proves the claim and completes the proof of (3).

Let $G_2' = G|(B_2 \cup C_2)$. We claim that $\alpha(G_2') = |L_2|$. For suppose that $G_2'$ has an independent set $L_2'$ with $|L_2'| > |L_2|$. Then $L_1 \cup L_2'$ is an independent set of $G$ of size greater than $|L| = \alpha(G)$, a contradiction. This proves the claim. From the claim and (3), it follows that $\alpha(G_1') + \alpha(G_2') = \alpha(G)$.

Next, we show that $G_i'$ has a clique minor of size $\nu(G_i')$ for $i = 1, 2$. For $G_2'$, this follows from the minimality of $G$ since $G_2'$ is tame. For $G_1'$, if $\alpha(G_1') \geq 3$, then $G_1'$ has a clique minor of size $\nu(G_1')$ by the minimality of $G$. So we may assume that $\alpha(G_1') = 2$. In this case, $C_1$ is a clique and therefore $G_1'$ is the union of three cliques, namely $C_1, A_1 \cup A_2$, and $B_1$. Hence, by 3.1, $G_1'$ has a clique minor of size $\nu(G_1')$. Now 6.3 follows from 3.4. ■

# 7 Proof of main result and tightness

We are now ready to prove the main result of this paper.

**Proof of 1.2.** Let $G$ be tame and suppose that $G$ is a minimum counterexample to 1.2. If $G$ is not connected, then by the minimality of $G$, every component $C$ of $G$ has a clique minor of size $\nu(G|C)$ and so 1.2 follows from 3.4. We may therefore assume that $G$ is connected. By 3.2 $\alpha(G) \geq 3$, by 3.9 $G$ does not admit a $W$-join, by 4.1 and 5.3, $G$ is not an inflation of a member of $\mathcal{I}$ or of an antiprismatic graph, and by 6.3, $G$ does not admit a nontrivial strip-structure as in 2.1. Hence, by 2.1, $G$ is either a circular interval graph or $V(G)$ is the union of three cliques. Since circular interval graphs are quasi-line graphs, $G$ is not a circular interval graph by [4]. It follows that $V(G)$ is the union of three cliques. But then $G$ has a clique (and hence a clique minor) of size $\frac{V(G)}{3} = \nu(G)$, a contradiction. Therefore, there is no minimal counterexample to 1.2, and so there is no counterexample at all. This proves 1.2. ■

Clearly, 1.2 implies 1.1.

Finally, we show that the results in this paper are tight. We provide an infinite family of claw-free graphs with stability number at least three such that their largest clique minor has size $\frac{n}{\alpha}$ (where $n$ is the number of vertices and $\alpha$ the independence number).

Let $H$ be a path on $k$ vertices for some positive integer $k$ and let $V(H) = \{v_1, \ldots, v_k\}$. Let $G$ be an inflation of $H$ such that $|X_{v_i}| = m$ for all $i = 1, \ldots k$ and some positive integer $m$. Then for even $k$, $\nu(G) = 2m$ and for $k \geq 6$, $\alpha(G) \geq 3$. We show by induction on $k$ that $G$ has no clique minor of size larger than $2m$. In fact, we prove that this is true for $k \geq 2$. For $k = 2$, $G$ is a clique of size $2m$ and so the result holds. So suppose that $k > 1$. Then $X = X_{v_2}$ is a clique cutset of $G$ of size $m$. Let $\mathcal{S}$ be a set of connected, disjoint subgraphs of $G$ that pairwise touch, and subject to that maximal. If every member of $\mathcal{S}$ meets $X$, then $|\mathcal{S}| \leq m$. So we may assume that some member of $\mathcal{S}$ does not meet $X$. But then since $X$ is a cutset, either every member meets $X_1 \cup X$ or every member meets $X \cup X_3 \cup \cdots \cup X_k$. Because $X$ is a clique, we may assume that every member is contained entirely in $X_1 \cup X$ or every member is contained entirely in $X \cup X_3 \cup \cdots \cup X_k$. But now by induction it follows that $|\mathcal{S}| \leq 2m$.

# 8 Conclusion

Recently, the author and Maria Chudnovsky proved an approximate version of Hadwiger's conjecture for claw-free graphs in [5]. More specifically, they showed that if $G$ is a claw-free graph with chromatic number $\chi$ then $G$ has a clique minor of size $\frac{2}{3}\chi$. The result of this paper is neither strictly stronger nor strictly weaker than the result of [5] (it is stronger when $\frac{n}{\alpha}$ is close to $\chi$ and weaker when the two quantities are far apart). Rather, the two results complement each other. However, one advantage of the result of this paper is that it is tight.

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