TRANSLATING SOLUTIONS TO LAGRANGIAN MEAN CURVATURE FLOW

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Abstract. We prove some non-existence theorems for translating solutions to Lagrangian mean curvature flow. More precisely, we show that translating solutions with an L^2 bound on the mean curvature are planes and that almost-calibrated translating solutions which are static are also planes. Recent work of D. Joyce, Y.-I. Lee, and M.-P. Tsui, shows that these conditions are optimal.

1. Introduction

It was shown in [5] that finite-time singularities are, in some sense, unavoidable. More precisely, the first author gave examples of Lagrangians in $\mathbb{C}^2$ having the Lagrangian angle as small as we want and for which the Lagrangian mean curvature flow develops a finite-time singularity. Thus, if one aims to use Lagrangian mean curvature flow in order to understand the existence problem for Special Lagrangians (i.e. Lagrangians which are minimal surfaces) it is crucial to understand how finite-time singularities form. The next example shows that this is a rather non-trivial problem.

Example 1.1. Let γ_0 be the curve in $\mathbb{C}$ given in Figure 1.

The curve can be made so that, under curve-shortening flow $(\gamma_t)_{t\geq 0}$, the small loop collapses at time T and γ_T becomes a curve with a cusp point. Moreover, γ_0 can be chosen so that the angle that the tangent vector makes with the x-axis has an oscillation not much bigger than π. Let L_t be the Lagrangian surface in $\mathbb{C}^2$ given by

$$L_t := \gamma_t \times \mathbb{R} \subset \mathbb{C} \times \mathbb{C}.$$

Then, L_0 is a zero-Maslov class Lagrangian with oscillation of the Lagrangian angle as close to π as we want and which develops a singularity at time T. The singular set is a line of cusp-points and hence has Hausdorff dimension one.

This example shows that in order to develop a regularity theory for the flow, i.e., show that singularities for Lagrangian mean curvature flow are isolated, we need to require the oscillation of the Lagrangian angle to be strictly smaller than π (*almost-calibrated*).

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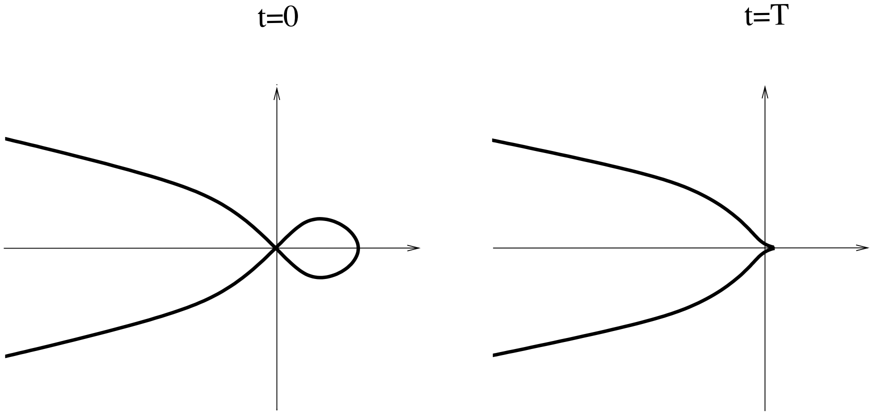

FIGURE 1. Curve γ_0 and γ_T.

So far, the only known evidence that such regularity theory is possible was given in [5]. Their, assuming the initial condition is a rational and almost-calibrated Lagrangian, the first author showed that if one rescales the flow around a fixed point in space-time, connected components of this rescaled flow converge to an area-minimizing union of planes. The fact that the rescaled flow converges to a union of planes is an almost trivial consequence of Huisken's monotonicity formula and so the interesting part is that the configuration of planes needs to be area-minimizing. Without this property it would be hopeless to expect any regularity theory.

Nonetheless, we should point out that the property mentioned above is not sufficient to develop a regularity theory. One needs to understand dilations of the flow where the point at which we center the dilation changes with the scale (called *Type II dilations*). From general theory, it follows that Type II dilations converge to an eternal solution with second fundamental form uniformly bounded. If singularities are indeed isolated the expectation is that this eternal solution has vanishing mean curvature. We now describe heuristically what could happen regarding Example 1.1. If we rescale around the fixed point in space-time at which the singularity is developing, the rescaled flow converges to a plane with multiplicity two. On the other hand, if we rescale the flow around the point of highest second fundamental form at some time t_1 close to T and choose the scale so that the second fundamental form for the new solution becomes bounded by one, the rescaled flow converges to the eternal solution given by

$$(1) \qquad L_t := \{(-\log\cos y_1 + t, y_1, x_2, 0) \mid -\pi/2 < y_1 < \pi/2,\, x_2 \in \mathbb{R}\}.$$

This solution is called the *grim-reaper* and is an example of translating solutions to mean curvature flow.

Definition 1.2. A Lagrangian L is a *translating solution* to Lagrangian mean curvature flow if we can find an ambient vector e_1 so that

$$L_t := L + te_1$$

is a solution to mean curvature flow.

Remark 1.3. Without loss of generality, we assume that $e_1 = (1, 0, 0, 0)$. We can always achieve this by scaling L and then choosing a suitable coordinate system.

In a surprising new result, Dominic Joyce, Yng-ing Lee, and Mao-Pei Tsui [4] found translating solutions to Lagrangian mean curvature flow with oscillation of the Lagrangian angle arbitrarily small. They are described as follows. Let w be a curve in $\mathbb{C}$ such that

$$w_t := \sqrt{2t}w \quad \text{for} \quad t > 0$$

is a solution to curve shortening flow in $\mathbb{C}$. This curve can be chosen in a way that the angle θ that the tangent vector makes with the x-axis has arbitrarily small oscillation. Set

$$L := \left\{ \left(\frac{|w|^2(y) - x^2}{2} - i\theta(y), xw(y) \right), \,|\, x, y \in \mathbb{R} \right\} \subset \mathbb{C} \times \mathbb{C}. \tag{2}$$

Using the fact that the curvature of w satisfies

$$\vec{k} = w^{\perp},$$

it is a straightforward computation to check that L is Lagrangian and that

$$L_t = L + t(1, 0, 0, 0)$$

is a solution to Lagrangian mean curvature flow. Moreover, the Lagrangian angle of L coincides with θ and hence its oscillation can be made arbitrarily small.

The main purpose of this paper is to give conditions that exclude the existence of nontrivial translating solutions to Lagrangian mean curvature flow. In order to do so, we need one more definition.

Let $(L_t)_{-\infty<t<\infty}$ be a eternal solution to Lagrangian mean curvature flow in $\mathbb{C}^2$ which is almost-calibrated. Given a sequence $(\lambda_i)_{i\in\mathbb{N}}$ converging to zero, we can consider the sequence of *blow-downs*

$$L^i_s := \lambda_i L_{s/\lambda_i^2} \quad \text{where} \quad -\infty < s < \infty.$$

It follows from Theorem 3.1 that we can extract a subsequence L^i_s converging weakly to the same union of planes for every $s \leq 0$.

Definition 1.4. A eternal solution to Lagrangian mean curvature flow is called *static* if we can find a convergent sequence of blow-downs L^i_s that converges weakly to the same union of planes for *every* s.

In Theorem 3.1 we show that if L_0 is exact (see next section for definition), then L^i_s converges to a self-expander for $s > 0$ and if the self-expander has zero mean curvature, then the eternal solution is static.

Theorem A. *Let L be a translating solution for Lagrangian mean curvature flow in $\mathbb{C}^2$ for which we can find a constant C_1 such that*

$$(H)\qquad \begin{cases} \text{The first Betti number of } L \text{ is finite,} \\ \sup_L |\theta| \leq C_1 \\ \mathcal{H}^2(L \cap B_R(0)) \leq C_1 R^2 \quad \text{for all } R > 0, \\ \sup_L |A|^2 \leq C_1. \end{cases}$$

If

$$\int_L |H|^2 d\mu \quad \text{is finite}$$

or if

$$L \quad \text{is static} \quad \text{and} \quad \inf_L \cos\theta \geq \varepsilon_1$$

for some $\varepsilon_1 > 0$, then L is a plane.

The static condition is necessary for the following reason. Consider the translating solution $(L_t)_{-\infty<t<\infty}$ described in (2) and denote by $\tilde{w}$ the curve in $\mathbb{C}$ given by the union of w with $-w$. If $(\lambda_i)_{i\in\mathbb{N}}$ is a sequence converging to zero, the curves $\sqrt{\lambda_i}\tilde{w}$ converge to a union of two lines crossing at the origin which we denote by $\tilde{w}_0$. A simple computation shows that the sequence of blow-downs L^i_s converges to the union of two planes given by

$$\mathbb{R} \times \tilde{w}_0 \subseteq \mathbb{C} \times \mathbb{C}$$

for every $s \leq 0$ and to a self-expander given by

$$\sqrt{2s}\,(\mathbb{R} \times \tilde{w}) = \mathbb{R} \times \sqrt{2s}\tilde{w} \subseteq \mathbb{C} \times \mathbb{C}$$

for every $s > 0$.

The condition

$$\inf_L \cos\theta \geq \varepsilon_1$$

is necessary because otherwise the grim-reaper described in (1) would be a counterexample.

The paper is organized as follows. In Section 2 we introduce some basic notation and derive some simple identities for translating solutions and in Section 3 we prove a compactness theorem for blow-down sequence of a eternal solution.

In Section 4 we assume that, outside a compact set, L can be decomposed into N components $L_1, \ldots, L_N$, where each L_j is the graph of a multivalued function defined over a plane P_j minus a disc, and show that in this case L needs to be a plane (Theorem 4.3). The argument consists in using barriers to show that on each component L_j the Lagrangian angle converges to a constant sufficiently fast and this will imply that, by choosing R sufficiently large, $L_j \setminus B_R(0)$ can be made as close to a plane as we want. In particular,

$$\lim_{R\to\infty} \oint_{\partial(L\cap B_R(0))} \langle \nu, e_1 \rangle d\sigma = 0,$$

where ν denotes the exterior unit normal to $\partial(L \cap B_R(0))$ in L. The result will follow because Proposition 2.1 (ii) implies that

$$\oint_{\partial(L\cap B_R(0))} \langle \nu, e_1\rangle d\sigma = \int_{L\cap B_R(0)} |H|^2 d\mu$$

and so L will have zero mean curvature. In that section we also prove a lemma (Lemma 4.6) that gives us conditions under which a translating solution to Lagrangian mean curvature flow admits a graphical decomposition.

In Section 5 we show that if L has the L^2-norm of $|H|$ bounded or if L is static and almost-calibrated, then L satisfies the conditions specified in Lemma 4.6 and hence admits a graphical decomposition.

1.1. **Open questions.** We propose two questions whose answer could provide some valuable insight on whether it is reasonable to expect any good behavior for singularities of Lagrangian mean curvature flow.

The first question is whether the translating solutions found by D. Joyce, Y.-I. Lee, and M.-P. Tsui, can arise as a blow-up of a finite time singularity fort Lagrangian mean curvature flow.

If one aims to develop a regularity theory for the flow, it is absolutely necessary to answer this question. This relation has been been observed before in codimension one mean curvature flow and Ricci flow. In the first case, Brian White used his work on mean convex solutions to mean curvature flow [9] to show that no grim-reaper appears as the limit of a sequence of rescaled flows [10, Corollary 4]. In the second case, one of the first breakthroughs of Perelman [6, Section 4] was to show that the cigar soliton does not arise as a finite-time singularity model.

If one could show that, assuming the initial condition for the flow is an exact and almost-calibrated Lagrangian, no blow-down of a Type II rescale can give rise to a self expander, then the question above would be solved for a large class of initial conditions.

The second question addresses the issue of uniqueness of translating solutions. Suppose that $(L_t)_{t\in\mathbb{R}}$ and $(L'_t)_{t\in\mathbb{R}}$ are two translating solutions which are almost calibrated and exact. If after blow-down they produce the same self-expander, do they have to differ only by a rigid motion?

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2. Definitions and basic identities

Let J and ω denote, respectively, the standard complex structure on $\mathbb{C}^2$ and the standard symplectic form on $\mathbb{C}^2$. We consider also the closed complex-valued 2-form given by

$$\Omega \equiv dz_1 \wedge dz_2$$

where $z_j = x_j + iy_j$ are complex coordinates of $\mathbb{C}^2$, and the Liouville form

$$\lambda = \sum_{j=1}^{2} x_j dy_j - y_j dx_j.$$

We denote by $\mathcal{H}^1$ and $\mathcal{H}^2$ the one dimensional and two dimensional Hausdorff measures in $\mathbb{C}^2$ respectively.

A smooth 2-dimensional submanifold L in $\mathbb{C}^2$ is said to be *Lagrangian* if $\omega_L = 0$ and this implies that

$$\Omega_L = e^{i\theta}\mathrm{vol}_L,$$

where vol_L denotes the volume form of L and θ is a multivalued function called the *Lagrangian angle.* When the Lagrangian angle is a single valued function the Lagrangian is called *zero-Maslov class* and if

$$\cos\theta \geq \varepsilon_0$$

for some positive ε_0, then L is said to be *almost-calibrated.* Furthermore, if $\theta \equiv \theta_0$, then L is calibrated by

$$\mathrm{Re}\left(e^{-i\theta_0}\Omega\right)$$

and hence area-minimizing. In this case, L is referred as being *Special Lagrangian.* The Lagrangian L is said to be *exact* if the Liouville form is an exact form on L.

Finally, the relation between the Lagrangian angle and the mean curvature is given by

$$H = J\nabla\theta.$$

Given a point x_0 in $\mathbb{C}^2$ and a time T, the backwards heat kernel is defined as

$$\Phi_{x_0,T}(x,t) = \frac{\exp\left(-\frac{|x-x_0|^2}{4(T-t)}\right)}{4\pi(T-t)}.$$

When x_0 is the origin and $T=0$, we denote it by Φ. When it is clear from the context at which instant t we are evaluating $\Phi_{x_0,T}(x,t)$, we denote it simply by $\Phi_{x_0,T}$. Moreover, $\mathbf{x}$ and $\mathbf{x}^\perp$ stand for the position vector associated with the point x in $\mathbb{C}^2$ and its projection on the normal space of T_xL respectively.

Throughout this paper, $(L_t)_{-\infty<t<\infty}$ will be a translating solution to Lagrangian mean curvature flow in $\mathbb{C}^2$, where L satisfies hypothesis (H). We denote by $|e_1^\perp|$ the projection of $e_1 = (1,0,0,0)$ on the normal space of L. The intrinsic ball of radius r around a point x in L is defined by $\widehat{B}_r(x)$ and we fix $r_0 < 1$ to be such that $\widehat{B}_{r_0}(x)$ is simply connected for every x in L.

Proposition 2.1. *The following equations hold on L.*

(i) *There is a constant θ_0 such that*

$$\theta(x) = -\langle Je_1, \mathbf{x}\rangle + \theta_0 \qquad \text{and} \qquad |H| = |e_1^\perp|;$$

(ii)
$$\Delta x_1 = |H|^2;$$

(iii)
$$\Delta\theta + \langle \nabla\theta, e_1\rangle = 0.$$

Proof. Because $(L_t)_{-\infty<t<\infty}$ is a translating solution to mean curvature flow we have that
$$H = e_1^{\perp}$$
and thus
$$\nabla\theta = -Je_1^{\perp} = -(Je_1)^{\top}.$$
This implies the first property. The second one follows from
$$\Delta x_1 = \langle H, e_1\rangle = |H|^2$$
and the third property is a consequence of
$$\Delta\theta = -\mathrm{div}(Je_1)^{\top} = -\langle H, Je_1\rangle = -\langle \nabla\theta, e_1\rangle.$$
$\square$

Given a sequence $(\lambda_i)_{i\in\mathbb{N}}$ converging to zero, we define the sequence of blow-downs
$$L^i_s := \lambda_i L_{s/\lambda_i^2} \quad \text{where} \quad -\infty < s < \infty.$$
Finally, we we use the notation
$$\{\theta = \alpha\} := \{x \in L \,|\, \theta(x) = \alpha\}$$
and
$$\{|\theta - \alpha| \leq \delta\} := \{x \in L \,|\, |\theta(x) - \alpha| \leq \delta\}.$$

3. BLOW-DOWN THEOREM

Let $(\Sigma_t)_{-\infty<t<\infty}$ be an eternal solution to Lagrangian mean curvature flow for which we can find a constant D such that, for every t,
$$\begin{cases} \mathcal{H}^2(\Sigma_t \cap B_R(0)) \leq DR^2 \quad \text{for all } R > 0, \\ \cos(\theta_t) \geq D^{-1}. \end{cases}$$
Given a sequence $(\lambda_i)_{i\in\mathbb{N}}$ converging to zero, consider the blow-downs
$$\Sigma^i_s := \lambda_i \Sigma_{s/\lambda_i^2} \quad \text{for} \quad -\infty < s < \infty.$$

Theorem 3.1. *There exist a finite set*
$$\{\bar\theta_1, \ldots, \bar\theta_N\}$$
and Lagrangian planes
$$P_1, \ldots, P_N$$
such that, after passing to a subsequence, we have for every smooth function ϕ compactly supported, every f in $C^2(\mathbb{R})$, and every $s \leq 0$
$$\lim_{i\to\infty} \int_{\Sigma^i_s} f(\theta_{i,s})\phi\, d\mu = \sum_{j=1}^{N} m_j f(\bar\theta_j)\mu_j(\phi), \tag{3}$$

where μ_j *and* m_j *denote the Radon measure of the support of* P_j *and its multiplicity respectively. The set*

$$\{\bar{\theta}_1, \ldots, \bar{\theta}_N\}$$

does not depend on the sequence of rescales chosen.

If Σ_0 *is exact, there exists an integer rectifiable* 2*-varifold* Σ_1^{∞} *satisfying*

$$H = \frac{\mathbf{x}^{\perp}}{2}$$

such that, after passing to a subsequence, Σ_s^i *converges as Radon measures to* $\sqrt{s}\Sigma_1^{\infty}$ *for every* $s > 0$.

If Σ_1^{∞} *is stationary, then identity* (3) *holds for* every s *and hence* (Σ_t) *is static.*

Remark 3.2.
- (i) Whenever identity (3) holds we say that Σ_s^i converges weakly to Σ_0^{∞}.
- (ii) In Section 1 we saw that the last property of the theorem does not hold when L is the grim-reaper in $\mathbb{C}^2$. Hence, we see that the almost-calibrated condition is necessary in order for the convergence to hold when $s = 0$.

Proof. The first property was essentially proven in [5, Theorem A]. The ideas apply with no modification.

From the compactness for integral Brakke motions [2, Section 7.1] we know that, after passing to a subsequence, $(\Sigma_s^i)_{-\infty<s<\infty}$ converges to an integral Brakke motion $(\Sigma_s^{\infty})_{-\infty<s<\infty}$. Moreover, Federer and Fleming compactness for integral currents implies that Σ_0^i converges to a current Σ. The fact that Σ is almost-calibrated implies that the support of Σ equals the support of Σ_0^{∞} because, for every $\phi \geq 0$,

$$\int_{\Sigma_0^{\infty}} \phi \, d\mu \leq D \lim_{i\to\infty} \int_{\Sigma_0^i} \phi \cos\theta \, d\mu = D \int_{\Sigma} \phi \operatorname{Re} \Omega.$$

From the monotonicity formula [1] we have that, for every x_0 in $\mathbb{C}^2$, every positive T, and every $s < 0$

$$\int_{\Sigma_0^i} \Phi_{x_0,T}(\cdot, 0) \, d\mu \leq \int_{\Sigma_s^i} \Phi_{x_0,T}(\cdot, s) \, d\mu$$

and thus

$$\int_{\Sigma_0^{\infty}} \Phi_{x_0,T}(\cdot, 0) \, d\mu \leq \sum_{j=1}^{N} m_j \int_{P_j} \Phi_{x_0,T}(\cdot, s) \, d\mu.$$

Denote the density function of Σ_0^{∞} and Σ_{-1}^{∞} by $\Theta_0(x)$ and $\Theta(x)$ respectively. Make T go to zero so that the left-hand side converges to $\Theta_0(x_0)$ for almost all x_0. After that, make s converge to zero to obtain $\Theta(x_0) \geq \Theta_0(x_0)$ and so the support of Σ is contained on the support of $\Sigma^{\infty}-1$. Because $\partial\Sigma = 0$, the Constancy Theorem [7, Theorem 26.27] implies that the support of Σ coincides with $\Sigma^{\infty}-1$. We now argue that the density functions also coincide.

We know that
$$\frac{d\cos\theta_t}{dt} = \Delta\cos\theta_t + \cos\theta_t|H|^2$$
and thus, for every $T > 0$, x_0 in $\mathbb{C}^2$, and $s < 0$, we have from the monotonicity formula that

$$\begin{aligned}
&\frac{d}{ds}\int_{\Sigma^i_s}\cos\theta_{i,s}\Phi_{x_0,T}\,d\mu\\
&\qquad = \int_{\Sigma^i_s}\cos\theta_{i,s}\left(|H|^2 - \left|H + \frac{(\mathbf{x}-\mathbf{x}_0)^\perp}{2(T-s)}\right|^2\right)\Phi_{x_0,T}\,d\mu\\
&\qquad\qquad \geq -\int_{\Sigma^i_s}\cos\theta_{i,s}\left(\delta|H|^2 + C\left|\frac{(\mathbf{x}-\mathbf{x}_0)^\perp}{2(T-s)}\right|^2\right)\Phi_{x_0,T}\,d\mu
\end{aligned}$$

where $C = C(\delta)$. The evolution equation for $\theta_{i,s}$ implies that for all $s < 0$

$$\begin{aligned}
\lim_{i\to\infty}\int_s^0\int_{\Sigma^i_t}2|H|^2\Phi_{x_0,T}d\mu\,dt &\leq \int_{\Sigma^i_s}\theta^s_{i,s}\Phi_{x_0,T}d\mu\\
&= \sum_{j=1}^N m_j\bar\theta_j^2\int_{P_j}\Phi_{x_0,T}(\cdot,s)\,d\mu \leq \sum_{j=1}^N m_j\bar\theta_j^2 := B
\end{aligned}$$

and it is not hard to see that
$$\lim_{i\to\infty}\int_s^0\int_{\Sigma^i_t}\left|\frac{(\mathbf{x}-\mathbf{x}_0)^\perp}{2(T-t)}\right|^2\Phi_{x_0,T}d\mu\,dt = 0.$$

Thus

$$\begin{aligned}
\int_\Sigma\Phi_{x_0,T}(\cdot,0)\operatorname{Re}\Omega &= \lim_{i\to\infty}\int_{\Sigma^i_0}\cos\theta_{i,0}\Phi_{x_0,T}\,d\mu\\
&\geq -\delta B + \lim_{i\to\infty}\int_{\Sigma^i_s}\cos(\theta_{i,s})\Phi_{x_0,T}\,d\mu\\
&= -\delta B + \sum_{j=1}^N m_j\cos\bar\theta_j\int_{P_j}\Phi_{x_0,T}(\cdot,s)\,d\mu.
\end{aligned}$$

Making δ, T, and s converge to zero, and using the almost-calibrated condition, we obtain that for almost all x_0
$$\Theta_0(x_0) \geq \Theta(x_0)$$
and so Σ coincides with Σ^∞_{-1}. As a result, we have that for every T positive
$$\lim_{i\to\infty}\int_{\Sigma^i_{-1}}\Phi_{0,T}(\cdot,-1)d\mu = \lim_{i\to\infty}\int_{\Sigma^i_0}\Phi_{0,T}(\cdot,0)d\mu$$
and so the monotonicity formula implies that
$$\lim_{i\to\infty}\int_{-1}^0\int_{\Sigma^s_i}(|H|^2 + |\mathbf{x}^\perp|^2)\exp(-|x|^2)d\mu ds = 0.$$

We can then argue as in [5, Theorem A] and conclude that identity (3) holds for $s = 0$.

Recall that Σ_0^i converges as Radon measures and as currents to a union of planes with possible multiplicities. Because the Grassmanian of 2-planes in $\mathbb{R}^4$ can be parametrized by the self-dual and anti self-dual two forms of $\mathbb{R}^4$, we have that

$$\lim_{i\to\infty}\int_{\Sigma_0^i\cap B_R(0)}|\mathbf{x}^\perp|^2 d\mu = 0$$

for every positive R. Moreover, the fact that Σ_0 is exact implies the existence of β_s^i defined on Σ_s^i for which

$$d\beta_s^i = \lambda \quad \text{and} \quad |\nabla \beta_s^i| = |\mathbf{x}^\perp|.$$

Hence, because Σ_0^i is connected, we can apply Proposition B.1 and conclude the existence of $\bar{\beta}$ such that, after passing to a subsequence,

$$\lim_{i\to\infty}\int_{\Sigma_0^i\cap B_R(0)}(\beta_0^i - \bar{\beta})^2 d\mu = 0.$$

According to [5, Section 6], we have that

$$\frac{d}{ds}(\beta_s^i + 2s\theta_s^i)^2 = \Delta(\beta_s^i + 2s\theta_s^i)^2 - 2|\mathbf{x}^\perp - 2sH|^2$$

and so we can apply Huisken's monotonicity formula to conclude that, for every positive T,

$$\lim_{i\to\infty}\int_0^T\int_{\Sigma_s^i} 2|\mathbf{x}^\perp - 2sH|^2\Phi_{0,T}d\mu ds \le \lim_{i\to\infty}\int_{\Sigma_0^i}(\beta_0^i - \bar{\beta})^2\Phi_{0,T}d\mu = 0. \tag{4}$$

We want to show that, for every compactly supported function ϕ in $\mathbb{C}^2$,

$$\frac{1}{s}\int_{\Sigma_s^\infty}\phi\left(x/\sqrt{s}\right)d\mu$$

is constant as a function of s for every $s > 0$.

A standard computation shows that

$$\frac{d}{ds}\left(\frac{1}{s}\int_{\Sigma_s^\infty}\phi\left(x/\sqrt{s}\right)d\mu\right) = -\frac{1}{s^2}\int_{\Sigma_s^\infty}\phi d\mu - \frac{1}{2s^{5/2}}\int_{\Sigma_s^\infty}\langle D\phi,\mathbf{x}\rangle d\mu + \frac{1}{s^{3/2}}\int_{\Sigma_s^\infty}\langle D\phi, H\rangle d\mu - \frac{1}{s}\int_{\Sigma_s^\infty}\phi|H|^2 d\mu.$$

Due to

$$\Delta|x|^2 = 4 + 2\langle \mathbf{x}, H\rangle$$

we obtain that

$$2\int_{\Sigma_s^i}\phi\left(x/\sqrt{s}\right)d\mu = -\int_{\Sigma_s^i}\langle\mathbf{x}, H\rangle\phi d\mu - \frac{1}{\sqrt{s}}\int_{\Sigma_s^i}\langle\mathbf{x}^\top, D\phi\rangle d\mu.$$

Hence,

$$\frac{d}{ds}\left(\frac{1}{s}\int_{\Sigma_s^\infty}\phi\left(x/\sqrt{s}\right)d\mu\right)=\frac{1}{s^{3/2}}\int_{\Sigma_s^\infty}\left\langle D\phi, H-\frac{x^\perp}{2s}\right\rangle d\mu$$
$$+\frac{1}{s}\int_{\Sigma_s^\infty}\phi\left\langle H,\frac{x^\perp}{2s}-H\right\rangle d\mu.$$

For every $0<a<b$ and $R>0$, we have that

$$\int_a^b\int_{\Sigma_s^i}|D\phi|^2 d\mu ds \quad\text{and}\quad \int_a^b\int_{\Sigma_s^i\cap B_R(0)}|H|^2 d\mu ds$$

are uniformly bounded. Therefore, (4) implies that

$$\frac{1}{s}\int_{\Sigma_s^\infty}\phi\left(x/\sqrt{s}\right)d\mu$$

is indeed independent of s for all $s>0$ and this is equivalent to $\Sigma_s^\infty=\sqrt{s}\Sigma_1^\infty$ for all $s>0$.

We now show the last property. Because Σ_1^∞ has $\mathbf{x}^\perp=0$, we obtain that for every $0<a<b$ and every $R>0$

$$\lim_{i\to\infty}\int_a^b\int_{\Sigma_s^i\cap B_R(0)}|\mathbf{x}^\perp|^2 d\mu ds.$$

Hence, identity (4) implies that

$$\lim_{i\to\infty}\int_a^b\int_{\Sigma_s^i}(|H|^2+|\mathbf{x}^\perp|^2)\exp(-|x|^2)d\mu ds=0$$

and so, assuming without loss of generality that

$$\int_{\Sigma_1^i}(|H|^2+|\mathbf{x}^\perp|^2)\exp(-|x|^2)d\mu ds=0,$$

[5, Proposition 5.1] implies that Σ_1^∞ is a union of Lagrangian planes with multiplicities. In order to prove the result, it suffices to show that Σ_1^∞ equals Σ_0^∞.

Huisken's monotonicity formula implies that for every $T>0$ and $0<s<T$

$$\int_{\Sigma_0^\infty}\Phi_{x_0,T}d\mu\le\int_{\Sigma_s^\infty}\Phi_{x_0,T}(\cdot,s)d\mu=\int_{\Sigma_1^\infty}\Phi_{x_0,T}(\cdot,s)d\mu.$$

Thus making s converge to T and then T converge to zero, we obtain that

$$\Theta_0(x_0)\ge\Theta_1(x_0),$$

where Θ_1 denotes the density function of Σ_1^∞. Hence the support of Σ_1^∞ is contained in the support of Σ_0^∞.

Choosing x_0 to be in exactly one P_{j_0}, we can apply Huisken's monotonicity formula to $\cos\theta^i_s$ and conclude that for any $0 < s < T$

$$\int_{\Sigma^\infty_1} \cos(\bar\theta_j)\Phi_{x_0,T}(\cdot, s)\, d\mu = \lim_{i\to\infty} \int_{\Sigma^i_s} \cos(\theta_{i,s})\Phi_{x_0,T}(\cdot, s)\, d\mu$$
$$\geq \lim_{i\to\infty} \int_{\Sigma^i_0} \cos(\theta_{i,s})\Phi_{x_0,T}(\cdot, 0)\, d\mu = \sum_{j=1}^{N} m_j \cos(\bar\theta_j) \int_{P_j} \Phi_{x_0,T}(\cdot, 0)\, d\mu$$
$$\geq \cos(\bar\theta_{j_0}) m_{j_0}.$$

Making s converge to T, the almost-calibrated condition implies that

$$\Theta_0(x_0) \leq \Theta_1(x_0).$$

□

4. GRAPHICAL IMPLIES FLATNESS

We start by defining what it means for a Lagrangian L to have a graphical decomposition. Recall that r_0 was chosen (see Section 2) so that $\widehat{B}_{r_0}(x)$ is simply-connected for all x in L.

Definition 4.1. A Lagrangian L is said to admit a *graphical decomposition* if, outside a compact set, L can be decomposed into N connected components L_j, $j = 1, \ldots, N$, having the following property.

For each $j = 1, \ldots, N$ there are constants $\bar\theta_j$, R_j, S_j, n_j positive integer, and a Lagrangian plane P_j

$$P_j := \{(u, 0, v\cos\bar\theta_j, v\sin\bar\theta_j) \mid (u, v) \in \mathbb{R}^2\}$$

such that

(i) For every x in L_j, $\widehat{B}_{s_0}(x)$ can be written as the graph of a function defined over P_j with its derivatives bounded by S_j;

(ii) The projection of L_j on P_j

$$\mathrm{Proj}_{P_j} \colon L_j \longrightarrow P_j \setminus B_{R_j}(0)$$

is a n_j-fold covering map;

(iii)

$$\lim_{R\to\infty} \sup_{L_j\setminus B_R(0)} |\theta - \bar\theta_j| = 0.$$

In particular,

$$\lim_{R\to\infty} \sup_{L\setminus B_R(0)} |H| = 0.$$

Remark 4.2. In case L is not embedded, property (ii) in Definition 4.1 should be interpreted as follows. If F denotes the immersion of the surface L in $\mathbb{C}^2$, then $\mathrm{Proj}_{P_j} \circ F$ is a n_j-fold covering map.

In this section we show that any translating solution L to Lagrangian mean curvature flow which admits a graphical decomposition is a plane.

Theorem 4.3. *If L is a translating solution to Lagrangian mean curvature flow with uniformly bounded second fundamental form and admitting a graphical decomposition, then L is a plane.*

Proof. It suffices to show that the mean curvature of L is zero because, in that case, we have from Proposition 2.1 that L is a Special Lagrangian contained in some hyperplane and therefore a plane. The idea consists in finding a sequence of compact sets K_n exhausting L for which

$$\lim_{n\to\infty}\oint_{\partial K_n}\langle \nu, e_1\rangle d\sigma = 0,$$

where ν denotes the exterior unit normal to ∂K_n in L. The theorem follows because, due to Proposition 2.1 (ii),

$$\oint_{\partial K_n}\langle \nu, e_1\rangle d\sigma = \int_{K_n}|H|^2 d\mu.$$

The following barrier will be needed to prove the main theorem.

Lemma 4.4. *For every $\alpha < 1/3$, there is a constant $R_0 = R_0(\alpha)$ such that, for every constants δ and B, the function*

$$V_{\delta,B} := B|x|^{-\alpha}\exp(-|x|/2) + \delta\exp(x_1/2)$$

satisfies

$$\Delta V_{\delta,B} \le V_{\delta,B}\frac{|H|^2+1}{4} \quad \textit{for all} \quad |x| \ge R_0.$$

Proof. Denote by ∂_r, $\partial_r^\top$, and $\partial_r^\perp$ the radial vector, its tangential projection on L, and its projection on the normal bundle of L respectively. Set

$$f(|x|) := |x|^{-\alpha}\exp(-|x|/2).$$

Away from the origin, we have

$$\Delta|x| = \langle\partial_r, H\rangle + \frac{2}{|x|} - \frac{|\partial_r^\top|^2}{|x|}$$

and so, because

$$f' = -\alpha\frac{f}{|x|} - \frac{f}{2} \quad \text{and} \quad f'' = \alpha(\alpha+1)\frac{f}{|x|^2} + \alpha\frac{f}{|x|} + \frac{f}{4},$$

we obtain that

$$\Delta f = f\left(\frac{|\partial_r^\top|^2}{4} - \frac{\langle\partial_r, H\rangle}{2}\right) + \frac{f}{|x|}\left(\alpha|\partial_r^\top|^2 - 1 - \alpha\langle\partial_r, H\rangle + \frac{|\partial_r^\top|^2}{2}\right) + \frac{f}{|x|^2}\left(\alpha(\alpha+2)|\partial_r^\top|^2 - 2\alpha\right).$$

Hence

$$\begin{aligned}\Delta f &\leq f\frac{|H|^2+1}{4}+\frac{f}{|x|}\left(\frac{\alpha|H|^2}{2}+\alpha-\frac{1}{2}\right)+\frac{f}{|x|^2}\alpha^2\\ &\leq f\frac{|H|^2+1}{4}+\frac{f}{|x|}\frac{3\alpha-1}{2}+\frac{f}{|x|^2}\alpha^2.\end{aligned}$$

Therefore, we can choose $R_0 = R_0(\alpha)$ so that for all $|x| \geq R_0$ we have

$$\Delta f \leq f\frac{|H|^2+1}{4}.$$

Proposition 2.1 (ii) implies that

$$\Delta \exp(x_1/2) = \exp(x_1/2)\frac{|H|^2+1}{4}$$

and so the proposition follows. □

This proposition implies the following decay for $|\theta-\bar{\theta}_j|$ on each component L_j, $j = 1, \ldots, N$ given by the graphical decomposition.

Lemma 4.5. *For each $\alpha < 1/3$, there are constants B and R_0 such that, on each component L_j, $j = 1, \ldots, N$, we have*

$$|\theta(x)-\bar{\theta}_j| + |\nabla\theta|(x) \leq B|x|^{-\alpha}\exp(-|x|/2 - x_1/2) \quad \text{for all} \quad |x| \geq R_0.$$

Proof. For each $j = 1, \ldots, N$, set

$$u_j := (\theta - \bar{\theta}_j)\exp(x_1/2).$$

From Proposition 2.1 we know that

$$\Delta u_j = u_j\frac{|H|^2+1}{4}$$

and thus, we obtain from Lemma 4.4 that

$$\Delta(V_{\delta,B}-u_j) \leq (V_{\delta,B}-u_j)\frac{|H|^2+1}{4} \quad \text{for all} \quad |x| \geq R_0,$$

where R_0 is chosen large enough so that $\partial L_j \subseteq B_{R_0}(0)$ and the constant B is chosen so that, for every $j = 1, \ldots, N$, we have

$$x \in L_j \cap \partial B_{R_0}(0) \implies B|x|^{-\alpha}\exp(-|x|/2) > |u_j(x)|.$$

Note that, for all R sufficiently large, we have from Definition 4.1 (iii)

$$\sup_{L_j\cap\partial B_R(0)} |u_j \exp(-x_1/2)| \leq \delta$$

and thus

$$\sup_{L_j\cap\partial B_R(0)} (V_{\delta,B}-u_j) > 0 \quad \text{for every} \quad j = 1, \ldots, N.$$

Applying the maximum principle to $L_j\cap(B_R(0)\backslash B_{R_0}(0))$ for all R sufficiently large, we have that

$$u_j(x) \leq B|x|^{-\alpha}\exp(-|x|/2) + \delta\exp(x_1/2) \quad \text{for all} \quad |x| \geq R_0.$$

As a result, after making δ tend to zero, we obtain

$$\theta(x) - \bar{\theta}_j \leq B|x|^{-\alpha} \exp(-|x|/2 - x_1/2) \quad \text{for all} \quad |x| \geq R_0.$$

The correspondent estimate for $|\theta(x) - \bar{\theta}_j|$ follows in the same way by considering the function $V_{\delta,B} + u_j$. The estimate for $|\nabla\theta|$ for is a consequence of Proposition 2.1 (i) and interior Schauder estimates. ☐

The graphical decomposition implies the existence of r_1 so that, for every p in L_j, $j = 1, \ldots, N$ the projection of $\widehat{B}_{r_0}(p)$ on P_j contains $B_{r_1}(\bar{p}) \cap P_j$, where $\bar{p}$ stands for the projection of p on P_j. Thus, after an appropriate change of coordinates, a neighborhood of p can be described as

$$(u, v, \partial_u f, \partial_v f) \quad \text{with} \quad (u, v) \in B_{r_1}(\bar{p})$$

for some function f with $|\mathrm{Hess}\, f|$ uniformly bounded, where the coordinate x_1 equals u and the coordinate y_1 equals $\partial_u f$. Furthermore, a direct computation shows that we can find a constant D depending only on the constants S_j (see Definition 4.1 (i)) for which one of the eigenvalues of Hess f satisfies

$$|\lambda| \leq D|e_1^{\perp}| \quad \text{on} \quad B_{r_1}(\bar{p}).$$

Thus, we obtain from Lemma 4.5 that, provided $|p|$ is large enough, one of the eigenvalues λ has the decay

$$|\lambda| \leq B|x|^{-\alpha} \exp(-|x|/2 - u/2) \tag{5}$$

for some constant B.

An explicit computation shows that the function f satisfies the equation

$$\arctan \lambda_1 + \arctan \lambda_2 = \theta(x) - \bar{\theta}_j,$$

where λ_1 and λ_2 are the eigenvalues of Hess f. As a result, Lemma 4.5 and estimate (5) imply that, provided $|p|$ is large enough,

$$|\mathrm{Hess}\, f| \leq B|x|^{-\alpha} \exp(-|x|/2 - u/2), \tag{6}$$

where $\alpha < 1/3$ and B depends on α.

On each of the connected components L_j, denote by $\gamma_{j,r}$ the lift to L_j of the path on P_j given by

$$c_j(t) := (r\cos t, r\sin t) \qquad 0 \leq t \leq 2n_j\pi,$$

where the variable r will be made as large as we want.

There is $t_0 = t_0(r, r_1)$ such that, for every $t_1 \leq 2n_j\pi$, we can find a function f for which

$$\gamma_{j,r}(t) = (r\cos t, r\sin t, \partial_u f, \partial_v f) \quad \text{for all } t_1 - t_0 \leq t \leq t_1 + t_0,$$

where, due to (6), the function f restricted to this portion of $\gamma_{j,r}$ satisfies

$$|\mathrm{Hess}\, f|(t) \leq Br^{-\alpha} \exp(-r/2 - r\cos t/2).$$

Using an obvious abuse of notation, the tangent vector $\gamma'_{j,r}(t)$ is given by

$$\gamma'_{j,r}(t) = r(\partial_t, (\mathrm{Hess}\, f)(\partial_t)), \quad \text{where} \quad \partial_t := (-\sin t, \cos t)$$

and, denoting by $\bar{\nu}$ the vector in $\mathbb{R}^2$ for which

$$\nu = (\bar{\nu}, (\operatorname{Hess} f)(\bar{\nu})),$$

then

$$\langle \bar{\nu}, \partial_t + (\operatorname{Hess} f)^*(\operatorname{Hess} f)(\partial_t)\rangle = 0 \quad \text{and} \quad 1 = |\bar{\nu}|^2 + |(\operatorname{Hess} f)(\bar{\nu})|^2,$$

where $(\operatorname{Hess} f)^*$ denotes the transpose of $\operatorname{Hess} f$. Thus, provided we choose r sufficiently large, we can find a constant C such that

$$|\bar{\nu} - \partial_r| \le Cr^{-2\alpha} \exp(-r - r\cos t)$$

and

$$|\gamma'_{j,r}(t)| \le r(1 + Cr^{-2\alpha} \exp(-r - r\cos t)).$$

For this reason,

$$\int_{t_1-t_0}^{t_1+t_0} \langle \nu, e_1\rangle |\gamma'_{j,r}(t)|\, dt = \int_{t_1-t_0}^{t_1+t_0} r\cos t\, dt + Q,$$

where we can find a constant C such that

$$|Q| \le \int_{t_1-t_0}^{t_1+t_0} Cr^{1-2\alpha} \exp(-r - r\cos t)\, dt.$$

Therefore, we obtain from unique continuation that

$$\begin{aligned}\left|\oint_{\gamma_{j,r}} \langle \nu, e_1\rangle d\sigma\right| &\le \int_0^{2n_j\pi} Cr^{1-2\alpha} \exp(-r - r\cos t)\, dt \\ &= n_j C r^{1-2\alpha} \int_0^{2\pi} \exp(-r - r\cos t)\, dt.\end{aligned}$$

Choose δ so that

$$|\cos(y) + 1| \le (y - \pi)^2 \quad \text{for all } |y - \pi| \le \delta.$$

Then, we can find a positive constant $D = D(\delta)$ so that, for all r sufficiently large,

$$\begin{aligned}\int_0^{2\pi} \exp(-r - r\cos t)\, dt \le& \int_{\pi-\delta}^{\pi+\delta} \exp(-r(t-\pi)^2)\, dt + 2\pi \exp(-Dr) \\ \le& r^{-1/2} \int_{-\sqrt{r}\delta}^{\sqrt{r}\delta} \exp(-s^2)\, ds + 2\pi \exp(-Dr) \\ \le& Dr^{-1/2}.\end{aligned}$$

Hence, provided we choose $\alpha > 1/4$ in Lemma 4.5, we have that

$$\lim_{r\to\infty} \left|\oint_{\gamma_{j,r}} \langle \nu, e_1\rangle d\sigma\right| = \lim_{r\to\infty} r^{1/2-2\alpha} = 0.$$

Property (i) and (ii) of Definition 4.1 imply that we can find a sequence of compact sets K_n exhausting L and such that

$$\partial K_n = \gamma_{1,r_n} \cup \cdots \cup \gamma_{N,r_n},$$

where $(r_n)_{n\in\mathbb{N}}$ is a sequence converging to infinity. This finishes the proof. □

The next lemma gives conditions under which a translating solution L satisfying (H) admits a graphical decomposition.

Lemma 4.6. *Assume that for almost all* $-C_1 < a < C_1$ *we have*

$$\{\theta = a\} \subset B_{R(a)}$$

for some positive $R(a) > 0$. *Then* L *admits a graphical decomposition.*

Proof. For every α, there is only one Lagrangian plane P_α with Lagrangian angle α and $|e_1^\perp| = 0$ which is given by

$$P_\alpha = \{(u, 0, v\cos\alpha, v\sin\alpha) \mid (u,v) \in \mathbb{R}^2\}. \tag{7}$$

Set ω_α to be the volume form of that plane extended by parallel translation to $\mathbb{C}^2$ and denote its Hodge-dual on L by $*\omega_\alpha$.

We claim that for every ε, there is δ_1 such that, for every x in L and every α, we have

$$\sup_{\widehat{B}_{r_0}(x)} |\theta - \alpha| \leq \delta_1 \quad \Longrightarrow \inf_{\widehat{B}_{r_0}(x)} *\omega_\alpha \geq \varepsilon. \tag{8}$$

A simple continuity argument shows the existence of δ_2 such that if Q is a Lagrangian plane with Lagrangian angle $\theta(Q)$, then for every α

$$|e_1^\perp| \leq \delta_2 \quad \text{and} \quad |\theta(Q) - \alpha| \leq \delta_2 \quad \Longrightarrow \quad \omega_\alpha(Q) \geq \varepsilon,$$

where $\omega_\alpha(Q)$ denotes the evaluation of ω_α on Q. Moreover, as we argue next, we can find $\delta_1 \leq \delta_2$ such that for all x in L

$$\sup_{\widehat{B}_{r_0}(x)} |\theta - \theta(x)| \leq \delta_1/2 \quad \Longrightarrow \quad \sup_{\widehat{B}_{r_0}(x)} |H| \leq \delta_2.$$

If not, we could find a sequence of translating solutions $(L_j)_{j\in\mathbb{N}}$ converging smoothly on compact sets to another translating solution L_∞ with Lagrangian angle constant on $\widehat{B}_{r_0}(0)$ and $|H|$ not identically zero on $\widehat{B}_{r_0}(0)$. This proves the desired claim.

Let $(x_i)_{i\in\mathbb{N}}$ be a sequence in L with $|x_i|$ going to infinity and $\theta(x_i)$ converging to some α. Set $\varepsilon = 1/2$ in identity (8) and choose $\delta < \delta_1/2$ so that

$$\{\theta = \alpha \pm \delta\} \subset B_{R(\alpha\pm\delta)}.$$

Set

$$R := \max\{R(\alpha+\delta), R(\alpha-\delta)\} + 2 \quad \text{and} \quad \Sigma := \{|\theta - \alpha| \leq \delta\}.$$

We have

$$\partial\Sigma \subseteq B_{R-2}(0)$$

and if x belongs to $\Sigma \setminus B_R(0)$, identity (8) implies that

$$\inf_{\widehat{B}_{r_0}(x)} *\omega_\alpha \geq 1/2.$$

Moreover, from [5, page 476], there exists a constant D such that

$$\inf_{x\in L,\, r\leq r_0} r^{-2}\mathcal{H}^2(\widehat{B}_r(x)) \geq D^{-1} \tag{9}$$

and so we can apply Lemma A.1 in order to conclude that, outside a compact set K,

$$\Sigma\setminus K = L_1\cup\cdots\cup L_{N_1}$$

where, for each $j = 1,\ldots,N_1$,

$$\mathrm{Proj}_{P_\alpha}\colon \Sigma_j \longrightarrow P_\alpha\setminus B_R(0)$$

is a n_j-fold covering map and, for every x in $\Sigma\setminus B_R(0)$, $\widehat{B}_{r_0}(x)$ can be written as the graph of a function defined over P_α with its derivatives uniformly bounded.

Take a unbounded sequence (y_i) in L_j such that $\theta(y_i)$ converges to some $\bar\theta_j$ and suppose there is another unbounded sequence p_i in L_j such that $\theta(p_i)$ converges to some β distinct from $\bar\theta_j$. Using the map Proj_{P_α} and the local graphical property we can find, for every a between β and $\bar\theta_j$, an unbounded sequence w_i in L_j such that $\theta(w_i) = a$. This contradicts our hypothesis and proves the third property of Definition 4.1. The first and second property of Definition 4.1 follow because, by choosing $R_j > R$, we can replace α by $\bar\theta_j$ on each L_j.

We can repeat the whole process but this time applied to $L\setminus\Sigma$. We only need to this finitely many times because

$$\lim_{R\to\infty}\mathcal{H}^2(L_j\cap B_R(0))R^{-2}\geq C$$

for some universal constant C. □

5. PROOF OF MAIN THEOREM

We now prove Theorem A. We start by showing

Theorem 5.1. *If L is a translating solution that satisfies (H), is almost-calibrated, and static, then L is a plane.*

Proof. In view of Lemma 4.6 and Theorem 4.3, it suffices to show

Proposition 5.2. *Let L be a translating solutions satisfying hypothesis (H), almost-calibrated, and static. Then for almost all $-\pi/2 < a < \pi/2$ we have*

$$\{\theta = a\}\subset B_{R(a)}$$

for some positive $R(a) > 0$.

Proof. The static condition implies the existence of a sequence of blow-downs

$$L^i_s := \lambda_i L_{s/\lambda_i^2} \quad\text{where}\quad \lim_{i\to\infty}\lambda_i = 0$$

converging weakly for every s to

$$L^\infty = m_1P_1+\cdots+m_NP_N,$$

where $P_1, \dots, P_N$ are Lagrangian planes with multiplicities $m_1, \dots, m_N$ and Lagrangian angles $\bar\theta_1, \dots, \bar\theta_N$.

We claim that

$$P_j := \{(u, 0, v\cos\bar\theta_j, v\sin\bar\theta_j) \,|\, (u, v) \in \mathbb{R}^2\}.$$

The reason is that the coordinate y_1 equals, up to a constant, $-\theta$ (Proposition 2.1 (i)) and so is bounded for every $\mathcal{L}_t$. Thus each plane P_j must have $e_1^\perp = 0$ because, for every $R < 0$ and $s \leq 0$,

$$\lim_{i\to\infty} \sup\{\, y_1 \,|\, x \in L^i_s \cap B_R(0)\} = 0.$$

Hence the almost-calibrated condition implies that P_j is uniquely determined by its Lagrangian angle.

From Theorem 3.1 we know that the set of limiting Lagrangian angles does not depend on the sequence of rescalings chosen and so any other convergent sequence of blow-downs $(\bar L^i_s)$ must converge to L^∞ for every $s \leq 0$. This observation will be used later.

Sard's Theorem implies that, for almost all $-\pi/2 < a < \pi/2$, the set $\{\theta = a\}$ is a smooth submanifold of L (possible empty).We argue that only finitely many curves contained in $\{\theta = a\}$ have finite length. Suppose that $(C_j)_{j\in\mathbb{N}}$ is a sequence of distinct curves contained in $\{\theta = a\}$ having finite length. Hypothesis (H) implies that, for some j_0, we can find integers $b_1, \dots b_{j_0}$ such that

$$b_1[C_1] + \cdots + b_{j_0}[C_{j_0}] = 0,$$

where $[C_j]$ denotes the homology class of C_j in $H_1(L, \mathbb{Z})$. Thus, there is a compact set $K \subseteq L$ such that

$$\partial K = b_1 C_1 + \cdots + b_{j_0} C_{j_0}.$$

From Proposition 2.1 (i) we know that θ cannot have any interior maximum or minimum and therefore, because θ equals a on ∂K, θ must be constant on K. Analytic continuation implies that θ is constant on L and this gives us a contradiction.

Due to Huisken's monotonicity formula and without loss of generality, we can assume that, after passing to a subsequence,

$$\lim_{i\to\infty} \int_{L^i_{-1}} |H|^2 \exp(-|x|^2) d\mu + \lim_{i\to\infty} \int_{L^i_1} |H|^2 \exp(-|x|^2) d\mu = 0.$$

Hence, from the coarea formula

$$\begin{aligned}\int_{-\pi/2}^{\pi/2} &\lambda_i \mathcal{H}^1(\{\theta = a\} \cap B_{\lambda_i^{-1}}(\lambda_i^{-2} e_1))da \\ &= \int_{-\pi/2}^{\pi/2} \mathcal{H}^1\{x \in L^i_{-1} \cap B_1(0) \,|\, \theta^i_{-1} = a\} da = \int_{L^i_{-1}\cap B_1(0)} |H| d\mu \\ &\leq \left(\int_{L^i_{-1}\cap B_1(0)} |H|^2 d\mu \right)^{1/2} \left(\mathcal{H}^2(L^i_{-1} \cap B_1(0)\right)^{1/2}\end{aligned}$$

and this implies that, for almost all $-\pi/2 < a < \pi/2$,

$$\lim_{i\to\infty} \lambda_i \mathcal{H}^1(\{\theta = a\} \cap B_{\lambda_i^{-1}}(\lambda_i^{-2} e_1)) = 0.$$

Likewise, we also obtain

$$\lim_{i\to\infty} \lambda_i \mathcal{H}^1(\{\theta = a\} \cap B_{\lambda_i^{-1}}(-\lambda_i^{-2} e_1)) = 0.$$

Choose a distinct from $\bar{\theta}_1, \cdots, \bar{\theta}_N$, such that

$$\begin{aligned}(10) \quad \lim_{i\to\infty} \lambda_i \mathcal{H}^1 &\left(\{\theta = a\} \cap B_{\lambda_i^{-1}}(\lambda_i^{-2} e_1) \right) \\ &+ \lim_{i\to\infty} \lambda_i \mathcal{H}^1 \left(\{\theta = a\} \cap B_{\lambda_i^{-1}}(-\lambda_i^{-2} e_1) \right) = 0,\end{aligned}$$

and such that $\{\theta = a\}$ is a smooth submanifold of L.

It suffices to show that there is no connected curve C with infinite length contained in $\{\theta = a\}$. If not, we could find an unbounded sequence (x_i) in C

$$x_i = t_i e_1 + u_i, \quad \text{where} \quad |x_i| = \lambda_i^{-2} \quad \text{and} \quad \langle u_i, e_1 \rangle = 0.$$

Lemma 5.3.

$$\liminf_{i\to\infty} |u_i|^2 t_i^{-1} > 0$$

Proof. Suppose for some subsequence

$$\lim i \to \infty |u_i|^2 t_i^{-1} = 0.$$

Then

$$\lim_{i\to\infty} |\lambda_i^{-2} - |t_i|| = \lim_{i\to\infty} |t_i| \left| \sqrt{1 + |u_i|^2 t_i^{-2}} - 1 \right| \leq \lim_{i\to\infty} |t_i|^{-1} |u_i|^2 = 0$$

and

$$\lim_{i\to\infty} \lambda_i^2 |x_i - t_i e_1|^2 = \lim_{i\to\infty} \lambda_i^2 |t_i||u_i|^2 |t_i|^{-1} = \lim_{i\to\infty} \frac{|t_i|}{\sqrt{t_i^2 + |u_i|^2}} |u_i|^2 |t_i|^{-1} = 0.$$

Thus, for every i sufficiently large, x_i belongs to either $B_{\lambda_i^{-1}/2}(\lambda_i^{-2} e_1)$ or $B_{\lambda_i^{-1}/2}(-\lambda_i^{-2} e_1)$ and so

$$\mathcal{H}^1(C \cap B_{\lambda_i^{-1}}(\pm\lambda_i^{-2} e_1)) \geq \pi \lambda_i^{-1}.$$

This contradicts identity (10). □

After passing to a subsequence we can assume that

$$\lim_{i\to\infty}|t_i||u_i|^{-2}=s_1\geq 0.$$

Moreover, we also assume without loss of generality that

$$\bar{L}^i_s:=|u_i|^{-1}L_{s|u_i|^2},\quad\text{where}\quad -\infty<s<\infty$$

converges for all s, the sequence of manifolds $L_i := L - x_i$ converges to a smooth translating solution L_∞ with $\theta_\infty(0) = a$, $t_i|u_i|^{-2}$ converges to $t_0 \geq 0$, and $v_i := u_i|u_i|^{-1}$ converges to a vector v perpendicular to e_1. The comments made at the beginning of this proof imply $\bar{L}^i_s$ converges to L^∞ for every $s \leq 0$.

Suppose $s_1 = 0$. For every β, $r > 0$ and $s < 0$

$$\begin{aligned}\int_{L_\infty}(\theta_\infty-\beta)^2\Phi_{0,r}(\cdot,0)d\mu &= \lim_{i\to\infty}\int_L(\theta-\beta)^2\Phi_{x_i,r}(\cdot,0)d\mu\\
&= \lim_{i\to\infty}\int_{L_{-t_i}}(\theta-\beta)^2\Phi_{u_i,r-t_i}(\cdot,-t_i)\\
&\leq \lim_{i\to\infty}\int_{L_{-t_i+s|u_i|^2}}(\theta-\beta)^2\Phi_{u_i,r-t_i}(\cdot,-t_i+s|u_i|^2)\\
&= \lim_{i\to\infty}\int_{\bar{L}^i_{-t_i|u_i|^{-2}+s}}(\theta-\beta)^2\frac{\exp\left(-\frac{|x-v_i|^2}{4(r|u_i|^{-2}-s)}\right)}{4\pi(r|u_i|^{-2}-s)}d\mu\\
&= \int_{L^\infty_s}(\theta^\infty_{s-t_0}-\beta)^2\frac{\exp\left(\frac{|x-v|^2}{4s}\right)}{-4\pi s}d\mu\\
&= \sum_{j=1}^N n_j(\bar{\theta}_j-\beta)^2\int_{P_j}\frac{\exp\left(\frac{|x-v|^2}{4s}\right)}{-4\pi s}d\mu.\end{aligned}$$

If v did not belong to any P_j, we could make s go to zero to conclude that the leftmost hand-side of the inequalities above is zero for every β, which is impossible. The fact that v is perpendicular to e_1 implies that it can belong at most to one P_{j_0} and thus, making r go to zero and then s go to zero, we see that

$$(a-\beta)^2\leq m_{j_0}(\bar{\theta}_{j_0}-\beta)^2$$

for every β. Therefore, $a = \bar{\theta}_{j_0}$ and so s_1 must be positive.

If $s_1 > 0$, then

$$\bar{L}^i_s=|u_i|^{-1}L_{s|u_i|^2}=\lambda_i^{-1}|u_i|^{-1}L^i_{s|u_i|^2\lambda_i^2}$$

converges weakly to $\sqrt{s_1}L^\infty_{s/s_1} = L^\infty$ for every $-\infty < s < \infty$ because

$$\lim_{i\to\infty}|u_i|^{-2}\lambda_i^{-2}=\lim_{i\to\infty}\sqrt{t_i^2|u_i|^{-4}+|u_i|^{-2}}=s_1.$$

As a result, we can argue as in the $s_1 = 0$ case and get a contradiction with the way a was chosen.

$\square$

$\square$

The proof of Theorem A will be completed after we show

Theorem 5.4. *If L is a translating solution that satisfies hypothesis (H) and has*

$$\int_L |H|^2 d\mu \le C_2$$

for some C_2, then L is a plane.

Proof. In light of Lemma 4.6 and Theorem 4.3, it suffices to show that for almost all $-C_1 < a < C_1$ we have

$$\{\theta = a\} \subset B_{R(a)}$$

for some positive $R(a) > 0$. Moreover, we can argue as in Proposition 5.2 and see that the fact that L has finite first Betti number implies that it is sufficient to show that for almost all $-C_2 \le s \le C_2$ there is no curve contained in $\{\theta = s\}$ having infinite length.

Suppose that such a smooth curve exists and denote it by C. Then, for all r sufficiently large, $C \cap \{|x| = r\}$ is not empty and so

$$\mathcal{H}^1(C \cap \{r \le |x| \le 2r\}) \ge r. \tag{11}$$

On the other hand, using the coarea formula, we have for all r

$$\begin{aligned}\int_{-C_1}^{C_1} \mathcal{H}^1(\{\theta = s\} \cap \{r \le |x| \le 2r\}) ds &= \int_{L\cap\{r\le|x|\le 2r\}} |H| d\mu \\ &\le \left(\int_{L\cap\{r\le|x|\le 2r\}} |H|^2 d\mu\right)^{1/2} \left(\mathcal{H}^2(L \cap \{r \le |x| \le 2r\})\right)^{1/2} \\ &\le \sqrt{C}_1 r \left(\int_{L\cap\{r\le|x|\le 2r\}} |H|^2 d\mu\right)^{1/2}.\end{aligned}$$

Note that

$$\lim_{r\to\infty} \int_{L\cap\{r\le|x|\le 2r\}} |H|^2 d\mu = 0$$

and thus, for almost all $-C_2 \le s \le C_2$, we can find a sequence r_i going to infinity such that

$$\lim_{i\to\infty} r_i^{-1}\mathcal{H}^1(\{\theta = s\} \cap \{r_i \le |x| \le 2r_i\}) = 0.$$

This contradicts (11). $\square$

Appendix A

Let τ be the volume form of a plane P extended by parallel translation to all of $\mathbb{C}^2$. We denote by Proj_P the projection onto the plane P and assume that P contains the line spanned by e_1. Given any surface Σ in $\mathbb{C}^2$, we denote by $*\tau$ the Hodge-dual of τ and if F denotes an immersion of Σ on $\mathbb{C}^2$, we also use Proj_P to represent $\mathrm{Proj}_P \circ F$.

In what follows, Σ will be a complete noncompact surface with smooth boundary such that $\widehat{B}_{r_0}(x)$ is simply-connected for all x in L,

$$\sup_{R>0} R^{-2}\mathcal{H}^2(\Sigma \cap B_R(0)) \leq D \quad \text{and} \quad \inf_{x\in\Sigma,\, r\leq r_0} r^{-2}\mathcal{H}^2(\widehat{B}_r(x)) \geq D^{-1}$$

for some constant D.

Lemma A.1. *Assume that we can find $R > 0$ and $\varepsilon > 0$ such that*

$$\partial\Sigma \subset B_R(0)$$

and

$$\inf_{\widehat{B}_{r_0}(x)} *\tau \geq \varepsilon \quad \textit{for all} \quad x \in \Sigma \setminus B_R(0).$$

Outside a compact set, Σ can be decomposed into k connected components Σ_j, $j = 1, \ldots, N$ having the following property.

(i) *For every x in $\Sigma \setminus B_R(0)$, $\widehat{B}_{r_0}(x)$ can be written as the graph of a function defined over P with its derivatives bounded by a constant depending only on ε;*

(ii)

$$\mathrm{Proj}_P \colon \Sigma_j \longrightarrow P \setminus B_R(0)$$

is a n_j-fold covering map.

Proof. The first property is an immediate consequence of the fact that $\widehat{B}_{r_0}(x)$ is simply-connected. Consider the map

$$\mathrm{Proj}_P \colon \Sigma \setminus B_R(0) \longrightarrow P.$$

The local graphical property combined with the uniform lower bounds on area densities implies that $\mathrm{Proj}_P^{-1}(B_R(0))$ is a compact subset of $\Sigma \setminus B_R(0)$.

Decompose $\mathrm{Proj}_P^{-1}(P \setminus B_R(0))$ into connected components $(\Sigma_j)_{j\in\mathbb{N}}$. The local graphical property implies the existence of an integer n_j so that

$$\mathrm{Proj}_P \colon \Sigma_j \longrightarrow P \setminus B_R(0)$$

is a n_j-fold covering map with

$$\mathrm{Proj}_P(\partial\Sigma_j) \subseteq \partial(P \setminus B_R(0)).$$

Moreover, there is a constant $C = C(\varepsilon)$ such that

$$\lim_{R\to\infty} R^{-2}\mathcal{H}^2(\Sigma_j \cap B_R(0)) \geq C$$

and so there can only exist finitely many connected components Σ_j. $\square$

Appendix B

The next proposition was proven in [5, Appendix A] with slightly different hypothesis. For that reason we will only indicate the modifications in the proof.

Proposition B.1. *Let (N^i) and (α_i) be a sequence of smooth Lagrangian surfaces in $\mathbb{R}^4$ and smooth functions on N^i respectively, such that (N^i) converges as Radon measure and as currents to a union of planes with positive integer multiplicities N. We assume that, for some $R > 0$, the following properties hold:*

a) *There exists a constant D_0 such that*

$$\mathcal{H}^2(N^i \cap B_{3R})) \leq D_0 R^2$$

and

$$\cos\theta^i \geq D_0^{-1}$$

for all $i \in \mathbb{N}$.

b)

$$\lim_{i\to\infty} \int_{N^i \cap B_{3R}(0)} |\nabla \alpha_i|^2 d\mu = 0.$$

c) *There exists a constant D_1 for which*

$$\sup_{N^i \cap B_{3R}(0)} |\nabla \alpha_i| + R^{-1} \sup_{N^i \cap B_3(0)} |\alpha_i| \leq D_1$$

for all $i \in \mathbb{N}$.

d) *For all $i \in \mathbb{N}$,*

$$N^i \cap B_{2R}(0) \quad \textit{is connected}$$

and

$$\partial(N^i \cap B_{3R}(0)) \subset \partial B_{3R}(0).$$

Then, there is a real number α such that, after passing to a subsequence, we have for all ϕ with compact support in $B_R(0)$ and all f in $C(\mathbb{R})$

$$\lim_{i\to\infty} \int_{N^i} f(\alpha_i)\phi d\mu = f(\alpha)\mu_N(\phi),$$

where μ_N denotes the Radon measure associated to N.

Proof. It suffices to find $\alpha \in \mathbb{R}$ and a sequence (ε_j) converging to zero such that, for some appropriate subsequence, we have for all $j \in \mathbb{N}$

$$\lim_{i\to\infty} \mathcal{H}^2(\{|\alpha_i - \alpha| \leq \varepsilon_j\} \cap B_R(0)) = \mathcal{H}^2(N \cap B_R(0)).$$

For the rest of this proof, $K = K(D_0, D_1, k)$ will denote a generic constant depending only on the mentioned quantities. Choose any sequence (x_i) in $N^i \cap B_R(0)$. After passing to a subsequence, we have that

$$\lim_{i\to\infty} x_i = x_0 \quad \text{and} \quad \lim_{i\to\infty} \alpha_i(x_i) = \alpha$$

for some $x_0 \in B_R(0)$ and $\alpha \in \mathbb{R}$. Furthermore, consider also a sequence (ε_j) converging to zero such that, for all $j \in \mathbb{N}$,

$$\lim_{i\to\infty} \mathcal{H}^1\big(\{\alpha_i = \alpha \pm \varepsilon_j\} \cap B_{3R}\big) = 0.$$

Such a subsequence exists because, by the coarea formula, we have

$$\lim_{i\to\infty}\int_{-\infty}^{\infty} \mathcal{H}^1\big(\{\alpha_i = s\} \cap B_{3R}\big)ds = \lim_{i\to\infty}\int_{N^i\cap B_{3R}} |\nabla\alpha_i| d\mu$$
$$\leq \lim_{i\to\infty} KR\left(\int_{N^i\cap B_{3R}} |\nabla\alpha_i|^2 d\mu\right)^{1/2} = 0.$$

Define

$$N^{i,\alpha,j} \equiv \{|\alpha_i - \alpha| \leq \varepsilon_j\}.$$

Standard compactness theorems imply that, after passing to a subsequence, we have convergence to a boundaryless integral current $\bar{N}^{\alpha,j}$ and to a Radon measure $N^{\alpha,j}$, both having their support contained in N. The almost-calibrated condition implies that the support of $N^{\alpha,j}$ coincides with the support of $\bar{N}^{\alpha,j}$ (the argument is the same as the one used in the beginning of the proof of Theorem 3.1). The Constancy Theorem [7, Theorem 26.27] implies that the support of $N^{\alpha,j}$ is a union of planes with multiplicities.

We note that N^i being almost calibrated (see [5, Lemma 7.1]) implies the existence of some constant D such that

$$\big(\mathcal{H}^2(A)\big)^{1/2} \leq D\mathcal{H}^1(\partial A),$$

where A is any open subset of N with rectifiable boundary. The rest of the proof follows exactly like it was done in [5, Proposition A. 1]. □

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