EXISTENCE AND UNIQUENESS OF CONSTANT MEAN CURVATURE FOLIATION OF ASYMPTOTICALLY HYPERBOLIC 3-MANIFOLDS

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Abstract. We prove existence and uniqueness of foliations by stable spheres with constant mean curvature for 3-manifolds which are asymptotic to Anti-de Sitter-Schwarzschild metrics with positive mass. These metrics arise naturally as spacelike timeslices for solutions of the Einstein equation with a negative cosmological constant.

1. Introduction

In 1996, Huisken and Yau [11] showed that metrics that arise as spacelike timeslices of solutions to the Einstein vacuum equation have a well defined center of mass. More precisely, they showed that metrics which are asymptotic to Schwarzschild metrics with positive mass outside a compact set admit, under some technical assumptions, a unique foliation by stable spheres with constant mean curvature. We prove an analogous result for a large class of metrics that arise naturally as spacelike timeslices for solutions to the Einstein equation with negative cosmological constant. More precisely, we show that metrics which are, outside a compact set, asymptotic to Anti–de Sitter–Schwarzschild metrics with positive mass admit, under some technical assumptions, a unique foliation by stable spheres with constant mean curvature and thus, they have a well defined center of mass.

We remark that even if the results proven here are formally analogue to the ones proven in [11], the fact that the manifolds we work with are asymptotically hyperbolic instead of asymptotically flat makes the underlying geometry rather different and hence new arguments are needed. For instance, in a forthcoming work, the first author constructs a solution to inverse mean curvature flow in a asymptotically hyperbolic manifolds which, independently of which coordinate system we choose, does not approach any coordinate spheres. It was proven by Huisken and Ilmanen that this phenomena cannot happen in asymptotically flat manifolds [9, Section 7].

Most of the work in this paper is devoted to prove uniqueness of foliations by stable spheres with constant mean curvature for asymptotically Anti–de Sitter–Schwarzschild metrics with positive mass. Note that the uniqueness of such foliations fails for the hyperbolic 3-space and so the positivity of the mass needs to be used. The central idea in the paper consists in combining the positivity of the mass with the Kazdan-Warner obstructions [12] in order to prove uniqueness. The relation between positive mass and the study of foliations near infinity for asymptotically flat manifolds was observed by Christodoulou and Yau in [6].

The existence of such foliations in the asymptotically flat setting was proven by Huisken and Yau [11] using a modified mean curvature flow and by Ye [18] using a perturbation method. For metrics asymptotic to Anti–de Sitter–Schwarzschild metrics, the existence of such foliations was proven by Rigger [15] using the mean curvature flow approach. The arguments in [18] can be adapted in a straightforward way in order to obtain the existence result for metrics asymptotic to Anti–de Sitter–Schwarzschild metrics. We include its proof in the last section for the sake completeness.

Before stating the main result we need to introduce some notation. Denote by g_0 the standard round metric on S^2. The Anti–de Sitter–Schwarzschild manifold with mass $m > 0$ is defined to be $(s_0, \infty) \times S^2$ with the metric

$$g_m = (1 + s^2 - m/s)^{-1} ds^2 + s^2 g_0,$$

where s_0 is the zero of $1 + s^2 - m/s$. Note that when $m = 0$ the Anti–de Sitter–Schwarzschild metric becomes the hyperbolic metric. After a change of coordinates, the metric can be written in the form

$$g_m = dr^2 + (\sinh^2 r + m/(3 \sinh r) + O(\exp(-3r)))g_0.$$

Our result will apply to metrics which are, outside a compact set, lower order perturbations of Anti–de Sitter–Schwarzschild metrics.

Definition 1.1. (M, g) is an asymptotically Anti–de Sitter–Schwarzschild manifold with mass m if, for some compact set K, $M - K$ is diffeomorphic to $\mathbb{R}^3$ minus a ball and, with respect to this diffeomorphism, the metric can be written (in spherical coordinates) as

$$g = dr^2 + (\sinh^2 r + m/(3 \sinh r))g_0 + Q,$$

where

$$|Q| + |\nabla Q| + |\nabla^2 Q| = O(\exp(-5r)).$$

Remark 1.2. A direct computation shows that the mass m coincides with the mass defined in [7] and [17] for asymptotically hyperbolic metrics.

With respect to the coordinates specified in the Definition 1.1, $M - K$ becomes equipped with a radial function r. Hence, given a foliation $(\Sigma_t)_{t \geq 0}$ of M, we can define the lower radius and the upper radius to be

$$\underline{r}_t = \sup\{r(x) \mid x \in \Sigma_t\} \quad \text{and} \quad \overline{r}_t = \inf\{r(x) \mid x \in \Sigma_t\}$$

respectively. Given a family of functions f_t defined on Σ_t, we use

$$f_t = O(\exp(-n\underline{r}_t)) \quad \text{and} \quad f_t = o(\exp(-n\underline{r}_t))$$

to denote that

$$\limsup_{t \to \infty}(|f_t| \exp(n\underline{r}_t)) < \infty \quad \text{and} \quad \limsup_{t \to \infty}(|f_t| \exp(n\underline{r}_t)) = 0$$

respectively.

A surface Σ with constant mean curvature is said to be stable if volume preserving variations do not decrease its area. A standard computation shows that stability is equivalent to the second variation operator

$$Lu \equiv -\Delta f - \left(|A|^2 + R(\nu,\nu)\right) f$$

having only nonnegative eigenvalues when restricted to functions with zero mean value, i.e.,

$$\int_{\Sigma_t} \left(|A|^2 + R(\nu,\nu)\right) f^2 d\mu \leq \int_{\Sigma_t} |\nabla f|^2 d\mu$$

for all functions f with $\int_{\Sigma_t} f d\mu = 0$.

The purpose of this paper is to show the following result.

Theorem 1.3. *Asymptotically Anti–de Sitter–Schwarzschild manifolds with positive mass admit a unique foliation* $(\Sigma_t)_{t\geq 0}$ *by stable spheres with constant mean curvature such that*

$$\lim_{t\to\infty} (\overline{r}_t - 6/5\underline{r}_t) = -\infty. \tag{1}$$

Remark 1.4. A condition similar to (1) was also assumed in [11] for metrics that are asymptotic Schwarzschild metrics. It is an interesting question if one can strengthen Theorem 1.3 by weakening condition (1) to the condition that $\underline{r}_t$ is sufficiently large. Its analogous version for metrics asymptotic to Schwarzschild metrics was solved in [14].

A contradiction argument implies the following corollary.

Corollary 1.5. *There are positive constants* C_0 *and* R_0 *depending only on* g *so that any stable sphere* Σ *with constant mean curvature satisfying*

$$|\Sigma| \geq C_0 \quad \textit{and} \quad \overline{r} - 6/5\underline{r} \leq -R_0$$

is unique.

Proof. Assume there are two distinct sequences $(\Sigma_i^1)_{i\in\mathbb{N}}$ and $(\Sigma_i^2)_{i\in\mathbb{N}}$ of stable spheres with constant mean curvature such that

$$\begin{aligned} &\lim_{i\to\infty} |\Sigma_i^1| = \lim_{i\to\infty} |\Sigma_i^2| = \infty, \\ &\lim_{i\to\infty} \overline{r}_i^1 - 6/5\underline{r}_i^1 = \lim_{i\to\infty} \overline{r}_i^2 - 6/5\underline{r}_i^2 = -\infty, \end{aligned}$$

and $H(\Sigma_i^1) = H(\Sigma_i^2)$ for every integer i. The proof of Theorem 1.3 also applies to the foliations

$$\Sigma_t^j \equiv \Sigma_i^j \quad \text{if } i \leq t < i+1,$$

where $j = 1, 2$, and so we obtain a contradiction. □

Acknowledgement The first author would like to express his gratitude to Alice Chang for many useful discussions.

2. Outline of the proof

We outline the proof of Theorem 1.3 in order to emphasize the main ideas over the technical aspects of the paper. Let $(\Sigma_t)_{t\geq 0}$ denote a foliation by stable spheres with constant mean curvature. For the sake of simplicity, we assume in this discussion that $\overline{r}_t - \underline{r}_t$ is uniformly bounded. We use the notation

$$\fint_{\Sigma_t} f\, d\mu \equiv |\Sigma_t|^{-1} \int_{\Sigma_t} f\, d\mu,$$

where $|\Sigma_t|$ stands for the surface area of Σ_t.

Section 3 is devoted to auxiliary computations. In Section 4 we follow the same argumentation done by Huisken and Yau in [11] and use the stability assumption in order to derive the following estimate for the mean curvature of Σ_t (Lemma 4.1)

$$H^2 = 4 + 16\pi/|\Sigma_t| + \fint_{\Sigma_t} O(\exp(-3r))d\mu \tag{2}$$

and the following integral estimate for the trace free part of the second fundamental form (Proposition 4.3)

$$\fint_{\Sigma_t} |\mathring{A}|^2 d\mu \leq O(\exp(-4\underline{r}_t)).$$

In Section 5 we study the intrinsic geometry of Σ_t. More precisely, we show that after pulling back by a suitable diffeomorphism from Σ_t to S^2, the metric

$$\hat{g}_t \equiv 4\pi/|\Sigma_t| g$$

can be written as

$$\exp(2\beta_t)g_0,$$

where g_0 denotes the standard round metric on S^2 and

$$\beta_t = O(\exp(-\underline{r}_t)).$$

This result implies that $\hat{g}_t$ is very "close" to being a round metric. The proof of this result (Theorem 5.1) has two steps. The first step consists in deriving a pointwise estimate for $|\mathring{A}|$ from the integral estimate (Proposition 5.3). In order to do so, we have to exploit the fact that the hyperbolic metric is conformal to the Euclidean metric on the unit ball and so the same is true, up to a term of low order, for the metric g. Therefore, denoting by $d\sigma$ the surface measure induced by the Euclidean metric on Σ_t, we have by conformal invariance that

$$\lim_{t\to\infty} \int_{\Sigma_t} \overline{|\mathring{A}|}^2 d\sigma = \lim_{t\to\infty} \int_{\Sigma_t} |\mathring{A}|^2\, d\mu = 0,$$

where the quantities measured with respect to the Euclidean metric are denoted with a bar. Because the Euclidean area of Σ_t converges to 4π (Proposition 4.2), the identity above and Gauss-Bonnet Theorem imply that the Euclidean mean curvature $\bar{H}$ has no concentration points. We can then use Michael-Simon Sobolev inequality and the equation satisfied by $|\mathring{A}|^2$ (with respect to the Euclidean metric) in order to apply the standard Moser iteration procedure and conclude that, up to lower order terms,

$$\sup_{\Sigma_t}|\mathring{A}|^2 \le C\int_{\Sigma_t}|\mathring{A}|^2\,d\sigma = C\fint_{\Sigma_t}|\mathring{A}|^2 d\mu \le O(\exp(-4\underline{r}_t)). \tag{3}$$

The second step consists in using Gauss equation (Lemma 3.2) which, combined with estimates (2) and (3), implies that the Gaussian curvature of Σ_t with respect to $\hat{g}_t$ satisfies

$$\hat{K}_t = (4\pi)^{-1}|\Sigma_t|((H^2-4)/4 - |\mathring{A}|^2/2) + O(\exp(-r)) = 1 + O(\exp(-r)).$$

The desired result follows from this estimate.

Section 6 contains the main estimate that makes uniqueness possible. Set

$$w_t(x) \equiv r(x) - \hat{r}_t, \quad \text{where} \quad |\Sigma_t| = 4\pi\sinh^2\hat{r}_t.$$

We want to show that w_t converges uniformly to zero (Theorem 6.1). With respect to the standard round metric g_0, the functions w_t satisfy the equation (see (14))

$$\Delta_0 w_t = \exp(-2w_t) - 1 + P,$$

where

$$\int_{S^2}|P|d\mu_0 = O(\exp(-r)).$$

Because we are assuming that $\overline{r}_t - \underline{r}_t$ is uniformly bounded, we have that w_t is bounded in $W^{1,2}$ with respect to g_0 and thus, we can take a sequence converging weakly to w_0 that satisfies

$$\Delta_0 w_0 = \exp(-2w_0) - 1. \tag{4}$$

It is well known that this equation can have many solutions. Therefore, we need to use the fact that the mass is nonzero in order to show that $w_0 = 0$. This is achieved through the Kazdan-Warner identity [12]. Because the metric $\hat{g}_t$ is “close” to being the round metric, this identity implies that, for each of the standard coordinate functions x_1, x_2, x_3 on S^2 (see (16)),

$$\int_{S^2} x_i\hat{K}_t d\mu_0 = O(\exp(-2\underline{r}_t)), \quad \text{for } i = 1, 2, 3.$$

On the other hand, a careful expansion of the terms involved in the Gauss equation shows that the Gaussian curvature of $\hat{g}_t$ is such that

$$|\Sigma_t|^{1/2}\,\hat{K}_t = (4\pi)^{-1}\,|\Sigma_t|^{3/2}\,(H^2-4)/4 \\ + m(4\pi)^{-1}\,|\Sigma_t|^{3/2}\,/\sinh^3 r + O(\exp(-r)).$$

Hence, because the mean curvature is constant, we obtain from the Kazdan-Warner identity that (Proposition 6.3)

$$m\int_{S^2} x_i\exp(-3w_t)d\mu_0 = O(\exp(-\underline{r}_t)), \quad \text{for } i = 1, 2, 3.$$

Therefore,

$$m\int_{S^2} x_i \exp(-3w_0)d\mu_0 = 0, \qquad \text{for } i = 1,2,3.$$

Recalling that w_0 is a solution to equation (4), the above identity implies that $w_0 = 0$. Standard techniques can then be used to show that w_t converges to zero uniformly.

In Section 7 we improve the rate of convergence of w_t to zero (Theorem 7.1). In order to do so, we redefine w_t to be

$$w_t(x) \equiv r(x) - \tilde{r}_t,$$

where $\tilde{r}_t$ is such that

$$H = 2\cosh\tilde{r}_t/\sinh\tilde{r}_t - m/\sinh^3\tilde{r}_t + o(\exp(-4\tilde{r}_t)).$$

Then, the equation satisfied by w_t improves to become

$$\Delta_0 w_t = \exp(-2w_t) - 1 + o(\exp(-2r)).$$

Using the orthogonality condition given by the Kazdan-Warner obstructions, standard elliptic estimates show that

$$|w_t|_{C^{2,\alpha}} = o(\exp(-\underline{r}_t)).$$

In Section 8.1, we use the strong approximation of Σ_t to a coordinate sphere in order to prove uniqueness of foliations by stable spheres with constant mean curvature. The main reason for this to work is that, with respect to the round metric on S^2, the linearization of the mean curvature on a coordinate sphere $\{|x| = r\}$ is the operator

$$L(f) = \Delta_0 f + (2 - 3m/\sinh r)f,$$

which is invertible is m is not zero. Finally, in Section 8.2 we adapt the arguments used in [18] and we show that, for all r sufficiently large, we can find a stable sphere with constant mean curvature which is a perturbation of $\{|x| = r\}$.

3. PRELIMINARIES

In this section we compute the relevant formulas needed throughout this paper. Before doing so, we need to introduce some notation. Σ is assumed to be a stable sphere with constant mean curvature. The radial vector is denoted by ∂_r and $\partial_r^\top$ stands for the tangential projection of ∂_r on $T\Sigma$, which has length denoted by $|\partial_r^\top|$. Finally, ν denotes the exterior unit normal to Σ.

We start by computing the asymptotic expansion of some geometric quantities depending on g_m, the Anti de Sitter- Schwarzschild metric with mass m. Let $\{e_1, e_2\}$ denote a g_m-orthonormal basis for the coordinate spheres $\{|x| = r\}$.

Lemma 3.1.

(i) *The mean curvature $H_m(r)$ of $\{|x| = r\}$ is such that*

$$H_m(r) = 2\cosh r/\sinh r - m/\sinh^3 r + O(\exp(-5r)).$$

(ii) *The scalar curvature is -6.*

(iii) *The Ricci curvature is such that*

$$R_m(\partial_r, \partial_r) = -2 - m/\sinh^3 r + O(\exp(-5r))$$
$$R_m(e_1, e_1) = R_m(e_2, e_2) = -2 + m/(2\sinh^3 r) + O(\exp(-5r))$$
$$R_m(e_1, e_2) = 0.$$

(iv) *The derivatives of the Ricci curvature with respect g_m are such that*

$$\nabla^m_{e_1} R_m(\partial_r, e_1) = \nabla^m_{e_2} R_m(\partial_r, e_2).$$

Proof. If we write g_m as $dr^2 + \psi^2(r)g_0$, then

$$H_m(r) = 2\psi'(r)/\psi(r)$$

and so the first formula follows. The second formula is just direct computation. The rotational symmetry of the metric implies that its Gaussian curvature is $\psi^{-2}(r)$ and that the second fundamental form of $\{|x| = r\}$ is trace free. Hence, we have from Gauss equation that

$$R_m(\partial_r, \partial_r) = -3 - \psi^{-2}(r) + H^2/4$$

and so the first identity in (iii) follows. The other two identities in (iii) are a consequence of (ii) and rotational symmetry. The last identity follows from the same type of arguments. □

The next lemma relates the mass m of a metric g with the Gaussian curvature K of a surface Σ.

Lemma 3.2. *The Gaussian curvature of Σ satisfies*

$$K = (H^2 - 4)/4 + m/\sinh^3 r - 3m|\partial_r^\top|^2/(2\sinh^3 r) - |\mathring{A}|^2/2 + O(\exp(-5r)).$$

Proof. Because g is C^2-perturbation of order $O(\exp(5r))$ of g_m, we have

$$R = -6 + O(\exp(-5r))$$

and

$$\begin{aligned} 2 + R(\nu, \nu) &= 2 + R_m(\nu, \nu) + O(\exp(-5r)) \\ &= -m/\sinh^3 r + 3m|\partial_r^\top|^2/(2\sinh^3 r) + O(\exp(-5r)). \end{aligned}$$

The result follows from Gauss equation

$$K = R/2 - R(\nu, \nu) + H^2/4 - |\mathring{A}|^2/2.$$

□

Next, we derive the equation satisfied by the trace-free part of the second fundamental form $\mathring{A}$.

Lemma 3.3. *The Laplacian of* $|\mathring{A}|^2$ *satisfies*

$$\begin{aligned}\Delta(|\mathring{A}|^2/2) = {}& \left(\frac{H^2-4}{2} - |\mathring{A}|^2\right)|\mathring{A}|^2 + |\nabla\mathring{A}|^2 \\ &+ \left(|\mathring{A}|^2 + |H||\partial_r^\top|^2|\mathring{A}|\right)O(\exp(-3r)) + |\mathring{A}|O(\exp(-5r)).\end{aligned}$$

Proof. We assume normal coordinates $x = \{x^i\}_{i=1,2}$ around a point p in the constant mean curvature surface Σ. The tangent vectors are denoted by $\{\partial_1, \partial_2\}$, the normal vector by ν, and the Einstein summation convention for the sum of repeated indices is used.

Simons' identity for the Laplacian of the second fundamental form A (see for instance [10]) implies that

$$\begin{aligned}\Delta(|\mathring{A}|^2/2) = {}& |\nabla\mathring{A}|^2 + H\mathrm{Tr}(\mathring{A}^3) + H^2|\mathring{A}|^2/2 - |\mathring{A}|^4 \\ &+ H\mathring{A}_{ij}R_{\nu i\nu j} - R_{\nu\nu}|\mathring{A}|^2 + 2R_{kikm}\mathring{A}_{mj}\mathring{A}_{ij} \\ &+ 2R_{kijm}\mathring{A}_{km}\mathring{A}_{ij} + \mathring{A}_{ij}(\nabla_k R_{\nu jik} + \nabla_i R_{\nu kjk}).\end{aligned}$$

From

$$R_{stuv} = -(\delta_{su}\delta_{tv} - \delta_{sv}\delta_{tu}) + O(\exp(-3r))$$

if follows that

$$2R_{kikm}\mathring{A}_{mj}\mathring{A}_{ij} + 2R_{kijm}\mathring{A}_{km}\mathring{A}_{ij} = -4|\mathring{A}|^2 + |\mathring{A}|^2 O(\exp(-3r)).$$

Hence, $\mathrm{Tr}(\mathring{A}^3) = 0$ implies that

$$\begin{aligned}\Delta(|\mathring{A}|^2/2) = {}& |\nabla\mathring{A}|^2 + \left(\frac{H^2-4}{2} - |\mathring{A}|^2\right)|\mathring{A}|^2 + HR_{\nu i\nu j}\mathring{A}_{ij} \\ &+ (\nabla_k R_{\nu jik} + \nabla_i R_{\nu kjk})\mathring{A}_{ij} + |\mathring{A}|^2 O(\exp(-3r)).\end{aligned}$$

On the other hand, if $\{v_1, v_2\}$ is an eigenbasis for $\mathring{A}$, we obtain from Lemma 3.1 that

$$\begin{aligned}R_{\nu i\nu j}\mathring{A}_{ij} &= \frac{|\mathring{A}|}{\sqrt{2}}(R(v_1,v_1) - R(v_2,v_2)) \\ &= \frac{|\mathring{A}|}{\sqrt{2}}(R_m(v_1,v_1) - R_m(v_2,v_2)) + |\mathring{A}|O(\exp(-5r)) \\ &= |\partial_r^\top|^2|\mathring{A}|O(\exp(-3r)) + |\mathring{A}|O(\exp(-5r))\end{aligned}$$

and

$$\begin{aligned}(\nabla_k R_{\nu jik} + \nabla_i R_{\nu kjk})\mathring{A}_{ij} &= \sqrt{2}|\mathring{A}|(\nabla_{v_1}R(\nu, v_1) - \nabla_{v_2}R(\nu, v_2)) \\ &= |\partial_r^\top|^2|\mathring{A}|O(\exp(-3r)) + |\mathring{A}|O(\exp(-5r)).\end{aligned}$$

Thus, the desired result follows. □

Finally, we derive the equation satisfied by the Laplacian of r on Σ. Throughout the rest of this paper, we will progressively estimate and explore all of its terms.

Proposition 3.4. *The Laplacian of* r *on* Σ *satisfies*

$$\Delta r = (4-2|\partial_r^\top|^2)\exp(-2r)+2-H \\ +(H-2)(1-\langle\partial_r,\nu\rangle)+(1-\langle\partial_r,\nu\rangle)^2+O(\exp(-3r))$$

or, being more detailed,

$$\Delta r = 2\cosh r/\sinh r - m/\sinh^3 r - H \\ +(H-2)(1-\langle\partial_r,\nu\rangle)+(1-\langle\partial_r,\nu\rangle)^2-2|\partial_r^\top|^2\exp(-2r) \\ +|\partial_r^\top|^2 O(\exp(-3r))+O(\exp(-5r)).$$

Proof. It suffices to prove the second identity. Using again normal coordinates $\{x_1,x_2\}$, we have from Lemma 3.1

$$\begin{aligned}\mathrm{div}_\Sigma\partial_r &= \langle\nabla^m_i\partial_r,\partial_i\rangle+O(\exp(-5r))\\ &= H^m(r)-H^m(r)|\partial_r^\top|^2/2+O(\exp(-5r))\\ &= (2-|\partial_r^\top|^2)\cosh r/\sinh r - m/\sinh^3 r\\ &\quad +|\partial_r^\top|^2 O(\exp(-3r))+O(\exp(-5r)).\end{aligned}$$

On the other hand,

$$\mathrm{div}_\Sigma\partial_r = \Delta r+\langle\partial_r,\nu\rangle H$$

and the result follows from the easily checked identity

$$|\partial_r^\top|^2\cosh r/\sinh r+\langle\partial_r,\nu\rangle H = H-(H-2)(1-\langle\partial_r,\nu\rangle) \\ -(1-\langle\partial_r,\nu\rangle)^2+2|\partial_r^\top|^2\exp(-2r)+|\partial_r^\top|^2 O(\exp(-4r)).$$

□

4. INTEGRAL ESTIMATES

We use the stability condition in the same spirit as in [11] in order to estimate the mean curvature H and the L^2 norm of $|\mathring{A}|$. In the next section, we combine these integral estimates with Lemma 3.3 in order to obtain pointwise estimates for $|\mathring{A}|$. From this section on,

$$(\Sigma_t)_{t\geq 0}$$

denotes a foliation by stable spheres with constant mean curvature satisfying condition (1). We omit the index t in the notation whenever it becomes obvious that we are referring to quantities depending on Σ_t.

Lemma 4.1. *For each* t*, we have that*

$$H^2 = 4+16\pi/|\Sigma_t|+\fint_{\Sigma_t} O(\exp(-3r))d\mu$$

or, equivalently,

$$H = 2+4\pi/|\Sigma_t|+\fint_{\Sigma_t} O(\exp(-3r))d\mu.$$

Proof. The stability condition implies that, after a clever choice of test function (see [11, Proposition 5.3]),

$$8\pi \geq \int_{\Sigma_t} |A|^2 + R(\nu,\nu) d\mu = \int_{\Sigma_t} |\mathring{A}|^2 + \left(H^2 - 4\right)/2 + R(\nu,\nu) + 2\, d\mu$$

and so, because

$$R(\nu,\nu) = -2 + O(\exp(-3r)),$$

we have

$$H^2 \leq 4 + 16\pi/|\Sigma_t| + \fint_{\Sigma_t} O(\exp(-3r))d\mu.$$

On the other hand, from Lemma 3.2 and Gauss-Bonnet Theorem we obtain that

$$H^2 = 4 + 16\pi/\left|\Sigma\right| + 2\fint_{\Sigma_t} |\mathring{A}|^2 d\mu + \fint_{\Sigma_t} O(\exp(-3r))d\mu,$$

and thus the result follows. ☐

This lemma combined with the equation for Δr gives us these first estimates regarding Σ_t.

Proposition 4.2. *The following identities hold:*

(i)

$$\int_{\Sigma_t} \exp\left(-2r\right)d\mu = \pi + O(\exp(-\underline{r}_t)).$$

In particular, there is a constant C so that

$$C^{-1}\exp(2\underline{r}_t) \leq |\Sigma_t| \leq C\exp(2\overline{r}_t).$$

(ii)

$$\int_{\Sigma_t} (1 - \langle \partial_r, \nu\rangle)^2 d\mu = O(\exp(-\underline{r}_t)).$$

(iii) *For every positive integer j*

$$\int_{\Sigma_t} j|\partial_r^\top|^2 \exp(-jr)d\mu = O(\exp(-j\underline{r}_t)).$$

Proof. Integrating the first identity in Proposition 3.4 and using Lemma 4.1 we obtain

$$\begin{aligned} 4\pi + \int_{\Sigma_t} O(\exp(-3r))d\mu &= \int_{\Sigma_t} (4 - 2|\partial_r^\top|^2)\exp(-2r)d\mu \\ &\quad + \fint_{\Sigma_t} 4\pi(1 - \langle\partial_r,\nu\rangle)d\mu + \int_{\Sigma_t} (1 - \langle\partial_r,\nu\rangle)^2 d\mu. \end{aligned} \tag{5}$$

Hence, both quantities in (i) and (ii) are uniformly bounded because

$$\int_{\Sigma_t} O(\exp(-3r))d\mu = O(\exp(2\overline{r}_t - 3\underline{r}_t)).$$

Integrating the identity

$$\Delta(\exp(-jr)/j) = -\exp(-jr)\Delta r + j\exp(-jr)|\partial_r^\top|^2$$

we obtain, after using Lemma 4.1 and Proposition 3.4,

$$\begin{aligned}\int_{\Sigma_t} j|\partial_r^\top|^2\exp(-jr)d\mu &= \int_{\Sigma_t}(4-2|\partial_r^\top|^2)\exp(-(j+2)r)d\mu\\ &\quad -4\pi\fint_{\Sigma_t}\exp(-jr)d\mu + \fint_{\Sigma_t}4\pi(1-\langle\partial_r,\nu\rangle)\exp(-jr)d\mu\\ &\quad +\int_{\Sigma_t}(1-\langle\partial_r,\nu\rangle)^2\exp(-jr)d\mu + \int_{\Sigma_t}O(\exp(-(3+j)r))d\mu\\ &\qquad + \fint_{\Sigma_t}O(\exp(-3r))d\mu\int_{\Sigma_t}O(\exp(-3r))d\mu.\end{aligned}$$

As a result, the third identity follows. Next, we use (iii) and the Euclidean isoperimetric identity to prove both (i) and (ii).

With respect to the the unit ball model for the hyperbolic metric, the metric g can be written as

$$g = \psi^2\left(dx^2+dy^2+dz^2\right) + O(\psi^{-3}),$$

where

$$\psi(x) \equiv \frac{2}{1-|x|^2} \quad\text{and}\quad \sinh r = \frac{2|x|}{1-|x|^2}.$$

If $\underline{s}_t$ denotes the radius of the largest Euclidean ball centered at the origin that is contained in the interior of Σ_t, then $\sinh\underline{r}_t = 2\underline{s}_t/(1-\underline{s}_t{}^2)$. The Euclidean isoperimetric inequality implies that

$$\text{area}(\Sigma_t) \geq (36\pi)^{1/3}\text{volume}(\{|x| = \underline{s}_t\})^{2/3} = 4\pi\underline{s}_t^2,$$

where the area and volume are measured with respect to the Euclidean metric. Thus, denoting by $d\sigma$ the surface measure induced by the Euclidean metric, we have

$$\begin{aligned}(6)\quad \int_{\Sigma_t}4\exp(-2r)d\mu &\geq \int_{\Sigma_t}\sinh^{-2}r d\mu - \int_{\Sigma_t}\exp(-2r)\sinh^{-2}r d\mu\\ &= \int_{\Sigma_t}|x|^{-2}d\sigma + O(\exp(-2\underline{r}_t)) \geq 4\pi - 4\pi(1-\underline{s}^2) + O(\exp(-2\underline{r}_t))\\ &= 4\pi + O(\exp(-\underline{r}_t)).\end{aligned}$$

On the other hand,

$$\fint_{\Sigma_t}(1-\langle\partial_r,\nu\rangle)d\mu \leq |\Sigma_t|^{-1/2}\left(\int_{\Sigma_t}(1-\langle\partial_r,\nu\rangle)^2d\mu\right)^{1/2} = O(\exp(-r_t))$$

and, recalling that both quantities in (i) and (ii) are uniformly bounded, we obtain from (5) that

$$4\pi + O(\exp(-\underline{r}_t)) = \int_{\Sigma_t}4\exp(-2r)d\mu + \int_{\Sigma_t}(1-\langle\partial_r,\nu\rangle)^2d\mu.$$

Therefore, (i) and (ii) follow from this estimate combined with (6). □

Next, we use the stability of Σ_t in the same way as in [11, Section 5] in order to obtain integral estimates for $\mathring{A}$.

Proposition 4.3. *The following estimate holds*

$$\fint_{\Sigma_t}|\mathring{A}|^2d\mu+\int_{\Sigma_t}|\mathring{A}|^4d\mu+\int_{\Sigma_t}|\nabla\mathring{A}|^2d\mu\le\int_{\Sigma_t}|\partial_r^\top|^2O(\exp(-4r))d\mu\\ +\int_{\Sigma_t}O(\exp(-7r))d\mu.$$

In particular,

$$\fint_{\Sigma_t}|\mathring{A}|^2d\mu\le O(\exp(-4\underline{r}_t)).$$

Proof. Integrating the identity in Lemma 3.3 and using Lemma 4.1 we obtain

$$8\pi\fint_{\Sigma_t}|\mathring{A}|^2d\mu+\int_{\Sigma_t}|\nabla\mathring{A}|^2d\mu=\int_{\Sigma_t}\left(|\mathring{A}|^2+|\mathring{A}||\partial_r^\top|^2\right)O(\exp(-3r))d\mu\\ +\int_{\Sigma_t}|\mathring{A}|^4d\mu+\int_{\Sigma_t}|\mathring{A}|^2d\mu\fint_{\Sigma_t}O(\exp(-3r))d\mu+\int_{\Sigma_t}|\mathring{A}|O(\exp(-5r))d\mu.$$

We now argue that, for every fixed $\varepsilon>0$,

$$(8\pi+o(1))\fint_{\Sigma_t}|\mathring{A}|^2d\mu+\int_{\Sigma_t}|\nabla\mathring{A}|^2d\mu=(1+\varepsilon/2)\int_{\Sigma_t}|\mathring{A}|^4d\mu\\ +\int_{\Sigma_t}|\partial_r^\top|^2O(\exp(-4r))d\mu+\int_{\Sigma_t}O(\exp(-7r))d\mu. \tag{7}$$

This is true because

$$\int_{\Sigma_t}|\mathring{A}|O(\exp(-5r))d\mu\le\int_{\Sigma_t}|\mathring{A}|^2O(\exp(-3\underline{r}_t))d\mu+\int_{\Sigma_t}O(\exp(-7r))d\mu,$$

$$|\mathring{A}||\partial_r^\top|^2O(\exp(-3r))\le\varepsilon/2|\mathring{A}|^4+|\partial_r^\top|^2O(\exp(-4r)),$$

and, due to Proposition 4.2,

$$\lim_{t\to\infty}\fint_{\Sigma_t}O(\exp(-3r))d\mu=0.$$

Before we use the stability of Σ_t let us first remark that, according to Lemma 3.1 and Lemma 4.1, we have for all t sufficiently large

$$|A|^2+R(\nu,\nu)\ge|\mathring{A}|^2+2+R(\nu,\nu)\ge|\mathring{A}|^2+O(\exp(-3r)).$$

Hence, the stability assumption implies that

$$\int_{\Sigma_t}|\mathring{A}|^2f^2d\mu\le\int_{\Sigma_t}|\nabla f|^2d\mu+\int_{\Sigma_t}f^2O(\exp(-3r))d\mu$$

for all functions f with $\int_{\Sigma_t} f d\mu = 0$. Using the test function (see [11, Section 5])

$$f = u - \fint_{\Sigma_t} u d\mu \equiv u - \bar{u}, \quad \text{with} \quad u = |\mathring{A}|,$$

we obtain

$$\int_{\Sigma_t} |\mathring{A}|^4 d\mu \leq \int_{\Sigma_t} |\nabla|\mathring{A}||^2 d\mu + 2\bar{u}\int_{\Sigma_t} |\mathring{A}|^3 d\mu \\ + \int_{\Sigma_t} (u-\bar{u})^2 O(\exp(-3r)) d\mu.$$

Looking at the proof of Lemma 4.1 we see that

$$\int_{\Sigma_t} |\mathring{A}|^2 d\mu = \int_{\Sigma_t} O(\exp(-3r)) d\mu = o(1)$$

and so, because

$$\bar{u}^2 \leq \fint_{\Sigma_t} |\mathring{A}|^2 d\mu,$$

we obtain for every $\varepsilon > 0$ fixed

$$\int_{\Sigma_t} |\mathring{A}|^4 d\mu \leq \varepsilon \int_{\Sigma_t} |\mathring{A}|^4 d\mu + o(1) \fint_{\Sigma_t} |\mathring{A}|^2 d\mu \\ + \int_{\Sigma_t} |\nabla|\mathring{A}||^2 d\mu + \int_{\Sigma_t} (u-\bar{u})^2 O(\exp(-3r)) d\mu. \tag{8}$$

We now estimate the last two terms in this inequality. If $\{e_1, e_2\}$ denotes an eigenbasis for $\mathring{A}$, one can easily check that

$$|\nabla|\mathring{A}||^2 = 2|\nabla\mathring{A}(e_1,e_1)|^2, \quad \nabla\mathring{A}(e_1,e_1) = -\nabla\mathring{A}(e_2,e_2)$$

and

$$|\nabla\mathring{A}|^2 = |\nabla|\mathring{A}||^2 + 2|\nabla\mathring{A}(e_1,e_2)|^2.$$

Using Codazzi equations and Lemma 3.1 we have that, for $i, j = 1, 2$,

$$\begin{aligned} \nabla_i\mathring{A}(e_i,e_j) &= \nabla_j\mathring{A}(e_i,e_i) + R(\nu, e_j) \\ &= \nabla_j\mathring{A}(e_i,e_i) + |\partial_r^\top| O(\exp(-3r)) + O(\exp(-5r)) \end{aligned}$$

and this implies that, for every $\varepsilon > 0$,

$$|\nabla_i\mathring{A}(e_i,e_j)|^2 \geq (1-\varepsilon/2)|\nabla_j\mathring{A}(e_i,e_i)|^2 \\ + |\partial_r^\top|^2 O(\exp(-6r)) + O(\exp(-10r)).$$

Therefore, we can estimate

$$2|\nabla\mathring{A}(e_1,e_2)|^2 \geq (1-\varepsilon)|\nabla|\mathring{A}||^2 + |\partial_r^\top|^2 O(\exp(-6r)) + O(\exp(-10r))$$

and obtain that

$$\begin{aligned}\int_{\Sigma_t}|\nabla\mathring{A}|^2d\mu \geq (2-\varepsilon)\int_{\Sigma_t}&|\nabla|\mathring{A}||^2d\mu\\ &+\int_{\Sigma_t}|\partial_r^\top|^2O(\exp(-6r))d\mu+\int_{\Sigma_t}O(\exp(-10r))d\mu.\end{aligned}$$

To handle the last term we remark that

$$\int_{\Sigma_t}(u-\bar{u})^2O(\exp(-3r))d\mu \leq o(1)⨍_{\Sigma_t}|\mathring{A}|^2d\mu$$

and hence, we can rewrite equation (8) as

$$\begin{aligned}(1-\varepsilon)\int_{\Sigma_t}|\mathring{A}|^4d\mu \leq 1/(2-\varepsilon)\int_{\Sigma_t}&|\nabla\mathring{A}|^2d\mu+o(1)⨍_{\Sigma_t}|\mathring{A}|^2d\mu\\ &+\int_{\Sigma_t}|\partial_r^\top|^2O(\exp(-6r))d\mu+\int_{\Sigma_t}O(\exp(-10r))d\mu.\end{aligned}$$

Multiplying this inequality by $(1+\varepsilon)/(1-\varepsilon)$ (with ε small) and adding to equation (7) we obtain

$$\begin{aligned}⨍_{\Sigma_t}|\mathring{A}|^2d\mu+\int_{\Sigma_t}|\mathring{A}|^4d\mu+\int_{\Sigma_t}|\nabla\mathring{A}|^2d\mu \leq \int_{\Sigma_t}&|\partial_r^\top|^2O(\exp(-4r))d\mu\\ &+\int_{\Sigma_t}O(\exp(-7r))d\mu.\end{aligned}$$

□

5. INTRINSIC GEOMETRY

We study the intrinsic geometry of $(\Sigma_t)_{t\geq 0}$, the foliation of stable spheres with constant mean curvature satisfying condition (1). More precisely, we show

Theorem 5.1. *After pulling back by a suitable diffeomorphism from Σ_t to S^2, the metric*

$$\hat{g}_t \equiv 4\pi\,|\Sigma_t|^{-1}\,g_t$$

can be written as

$$\exp(2\beta_t)g_0$$

with

$$\sup|\beta_t| = O(\exp(2\overline{r}_t-3\underline{r}_t)) \quad \textit{and} \quad \int_{S^2}|\nabla\beta_t|^2d\mu_0 = O(\exp(4\overline{r}_t-6\underline{r}_t)),$$

where the norms are computed with respect to g_0, the standard round metric on S^2.

We need to show that the Gaussian curvature $\widehat{K}_t$ of Σ_t (computed with respect to $\hat{g}_t$) converges to one with order $O(\exp(2\overline{r}_t-3\underline{r}_t))$. In order to do so, we know from Gauss equation (see Lemma 3.2) that we need to estimate $|\mathring{A}|$.

Lemma 5.2. $|\mathring{A}|^2$ *is uniformly bounded.*

Proof. Suppose there is a sequence t_i going to infinity and a sequence of points x_i in each Σ_{t_i} (denoted simply by Σ_i) such that

$$\sup_{\Sigma_i}|\mathring{A}| = |\mathring{A}|(x_i) \equiv 1/\sigma_i \quad \text{and} \quad \lim_{i\to\infty}|\mathring{A}|(x_i) = \infty.$$

Consider the sequence of ambient metrics $g_i = \sigma_i^{-2}g$ and denote the various geometric quantities with respect to g_i using an index i. Because $H_i = \sigma_i H$ converges to zero (see Lemma 4.1), we have that for all i sufficiently large

$$|A|_i^2 = H_i^2/2 + |\mathring{A}|_i^2 = \sigma_i^2(H^2/2 + |\mathring{A}|^2) \leq 2.$$

Thus, there exists a universal constant r_0 for which $B^i_{r_0}(x_i)\cap\Sigma_i$ is the graph over $T_{x_i}\Sigma_i$ of a function with gradient bounded by one (see, for instance, [5])). As a result, there is a uniform constant C such that, for all $s \leq r_0$,

$$C^{-1}s^2 \leq \int_{B^i_s(x_i)\cap\Sigma_i} d\mu_i \leq Cs^2.$$

Furthermore, the generalization of Michael-Simon Sobolev inequality proven in [8] states the existence of some other universal constant C such that, for every compactly supported function u,

$$\left(\int_{\Sigma_i} u^2\, d\mu_i\right)^{1/2} \leq C\left(\int_{\Sigma_i}|\nabla u|_i\, d\mu_i + \int_{\Sigma_i} H_i|u|\, d\mu_i\right).$$

Hence, because H_i converges to zero, we have that for all i sufficiently large and every compactly supported function u

$$\left(\int_{B^i_{r_0}(x_i)\cap\Sigma_i} u^2\, d\mu_i\right)^{1/2} \leq 2C\int_{B^i_{r_0}(x_i)\cap\Sigma_i}|\nabla u|_i\, d\mu_i.$$

Finally, because $|\mathring{A}|_i$ is uniformly bounded, it follows easily from Lemma 3.3 that

$$\Delta_i|\mathring{A}|_i^2 \geq -3|\mathring{A}|_i^2 + O(\exp(-3r)).$$

We have now all the necessary conditions to apply Moser's iteration argument (see, for instance, [13, Lemma 11.1.]) and obtain that, for some constant C,

$$\begin{aligned} 1 = |\mathring{A}|_i^2(x_i) &\leq C\int_{\Sigma_i}|\mathring{A}|_i^2 d\mu_i + O(\exp(-3\underline{r}_t)) \\ &= C\int_{\Sigma_{t_i}}|\mathring{A}|^2 d\mu + O(\exp(-3\underline{r}_t)). \end{aligned}$$

The last expression converges to zero by Proposition 4.3 and this gives us a contradiction. □

Next, we improve the estimate on $|\mathring{A}|^2$. The idea is to exploit the fact that the hyperbolic metric is conformal to the Euclidean metric on the unit ball. Therefore, each Σ_t inherits another induced metric. The proof consists in showing that, for this new induced metric, we can apply Moser's iteration argument for $|\mathring{A}|^2$ and thus bound the supremum by its L^2 norm (computed with respect to the Euclidean metric).

Proposition 5.3. *The following estimate holds*

$$\sup_{\Sigma_t}|\mathring{A}|^2 \leq O(\exp(2\overline{r}_t - 2\underline{r}_t))\int_{\Sigma_t}|\partial_r^\top|^2 O(\exp(-4r))d\mu + O(\exp(2\overline{r}_t - 7\underline{r}_t)).$$

In particular,

$$\sup_{\Sigma_t}|\mathring{A}|^2 \leq O(\exp(2\overline{r}_t - 6\underline{r}_t)).$$

Proof. Recall that, with respect to the the unit model for the hyperbolic metric, the metric g can be written as

$$g = \psi^2\left(dx^2 + dy^2 + dz^2\right) + O(\psi^{-3})$$

where

$$\psi(x) = \frac{2}{1-|x|^2} \quad \text{and} \quad \sinh r = \frac{2|x|}{1-|x|^2}.$$

The surfaces Σ_t converge pointwise to the sphere of radius one and, according to Proposition, 4.2 (i)

$$\lim_{t\to\infty}\int_{\Sigma_t}\exp\left(-2r\right)d\mu = \lim_{t\to\infty}\int_{\Sigma_t}d\sigma = \pi,$$

where $d\sigma$ denotes the surface measure induced by the Euclidean metric. The trace free part of the second fundamental form and the mean curvature with respect to the Euclidean metric are denoted by $\overline{|\mathring{A}|}$ and $\overline{H}$ respectively. Next, we argue that we have the necessary conditions to apply Moser's iteration on Σ_t with respect to the Euclidean metric.

We start with the remark that

$$\lim_{t\to\infty}\int_{\Sigma_t}\overline{|\mathring{A}|}^2 d\sigma = 0.$$

The reason is that, by conformal invariance of

$$\int_{\Sigma_t}|\mathring{A}|^2 d\mu,$$

the above identity is true if we substitute the Euclidean metric by $\delta \equiv \psi^{-2}g$. Nevertheless, this metric is a lower order perturbation of the Euclidean metric and so, using the formulas derived in [9, Section 7] (more precisely identity (7.10)) the remark follows. Therefore, Gauss-Bonnet Theorem implies that

$$\lim_{t\to\infty}\int_{\Sigma_t}\overline{H}^2 d\sigma = 16\pi.$$

We use this to argue that the mean curvature has no concentration points.

Lemma 5.4. *For every* $\varepsilon_0 > 0$ *there is a positive* r_0 *such that, for all* t *sufficiently large and all* x *in* Σ_t,

$$\int_{\Sigma_t \cap \overline{B}_{r_0}(x)} \overline{H}^2 d\sigma \leq \varepsilon_0.$$

Proof. Denote by $\mathcal{H}^2$ the 2-dimensional Hausdorff measure and consider the sequence of Radon measures given by

$$\eta_i(A) \equiv \int_{\Sigma_t \cap A} \overline{H}^2 d\sigma,$$

where A is any $\mathcal{H}^2$-measurable set. From Allard's compactness Theorem we can extract a sequence Σ_i that converges to the unit sphere in the varifold sense and, moreover, we can also assume that η_i converges to a Radon measure $\bar{\eta}$ supported on the unit sphere. Lower semicontinuity implies that, for every $\mathcal{H}^2$-measurable set A,

$$\bar{\eta}(A) \geq 4\mathcal{H}^2(A \cap \{|x| = 1\}). \tag{9}$$

On the other hand,

$$4\mathcal{H}^2(\{|x| = 1\}) = 16\pi = \lim_{i \to \infty} \int_{\Sigma_i} \overline{H}^2 d\sigma = \bar{\eta}(\{|x| = 1\}).$$

Therefore, an equality in (9) must hold for every $\mathcal{H}^2$-measurable set A and this implies the desired result. □

An immediate consequence of this lemma combined with the generalization of Michael-Simon Sobolev inequality proven in [8] is the existence of universal constants C and r_0 such that, for every Euclidean ball $\bar{B}_{r_0}$ centered at a point in $\{|x| = 1\}$ and for all t sufficiently large, we have both that

$$\left(\int_{\Sigma_t \cap \bar{B}_{r_0}} u^2 \, d\sigma \right)^{1/2} \leq C \int_{\Sigma_t \cap \bar{B}_{r_0}} |\overline{\nabla} u| \, d\sigma$$

for every compactly supported function u and that

$$\sqrt{\mathcal{H}^2(A)} \leq K \mathcal{H}^1(\partial A)$$

for every open subset A of $\Sigma_t \cap \bar{B}_{r_0}$ with rectifiable boundary. A standard argument implies the existence of a universal constant C for which

$$C^{-1} s^2 \leq \mathcal{H}^2\left(\widehat{B}_s(x)\right) \leq C s^2 \quad \text{for all} \quad s \leq r_0,$$

where x is in Σ_t and $\widehat{B}_s(x)$ denotes the intrinsic ball of radius s.

With respect to the Euclidean Laplacian, the equation for $|\mathring{A}|^2$ becomes (see Lemma 3.3)

$$\overline{\Delta} |\mathring{A}|^2 \geq -4\psi^2 |\mathring{A}|^2 |\mathring{A}|^2 + O(1) |\mathring{A}|^2 + |\partial_r^\top|^2 |\mathring{A}| O(\exp(-r)) \\ + O(\exp(-6r)).$$

The zeroth order terms are bounded and so, in order to apply Moser's iteration argument, we need to check that $\psi^2|\mathring{A}|^2$ is in L^p for some $p > 1$. Because $|\mathring{A}|$ is bounded, we have from Proposition 4.3

$$\begin{aligned}\int_{\Sigma_t} \psi^{2+\varepsilon}|\mathring{A}|^{2+\varepsilon} d\sigma &= \int_{\Sigma_t} \psi^{\varepsilon}|\mathring{A}|^{2+\varepsilon} d\mu + \int_{\Sigma_t} |\mathring{A}|^{2+\varepsilon} O(\exp(-(3-\varepsilon)r))d\mu \\ &= O(\exp((2+\varepsilon)\overline{r}_t - 4\underline{r}_t)),\end{aligned}$$

which is bounded for all ε sufficiently small. Hence, setting

$$\alpha_t \equiv \sup_{\Sigma_t} |\mathring{A}|,$$

Moser's iteration (see, for instance, [13, Lemma 11.1]) implies the existence of some constant C depending on ε small for which

$$\begin{aligned}\alpha_t^2 \leq C \int_{\Sigma_t} |\mathring{A}|^2 d\sigma + \alpha_t \left(\int_{\Sigma_t} |\partial_r^{\top}|^{2+2\varepsilon} O(\exp(-(1+\varepsilon)r)) d\sigma \right)^{1/(1+\varepsilon)} \\ + O(\exp(-6\underline{r}_t)).\end{aligned}$$

Therefore, Proposition 4.2 (iii) implies that

$$\alpha_t^2 \leq O(\exp(-2\underline{r}_t)) \int_{\Sigma_t} |\mathring{A}|^2 d\mu + O(\exp(-2(3+\varepsilon)/(1+\varepsilon)\underline{r}_t)).$$

The result follows from applying Proposition 4.3 and choosing ε appropriately small.

□

We can now prove Theorem 5.1.

Proof of Theorem 5.1. Lemma 3.2 and Lemma 4.1 imply that the Gaussian curvature of Σ_t with respect to $\hat{g}_t$ satisfies

$$\widehat{K}_t = 1 - |\Sigma_t|\,|\mathring{A}|^2/(8\pi) + O(\exp(2\underline{r}_t - 3\underline{r}_t)).$$

As a result, we obtain from Proposition 5.3 that

$$\widehat{K}_t = 1 + O(\exp(4\overline{r}_t - 6\underline{r}_t)) + O(\exp(2\overline{r}_t - 3\underline{r}_t)) = 1 + O(\exp(2\overline{r}_t - 3\underline{r}_t)).$$

Hence, because we are assuming condition (1), $\widehat{K}_t$ converges uniformly to one and this implies that, after pulling back by a diffeomorphism, the metric $\hat{g}_t$ can be written as $\exp(2\beta_t)g_0$ where, according to [4, Lemma 3.7], β_t converges uniformly to zero and, denoting the coordinate function on S^2 by x_1, x_2, and x_3,

$$\int_{S^2} x_j \exp(2\beta_t) d\mu_0 = 0 \quad \text{for } j = 1, 2, 3. \tag{10}$$

Using the smallness of β_t for t sufficiently large, we have that β_t satisfies the equation

$$\begin{aligned}\Delta_0\beta_t &= 1-\widehat{K}_t\exp(2\beta_t)\\ &= 1-\exp(2\beta_t)+O(\exp(2\overline{r}_t-3\underline{r}_t))\exp(2\beta_t)\\ &= -2\beta_t+f(\beta_t)+O(\exp(2\overline{r}_t-3\underline{r}_t))\exp(2\beta_t),\end{aligned}$$

where

$$f(s)\equiv 1+2s-\exp(2s)$$

is such that $|f(\beta_t)|=O(\beta_t^2)$. Therefore,

$$\begin{aligned}\bar{\beta}_t\equiv \fint_{S^2}\beta_t d\mu_0 &= \fint_{S^2} f(\beta_t)/2 d\mu_0+O(\exp(2\overline{r}_t-3\underline{r}_t))\\ &= O(|\beta_t|_2^2)+O(\exp(2\overline{r}_t-3\underline{r}_t))\end{aligned}\tag{11}$$

and, combining integration by parts with Cauchy's inequality,

$$\int_{S^2}|\nabla\beta_t|^2 d\mu_0\leq (2+o(1)+1/2)\int_{S^2}\beta_t^2 d\mu_0+O(\exp(4\overline{r}_t-6\underline{r}_t)).\tag{12}$$

On the other hand, we know that the L^2 norm of the projection of β_t on the kernel of Δ_0+2 has order $O(|\beta_t|_2^2)$ because, from (10), we have that for $j=1,2,3$,

$$2\int_{S^2}x_j\beta_t d\mu_0=\int_{S^2}x_j f(\beta_t)d\mu_0=O(|\beta_t|_2^2).$$

Hence, because β_t converges uniformly to zero,

$$\int_{S^2}|\nabla\beta_t|^2 d\mu_0\geq(6+o(1))\int_{S^2}(\beta-\bar{\beta}_t)^2 d\mu_0.$$

Combining this with (11) and (12) we obtain

$$\int_{S^2}\beta_t^2 d\mu_0=O(\exp(4\overline{r}_t-6\underline{r}_t))$$

and so, by (12), the estimate on the L^2-norm of the gradient follows. Finally, a simple computation shows that for some constant C

$$\Delta_0\beta_t^2\geq -C\beta_t^2-O(\exp(4\overline{r}_t-6\underline{r}_t)).$$

Thus, from Moser's iteration, we obtain that for another constant C

$$\sup_{\Sigma_t}\beta_t^2\leq C\int_{S^2}\beta_t^2 d\mu_0+O(\exp(4\overline{r}_t-6\underline{r}_t))=O(\exp(4\overline{r}_t-6\underline{r}_t)).$$

□

6. UNIQUE APPROXIMATION TO COORDINATE SPHERES

The purpose of this section is to show that a foliation by stable spheres $(\Sigma_t)_{t\geq 0}$ with constant mean curvature must approach the coordinate spheres $\{|x| = r\}$ when t goes to infinity. Such result is obviously false in hyperbolic space because we can always apply an isometry to our foliation that changes its center. Hence, we need to use the fact that the ambient manifold is an asymptotically hyperbolic space with nonzero mass. The connection between nonzero mass and uniqueness of the limit for any foliation will be made through the Kazdan-Warner identity. More precisely, we show

Theorem 6.1. *Let $(\Sigma_t)_{t\geq 0}$ be a foliation by stable spheres with constant mean curvature satisfying condition* (1)*. Then*

$$\lim_{t\to\infty}(\overline{r}_t - \underline{r}_t) = 0 \quad \textit{and} \quad \lim_{t\to\infty}\int_{\Sigma_t} |\partial_r^\top|^2 d\mu = 0.$$

Proof. Let

$$w_t(x) \equiv r(x) - \hat{r}_t, \quad \text{where} \quad |\Sigma_t| = 4\pi \sinh^2 \hat{r}_t.$$

We want to show that

$$\lim_{t\to\infty} w_t = 0 \quad \text{and} \quad \lim_{t\to\infty}\int_{\Sigma_t} |\nabla w_t|^2 d\mu = 0.$$

An immediate consequence of Proposition 4.2 and Theorem 5.1 is that

$$\begin{aligned}(13) \quad \lim_{t\to\infty}\int_{S^2} \exp(-2w_t) d\mu_0 &= \lim_{t\to\infty}\int_{\Sigma_t} 4\pi \exp(-2w_t)/|\Sigma_t| d\mu \\ &= \lim_{t\to\infty}\int_{\Sigma_t} 4\exp(-2r) d\mu = 4\pi.\end{aligned}$$

We start by deriving a sequence of preliminary results. Combining the first identity in Proposition 3.4 with both Lemma 4.1 and Theorem 5.1, we obtain that the Laplacian of w_t with respect to the standard round metric g_0 is given by

$$(14) \qquad \Delta_0 w_t = \exp(-2w_t) - 1 + P_t,$$

where

$$\begin{aligned}P_t = -2|\nabla_0 w_t|^2 \exp(-2r) &+ (4\pi)^{-1} |\Sigma_t| \exp(2\beta_t)(1 - \langle \partial_r, \nu\rangle)^2 \\ &+ \exp(2\beta_t)(1 - \langle \partial_r, \nu\rangle) + (\exp(-2w_t) + 1) O(\exp(2\overline{r}_t - 3\underline{r}_t)) \\ &+ O(\exp(-2\underline{r}_t)).\end{aligned}$$

Furthermore, Proposition 4.2 and (13) imply

Lemma 6.2.

$$\int_{\Sigma_t} |P| d\mu_0 = O(\exp(-\underline{r}_t)) + O(\exp(2\overline{r}_t - 3\underline{r}_t)).$$

Note that if $\hat{g}_t$ was the round metric and $P_t \equiv 0$, then solutions to equation (14) would correspond to constant scalar curvature metrics. Because there is no compactness for the set of such solutions, we cannot expect to derive apriori estimates solely from equation (14). In order to have good estimates we need to prevent $\exp(-2w_t)$ from concentrating at any point. This will be achieved because of the lemma we present next. It uses the Kazdan-Warner identity [12] combined with the fact that the mass of the ambient manifold is not zero and that Σ_t has constant mean curvature.

Lemma 6.3. *For each of the standard coordinate functions* x_1, x_2, x_3 *on* S^2 *we have*

$$\begin{aligned} m\int_{S^2} x_i \exp(-3w_t)d\mu_0 = O(\exp(2\overline{r}_t - 3\underline{r}_t))\int_{S^2} x_i \exp(-3w_t)d\mu_0 \\ + O(\exp(3\overline{r}_t - 4\underline{r}_t)) + O(\exp(5\overline{r}_t - 6\underline{r}_t)). \end{aligned}$$

In particular, for $i = 1, 2, 3$,

$$\lim_{t\to\infty}\int_{S^2} x_i \exp\left(-3w_t\right)d\mu_0 \left(\int_{S^2}\exp\left(-3w_t\right)d\mu_0\right)^{-1} = 0.$$

Proof. Recall that the Gaussian curvature of Σ_t with respect to $\hat{g}_t$ is denoted by $\widehat{K}_t$. Kazdan-Warner identity [12] says that, for the standard coordinate functions x_1, x_2, x_3 on S^2, the following identity holds for each $i = 1, 2, 3$,

$$\int_{S^2}\langle\nabla\widehat{K}_t, \nabla x_i\rangle\exp(2\beta_t)d\mu_0 = 0$$

or, equivalently,

$$\int_{S^2} x_i\widehat{K}_t\exp(2\beta_t)d\mu_0 - \int_{S^2}\widehat{K}_t\langle\nabla\beta_t, \nabla x_i\rangle\exp(2\beta_t)d\mu_0 = 0.$$

We saw in Section 5 that

$$\widehat{K}_t = 1 + O(\exp(2\overline{r}_t - 3\underline{r}_t))$$

and this implied that (see Theorem 5.1)

$$\beta_t = O(\exp(2\overline{r}_t - 3\underline{r}_t)) \quad \text{and} \quad \int_{S^2}|\nabla\beta_t|^2 d\mu_0 = O(\exp(4\overline{r}_t - 6\underline{r}_t)).$$

Thus,

$$\int_{S^2}|\langle\nabla\beta_t, \nabla x_i\rangle|d\mu_0 \le O(\exp(2\overline{r}_t - 3\underline{r}_t))$$

and so

$$\begin{aligned} \int_{S^2}\widehat{K}_t\langle\nabla\beta_t, \nabla x_i\rangle\exp(2\beta_t)d\mu_0 &= \int_{S^2}\langle\nabla\beta_t, \nabla x_i\rangle d\mu_0 + O(\exp(4\overline{r}_t - 6\underline{r}_t)) \\ &= \int_{S^2} 2\beta_t x_i d\mu_0 + O(\exp(4\overline{r}_t - 6\underline{r}_t)) = O(\exp(4\overline{r}_t - 6\underline{r}_t)), \end{aligned}$$

where the last equality comes from the fact that (recall proof of Theorem 5.1)

$$\int_{S^2} x_j \exp(2\beta_t) d\mu_0 = 0 \quad \text{for } j = 1, 2, 3. \tag{15}$$

Therefore, the Kazdan-Warner identity becomes

$$\int_{S^2} x_i \widehat{K}_t \exp(2\beta_t) d\mu_0 = O(\exp(4\overline{r}_t - 6\underline{r}_t)) \quad \text{for } i = 1, 2, 3. \tag{16}$$

From Lemma 3.2 we know that

$$\begin{aligned} 4\pi\widehat{K}_t = |\Sigma_t|(H^2 - 4)/4 + m|\Sigma_t|/\sinh^3 r - 3m|\partial_r^\top|^2 |\Sigma_t|/(2\sinh^3 r) \\ - |\mathring{A}|^2 |\Sigma_t|/2 + O(\exp(2\overline{r}_t - 5r)) \end{aligned}$$

and hence, because Σ_t has constant mean curvature, we obtain from (16) and (15) that

$$\begin{aligned} m\int_{S^2} x_i\, |\Sigma_t|^{3/2} \sinh^{-3} r \exp(2\beta_t)\, d\mu_0 \\ = \int_{\Sigma_t} 3/2m|\partial_r^\top|^2 |\Sigma_t|^{3/2} \sinh^{-3} r \exp(2\beta_t) d\mu_0 \\ + \int_{S^2} |\Sigma_t|^{3/2} |\mathring{A}|^2/2 \exp(2\beta_t)\, d\mu_0 \\ + O(\exp(3\overline{r}_t - 5\underline{r}_t)) + O(\exp(5\overline{r}_t - 6\underline{r}_t)). \end{aligned}$$

The first assertion in the result follows because, due to Theorem 5.1, Proposition 4.2, and Proposition 4.3,

$$\int_{\Sigma_t} 3m|\partial_r^\top|^2 |\Sigma_t|^{3/2}/(2\sinh^3 r) d\mu_0 = O(\exp(\overline{r}_t - 3\underline{r}_t)),$$

$$\int_{S^2} |\mathring{A}|^2 |\Sigma_t|^{3/2}\, d\mu_0 = O(\exp(3\overline{r}_t - 4\underline{r}_t)),$$

and

$$\begin{aligned} m\int_{S^2} x_i\, |\Sigma_t|^{3/2} / \sinh^3 r d\mu_0 = (4\pi)^{3/2} m \int_{S^2} x_i \exp(-3w_t) d\mu_0 \\ + O(\overline{r}_t - 3\underline{r}_t) + O(-4\underline{r}_t). \end{aligned}$$

The second assertion of the lemma follows from Hölder's inequality and condition (1). ☐

We use this lemma to prove

Proposition 6.4. *The functions w_t are uniformly bounded in $W^{1,2}$.*

Proof. Given a smooth function u on S^2, denote

$$S[u] \equiv \int_{S^2} |\nabla_0 u|^2 d\mu_0 - 2\int_{S^2} u d\mu_0.$$

This functional has the property that, given F a conformal transformation of S^2, then

$$S[u] = S[v] \quad \text{where} \quad F^*(\exp(-2u)g_0) = \exp(-2v)g_0.$$

This invariance can be used in the same way as in [4, Proposition 4.1] in order to show

Lemma 6.5. *$S[w_t]$ is bounded independently of t.*

The proof will be given in the Appendix. From integration by parts in (14) we obtain

$$\int_{S^2} |\nabla_0 w_t|^2 d\mu_0 - \int_{S^2} w_t d\mu_0 = -\int_{S^2} w_t \exp(-2w_t) d\mu_0 - \int_{S^2} w_t P_t d\mu_0. \tag{17}$$

The last term on the right hand side converges to zero because Lemma 6.2 implies that

$$\int_{S^2} |w_t P_t| d\mu_0 \leq |\overline{r}_t - \underline{r}_t| \left(O(\exp(-\underline{r}_t)) + O(\exp(2\overline{r}_t - 3\underline{r}_t)\right).$$

Set

$$V_t \equiv \int_{S^2} \exp\,(-2w_t) d\mu_0,$$

which we know that it converges to 4π. Using Jensen's inequality we obtain that, for all t sufficiently large,

$$\int_{S^2} |\nabla_0 w_t|^2 d\mu_0 - \int_{S^2} w_t\, d\mu_0 \leq V_t \log\left(V_t^{-1} \int_{S^2} \exp(-3w_t) d\mu_0\right) + 1. \tag{18}$$

We estimate the first term on the right-hand side using a slightly modified form of Aubin's inequality [1, Theorem 6].

Lemma 6.6 (Aubin)**.** *Given a smooth function u in S^2 such that for all $i = 1, 2, 3$*

$$\left|\int_{S^2} x_i \exp\,(-3u) d\mu_0\right| \left(\int_{S^2} \exp\,(-3u) d\mu_0\right)^{-1} \leq 1/2$$

then, for every $\delta > 0$ there is $C(\delta)$ so that

$$4\pi \log \fint_{S^2} \exp(-3u) d\mu_0 \leq \left(\frac{9}{8} + \delta\right) \int_{S^2} |\nabla_0 u|^2 d\mu_0 - 3\int_{S^2} u\, d\mu_0 + C(\delta).$$

Aubin's proof extends to this setting with obvious modifications. Lemma 6.3 implies that we can apply this result and so, combining it with (18), we have

$$C(\delta) \leq ((9/8 + \delta)V_t/(4\pi) - 1) \int_{S^2} |\nabla_0 w_t|^2 d\mu_0 - (3V_t/(4\pi) - 1) \int_{S^2} w_t d\mu_0$$

for some constant $C(\delta)$. Therefore, using Lemma 6.5, we obtain the existence of some constant $C(\delta)$ for which

$$C(\delta) \leq ((2\delta - 3/4)V_t/(4\pi) - 1) \int_{S^2} w_t\, d\mu_0$$

and thus, choosing δ sufficiently small, we get that

$$\int_{S^2} w_t \, d\mu_0$$

is uniformly bounded from above. We already know it is trivially bounded from below because V_t converges to 4π and so, we can apply Lemma 6.5 and conclude that

$$\int_{S^2} |\nabla_0 w_t|^2 d\mu_0$$

is uniformly bounded above. The result follows from Poincaré inequality. $\square$

For any sequence t_j going to infinity, we can extract a subsequence so that w_{t_j} converges weakly in $W^{1,2}$ to w_0. Furthermore, we know that from Trudinger's inequality [16] (see also [3, Corollary 1.8]), $\exp(-pw_{t_j})$ converges in L^1 to $\exp(-pw_0)$ for any p. Therefore, for $i = 1, 2, 3$,

$$m \int_{S^2} x_i \exp(-3w_0) d\mu_0 = 0 \quad \text{and} \quad \int_{S^2} \exp(-2w_0) d\mu_0 = 1. \tag{19}$$

On the other hand, it is easy to recognize from (14) that w_0 satisfies weakly

$$\Delta_0 w_0 = \exp(-2w_0) - 1$$

and so, the identities in (19) imply that $w_0 = 0$, i.e., w_t converges weakly in $W^{1,2}$ to zero. Hence, from Rellich's Theorem and integration by parts formula (17) we obtain

$$\lim_{t\to\infty} \int_{S^2} |\nabla_0 w_t|^2 d\mu_0 = \lim_{t\to\infty} \int_{S^2} w_t^2 d\mu_0 = 0.$$

We argue next that w_t converges to zero uniformly. We start with

Lemma 6.7.

$$\lim_{t\to\infty} \sup_{\Sigma_t} |\partial_r^\top|^2 = 0.$$

Proof. Note that we have from Proposition 4.2

$$\lim_{t\to\infty} \int_{\Sigma_t} |\partial_r^\top|^2 d\mu = 0.$$

Arguing like in the proof of Proposition 3.4 and using the fact that the norm of the second fundamental form of Σ_i is bounded, it is straightforward to see that

$$|\nabla\langle \nu, \partial_r\rangle| \le C|\partial_r^\top| + O(\exp(-3r))$$

for some constant C. Combining Proposition 3.4 with the Bochner formula for $|\nabla r|^2$ we obtain that, for some other constant C and for all t sufficiently large,

$$\begin{aligned}\Delta|\partial_r^\top|^2 &\ge -C|\partial_r^\top|^2 + 2|\nabla\nabla r|^2 - |\nabla\nabla r||\partial_r^\top|^2 O(\exp(-2r)) \\ &\quad + |\partial_r^\top| O(\exp(-3r)) \\ &\ge -C|\partial_r^\top|^2 + O(\exp(-6r)).\end{aligned}$$

Because the second fundamental form of Σ_t is bounded, we can invoke the same reasons as in Lemma 5.2 in order find r_0 so that, for all t sufficiently large and all x in Σ_t, we can apply Moser's iteration argument in $B_{r_0}(x)\cap\Sigma_t$ and conclude that

$$\sup_{\Sigma_t}|\partial_r^\top|^2 \leq C\int_{\Sigma_t}|\partial_r^\top|^2 d\mu + O(\exp(-6r)),$$

where C is a constant independent of t. □

An immediate consequence of this lemma, Proposition 4.2, and the identity $|\partial_r^\top|^2 = 1-\langle\nu,\partial_r\rangle^2 + O(\exp(-5r))$ is

Lemma 6.8. *For all t sufficiently large*

$$O(\exp(-5r)) + |\partial_r^\top|^2/3 \leq 1-\langle\nu,\partial_r\rangle \leq |\partial_r^\top|^2/2 + O(\exp(-5r)).$$

A simple computation using identity (14) shows that

$$\Delta_0 w_t^2 \geq 8\pi w_t(\exp(-2w_t)-1) + 2P_t w_t \tag{20}$$

and we want to write this inequality as

$$\Delta_0 w_t^2 \geq -Cw_t^2 - (\text{finite sum of positive terms } f_i),$$

where C is a t-independent constant and each term f_i has its L^p norm converging to zero for some $p>1$. We can then apply Moser's iteration argument and conclude that, for some constant C,

$$\sup w_t^2 \leq C\int_{S^2} w_t^2 d\mu_0 + C\sum_i |f_i|_{L^p}.$$

This implies uniform convergence of w_t to zero.

In what follows, C is a generic constant independent of t. The first term in (20) can be easily estimated as

$$w_t(\exp(-2w_t)-1) \geq -w_t^2/2 - (\exp(-2w_t)-1)^2/2,$$

where the term with the exponential converges to zero in any L^p-norm due to Trudinger's inequality [16]. Looking at the expression of $P_t\, w_t$ (see (14)), we have that the the first term and the third term cause no problem because

$$\begin{aligned}-|\nabla_0 w_t|^2\exp(-2r)w_t &= O(1)|\partial_r^\top|^2\exp(-2w_t)w_t + w_t O(\exp(-2\underline{r}_t))\\ &\geq -Cw_t^2 - C|\partial_r^\top|^4\exp(-4w_t) + O(\exp(-4\underline{r}_t))\end{aligned}$$

and also

$$(1-\langle\partial_r,\nu\rangle)w_t \geq -w_t^2/2 - (1-\langle\partial_r,\nu\rangle)^2/2 \geq -w_t^2/2 - |\partial_r^\top|^4/8.$$

The same sort of estimate works for the last two terms and to handle the second term we note that, by Proposition 4.2,

$$\int_{S^2}|\nabla_0 w|^2|\partial_r^\top|^2 d\mu_0 \leq 9\int_{\Sigma_t}(1-\langle\partial_r,\nu\rangle + O(\exp(-5r)))^2 d\mu = O(\exp(-\underline{r}_t)).$$

Hence, for some constant C,

$$\begin{aligned}\int_{S^2}(|\Sigma_t|\,(1-\langle\partial_r,\nu\rangle)^2|w_t|)^{1+\varepsilon}d\mu_0 &\leq C\int_{S^2}|\nabla_0 w|^{2+2\varepsilon}|\partial_r^\top|^{2+2\varepsilon}|w_t|^{1+\varepsilon}d\mu_0\\ &\leq C\int_{S^2}|\nabla_0 w|^{2+2\varepsilon}|\partial_r^\top|^{2-2\varepsilon}|w_t|^{1+\varepsilon}d\mu_0\\ &= C\,|\Sigma_t|^{\varepsilon}\int_{S^2}|\nabla_0 w|^2|\partial_r^\top|^2|w_t|^{1+\varepsilon}d\mu_0\\ &\leq |\overline{r}_t-\underline{r}_t|^{1+\varepsilon}O(\exp(2\varepsilon\overline{r}_t-\underline{r}_t))\end{aligned}$$

and, because we are assuming condition (1), we obtain that, for some sufficiently small ε,

$$\lim_{t\to\infty}\int_{S^2}(|\Sigma_t|\,(1-\langle\partial_r,\nu\rangle)^2|w_t|)^{1+\varepsilon}d\mu_0=0.$$

□

7. STRONG APPROXIMATION TO COORDINATE SPHERES

We will start by arguing that we can choose $\tilde{r}_t$ such that

$$H=2\cosh\tilde{r}_t/\sinh\tilde{r}_t-m/\sinh^3\tilde{r}_t+o(\exp(-4\tilde{r}_t)).$$

We will then show that, with respect to $\hat{g}_t$, Σ_t is $o(\exp(-\underline{r}_t))$ "close" to $\{|x|=\tilde{r}_t\}$. This is exactly the rate we need in order to prove uniqueness for the constant mean curvature foliation. Being more precise, rename the family of functions w_t on Σ_t by

$$w_t(x)\equiv r(x)-\tilde{r}_t.$$

This section is devoted to show

Theorem 7.1. *With respect to the metric $\hat{g}_t$,*

$$|w_t|_{C^{2,\alpha}}=o(\exp(-\underline{r}_t)).$$

Before proving Theorem 7.1 we need to argue that $\tilde{r}_t$ is well defined. In order to do so, the following proposition is important

Proposition 7.2. *The following identity holds for each Σ_t :*

$$\sup_{\Sigma_t}|\partial_r^\top|^2=o(\exp(-2\underline{r}_t)).$$

Proof. Direct computation implies the following gradient estimate

Lemma 7.3.

$$|\nabla|\partial_r^\top|^2|\leq(4\exp(-2r)-4\pi/\,|\Sigma_t|)|\partial_r^\top|+3|\partial_r^\top|^3+|\partial_r^\top|o(\exp(-2\underline{r}_t)).$$

Proof. Because g is a perturbation of the hyperbolic metric, if we decompose a unit tangent vector V as $\overline{V}+\beta\,\partial_r$, where $\overline{V}$ has no ∂_r component, then $|\beta|\leq|\partial_r^\top|+O(\exp(-3\underline{r}_t))$ and so

$$|\langle V-\overline{V},\partial_r^\top\rangle|\leq|\partial_r^\top|^3+|\partial_r^\top|^2O(\exp(-3\underline{r}_t)). \tag{21}$$

Denoting the connection with respect to the hyperbolic metric by $\overline{\nabla}$, we can estimate

$$\langle V, \nabla|\partial_r^\top|^2\rangle = 2\langle\overline{\nabla}_V\partial_r, \partial_r^\top\rangle - 2\langle\partial_r,\nu\rangle\langle\nabla_V\nu,\partial_r^\top\rangle + |\partial_r^\top|O(\exp(-3\underline{r}_t)).$$

Due to Theorem 6.1 and Proposition 5.3, we know that

$$\sup_{\Sigma_t}|\mathring{A}| = o(\exp(-2\underline{r}_t)).$$

Hence,

$$\begin{aligned}\langle V,\nabla|\partial_r^\top|^2\rangle &= 2(\cosh r/\sinh r)\,\langle\overline{V},\partial_r^\top\rangle - \langle\partial_r,\nu\rangle\langle V,\partial_r^\top\rangle H\\ &\quad + o(\exp(-2\underline{r}_t))|\partial_r^\top|\\ &\leq (2\cosh r/\sinh r - H)|\partial_r^\top| + H(1-\langle\partial_r,\nu\rangle)\langle\overline{V},\partial_r^\top\rangle\\ &\quad + H\langle\partial_r,\nu\rangle\langle\overline{V}-V,\partial_r^\top\rangle + o(\exp(-2\underline{r}_t))|\partial_r^\top|\\ &\leq (4\exp(-2r) - 4\pi/|\Sigma_t|)|\partial_r^\top| + 3|\partial_r^\top|^3 + |\partial_r^\top|o(\exp(-2\underline{r}_t)),\end{aligned}$$

where in the last inequality we used Lemma 4.1, Lemma 6.8, and (21). □

Choose p in Σ_t so that $|\partial_r^\top|(p)=0$. Given q in Σ_t, denote by $\gamma(s)$ the unit speed geodesic connecting p to q and let $f(s) = |\partial_r^\top|(\gamma(s))$. From Lemma 7.3 we have

$$\begin{aligned}\left|\frac{d}{ds}f^2\right| &\leq (|4\pi/|\Sigma_t| - 4\exp(-2r)| + o(\exp(-2\underline{r}_t)))f + 3f^3\\ &\leq \alpha_t^2 f + 3f^3,\end{aligned}$$

where

$$\alpha_t^2 = \sup_{\Sigma_t}|4\pi/|\Sigma_t| - 4\exp(-2r)| + o(\exp(-2\underline{r}_t)).$$

If we set

$$V(s) \equiv \alpha_t\tan((3/4)^{1/2}\alpha_t\, s)/\sqrt{3},$$

then

$$(V^2)' = \alpha_t^2 V + 3V^2$$

and so, while V is well defined, we have $f\leq V$ because $f(p)=0$. Moreover, $\mathrm{diam}(\hat{g}_t)$ converges to 2 and therefore

$$\begin{aligned}\lim_{t\to\infty}\alpha_t^2\mathrm{diam}^2(g_t) &= \lim_{t\to\infty}\alpha_t^2\,|\Sigma_t|\,\pi^{-1}\\ &\leq \lim_{t\to\infty}\sup_{\Sigma_t}|4 - 4\,|\Sigma_t|\,\pi^{-1}\exp\left(-2\underline{r}_t\right)| = 0,\end{aligned}$$

where the last equality follows from Theorem 6.1. Hence, we obtain for all t sufficiently large

$$|\partial_r^\top|(q) = f(q) \leq V(\mathrm{diam}(g_t)/2) \leq \alpha_t = o(\exp(-\underline{r}_t)).$$

□

The decay for $|\partial_r^\top|$ proved above combined with Lemma 6.8 allow us to write the second identity in Proposition 3.4 as

$$\Delta r = 2\cosh r/\sinh r - m/\sinh^3 r - H + P,$$

where $|P| = o(\exp(-4r))$. Thus, we obtain after integration

$$H = \fint_{\Sigma_t} 2\cosh r/\sinh r - m/\sinh^3 r d\mu + \fint o(\exp(-4r))d\mu$$

and so, the existence of $\tilde{r}_t$ satisfying $\underline{r}_t \le \tilde{r}_t \le \overline{r}_t$ follows easily.

We also remark that a careful inspection of the term P shows that

$$|\nabla P| = o(\exp(-5r)) \tag{22}$$

because, as can be seen from Lemma 6.8, Lemma 7.3, and Proposition 7.2, we have

$$|\nabla \langle \partial_r, \nu\rangle| = o(\exp(-3r)) \quad \text{and} \quad |\nabla |\partial_r^\top|^2| = o(\exp(-3r)).$$

Proof of Theorem 7.1. We know from Theorem 6.1 that w_t converges to zero uniformly and thus, the expansion for H implies

$$\begin{aligned}\Delta r &= 4\exp(-2\tilde{r}_t)(\exp(-2w_t) - 1) - 8m\exp(-3\tilde{r}_t)(\exp(-3w_t) - 1)\\ &\quad + o(\exp(-4r))\\ &= -8w_t\exp(-2\tilde{r}_t) + f(w_t)O\left(\exp(-2r)\right) + w_tO(\exp(-3r))\\ &\quad + o(\exp(-4r)),\end{aligned}$$

where

$$f(x) \equiv \exp(-2x) - 1 + 2x.$$

Hence, from Theorem 5.1, the above identity translates to

$$\Delta_0 w_t + (2\exp(2\hat{r}_t - 2\tilde{r}_t + 2\beta_t) + o(1))w_t = f(w_t)O\left(1\right) + o(\exp(-2r)).$$

When t goes to infinity the above operator converges to $\Delta_0 + 2$. Therefore, in order to get a good control of w_t we need to control the projection of w_t on the kernel of $\Delta_0 + 2$. This is achieved using the following improvement of Lemma 6.3

Lemma 7.4.

$$\int_{S^2} w_t x_i d\mu_0 = o(\exp(-\underline{r}_t)) + |w_t|_2^2 O(1), \quad i = 1, 2, 3.$$

Proof. We saw in Lemma 7.3 that $|\mathring{A}|^2$ has order $o(\exp(-4\underline{r}_t))$ and so, we obtain from Lemma 3.2 and Proposition 7.2 that

$$\widehat{K}_t = 1 + o(\exp(-\underline{r}_t)).$$

This implies that (check proof of Theorem 5.1)

$$\exp(2\beta_t) = 1 + o(\exp(-\underline{r}_t))$$

and thus, an inspection of the proof of Lemma 6.3 shows that

$$m\int_{S^2} \exp(-3w_t)x_i d\mu_0 = o(\exp(-\underline{r}_t)) \quad i = 1, 2, 3.$$

This implies the desired result. □

As a result, if we decompose w_t as $u_t + v_t$, where u_t is in the kernel of $\Delta_0 + 2$ and v_t is perpendicular to u_t, we obtain from the previous lemma that

$$|u_t|_{C^{2,\alpha}} = o(\exp(-\underline{r}_t)) + |w_t|_2^2 O(1)$$

and thus, because u_t converges to zero uniformly,

$$|u_t|_{C^{2,\alpha}} = o(\exp(-\underline{r}_t)) + |v_t|_2^2 O(1). \tag{23}$$

On the other hand, we have that v_t satisfies

$$\Delta_0 v_t + (2 + o(1))v_t = u_t o(1) + f(w_t)O\,(1) + o(\exp(-2r)). \tag{24}$$

Looking at the way the terms in this equation were derived and using (22) we see that

$$|u_t o(1)|_{C^{0,\alpha}} = o(1)|u_t|_{C^{0,\alpha}}, \quad |f(w_t)O\,(1)|_{C^{0,\alpha}} \le O(1)|w_t|^2_{C^{0,\alpha}},$$

and

$$|o(\exp(-2r))|_{C^{0,\alpha}} = o(\exp(-2\underline{r}_t)).$$

Hence, from Schauder estimates, we have that for some constant C

$$|v_t|_{C^{2,\alpha}} \le o(1)|u_t|_{C^{0,\alpha}} + C|w_t|^2_{C^{0,\alpha}} + C\sup|v_t| + o(\exp(-2\underline{r}_t))$$

Moreover, $|w_t|_{C^{0,\alpha}}$ converges to zero (see Proposition 7.2) and hence we have from (23)

$$|v_t|_{C^{2,\alpha}} \le \sup|v_t| + o(\exp(-r)).$$

Using the same arguments as in the proof of Theorem 5.1, we can see that the orthogonality condition of v_t, identity (23), and equation (24), imply the estimate

$$\int_{S^2} w_t^2 d\mu_0 = o(\exp(-2\underline{r}_t)).$$

A simple computation shows that, for some constant C,

$$\Delta_0 v_t^2 \ge -Cv_t^2 - Cu_t^2 + o(\exp(-2\underline{r}_t))$$

and thus, using Moser's iteration we obtain

$$\sup v_t^2 \le C\int_{S^2} w_t^2 d\mu_0 + C\sup u_t^2 + o(\exp(-2\underline{r}_t)) = o(\exp(-2\underline{r}_t)).$$

□

8. EXISTENCE AND UNIQUENESS OF CONSTANT MEAN CURVATURE FOLIATIONS

8.1. **Uniqueness.** We are now ready to prove the main theorem.

Theorem 8.1. *If the mass of the asymptotically Anti de Sitter- Schwarzschild metric is nonzero, then any two smooth foliations by stable spheres with constant mean curvature* $\left(\Sigma_t^1\right)_{t\geq 0}$ *and* $\left(\Sigma_t^2\right)_{t\geq 0}$ *for which*

$$\lim_{t\to\infty}(\overline{r}_t - 4/3\underline{r}_t) = -\infty \tag{25}$$

will coincide for t *sufficiently large.*

Proof. We can reparametrize the foliations so that $H(\Sigma_t^1) = H(\Sigma_t^2) = H_t$ for all t sufficiently large. The results in the previous section imply the existence of $\tilde{r}_t$ so that

$$H_t = 2\cosh\tilde{r}_t/\sinh\tilde{r}_t - m/\sinh^3\tilde{r}_t + o(\exp(-4\tilde{r}_t))$$

and for $i = 1, 2$, the functions

$$w_t^i \equiv r(x) - \tilde{r}_t$$

have $C^{2,\alpha}$ norm of order $o(\exp(-\underline{r}_t))$, where the norm is computed with respect to the rescaled metric $\hat{g}_t^i$. From Theorem 5.1 we know that, after pulling back $\hat{g}_t^1$ by a suitable diffeomorphism, the metric can be written as $\exp(2\beta_t)g_0$, where g_0 denotes the standard round metric on S^2. The following lemma improves the estimate on β_t.

Lemma 8.2. *The functions* β_t *can be chosen so that*

$$|\beta_t|_{C^{2,\alpha}} = o(\exp(-\underline{r}_t)).$$

Proof. A direct computation using the fact that each Σ_t^1 is the graph over $\{|x| = \tilde{r}_t\}$ of a function w_t^1 with $|w_t^1|_{C^{2,\alpha}} = o(\exp(-\underline{r}_t))$, reveals that $|\mathring{A}|^2_{C^{0,\alpha}} = o(\exp(-2\underline{r}_t))$, where the norm is computed with respect to the metric g_0. Combining Lemma 3.2 with the asymptotic expansion of H_t in terms of $\tilde{r}_t$, we obtain that the Gaussian curvature of $\hat{g}_t^1$ can be written as

$$\widehat{K}_t = 1 + S_t, \quad \text{where} \quad |S_t|_{C^{0,\alpha}} = o(\exp(-\underline{r}_t)).$$

The functions β_t were chosen so that, for each coordinate function x_1, x_2, and x_3,

$$\int_{S^2}\exp(2\beta_t)x_i d\mu_0 = 0, \quad i = 1, 2, 3$$

and they satisfy the equation

$$\begin{aligned}\Delta_0\beta_t &= 1 - \widehat{K}_t\exp(2\beta_t)\\ &= 1 - \exp(2\beta_t) + \exp(2\beta_t)o(\exp(-r)).\end{aligned}$$

One can then use the smallness of β_t and argue in the same way as in the proof of either Theorem 5.1 or Theorem 7.1 in order to prove the lemma. □

With respect to the coordinates (r, ω) in $\mathbb{R} \times S^2$ we consider, for each fixed t, the interpolation surfaces

$$\Sigma_{t,s} = \left\{ \tilde{r}_t + (1-s)w_t^1(\omega) + sw_t^2(\omega) \,|\, \omega \text{ in } S^2 \right\}, \quad 1 \le s \le 0$$

and set

$$F_t(s) \equiv H(\Sigma_{t,s}).$$

In what follows, we fix t large and suppress the index t in the notation for the sake of simplicity. From Taylor's formula, we have

$$F(1) = F(0) + F'(0) + P, \quad P \equiv \int_0^1 \int_0^1 sF''(su)duds,$$

where

$$F'(0) = \Delta_{\Sigma_1}\phi + (|A|^2 + R(\nu,\nu))_{\Sigma_1}\phi, \quad \phi \equiv w^2 - w^1$$

and, for some universal constant C,

$$|F''(s)| \le C(|\nabla^2\phi||\phi| + |\nabla\phi|^2 + |\phi|^2\exp(-2r))_{|\Sigma_{t,s}}.$$

We know that $F(1) = F(0)$ and that

$$\begin{aligned}(|A|^2 + R(\nu,\nu))_1 &= (H^2-4)/2 - m/\sinh^3\tilde{r} + O(\exp(-4r))\\ &= 8\pi/|\Sigma^1| - 3m/\sinh^3\tilde{r} + O(\exp(-4r)).\end{aligned}$$

Therefore, using Theorem 7.1, we have that, with respect to the metric g_0, ϕ satisfies the following equation

$$\Delta_0\phi + \exp(2\beta)(2 - 3m/\sinh\tilde{r} + O(\exp(-2r)))\phi \\ + |\Sigma^1|(4\pi)^{-1}\exp(2\beta)P = 0$$

or, alternatively,

$$\Delta_0\phi + (2 - 3m/\sinh\tilde{r})\phi = Q - |\Sigma^1|(4\pi)^{-1}\exp(2\beta)P,$$

where

$$Q \equiv (1-\exp(2\beta))(2 - 3m/\sinh\tilde{r})\phi - \exp(2\beta)O(\exp(-2r))\phi.$$

Consider the decomposition

$$\phi = \phi_0 + \phi_1, \quad Q = Q_0 + Q_1, \quad |\Sigma^1|\exp(2\beta)P = P_0 + P_1$$

where ϕ_0, Q_0, P_0 belong to the kernel of $\Delta_0 + 2$ and ϕ_1, Q_1, P_1 are orthogonal to ϕ_0, Q_0, P_0 respectively. Then

$$\Delta_0\phi_1 + (2 - 3m/\sinh\tilde{r})\phi_1 = Q_1 - P_1, \tag{26}$$

$$3m/\sinh\tilde{r}\,\phi_0 = P_0 - Q_0, \tag{27}$$

and it is immediate to recognize that, for some universal constant C,

$$|P_0|_{C^{0,\alpha}} + |P_1|_{C^{0,\alpha}} \le C|\phi|^2_{C^{2,\alpha}}$$

and

$$|Q_0|_{C^{0,\alpha}} + |Q_1|_{C^{0,\alpha}} \le o(\exp(-\underline{r}))|\phi|_{C^{0,\alpha}},$$

where the norms are computed with respect to g_0.

Applying Schauder estimates to equation (26) and using the fact that $|\phi|_{C^{2,\alpha}}$ converges to zero, we obtain

$$|\phi_1|_{C^{2,\alpha}} \leq C \sup_{\Sigma^1} |\phi_1| + C|\phi_0|^2_{C^{2,\alpha}} + o(\exp(-\underline{r}))|\phi_0|_{C^{0,\alpha}},$$

where C is some uniform constant. The orthogonality condition satisfied by ϕ_1 can be used in the same way as in the proof of Theorem 7.1 in order to show that

$$\sup_{\Sigma^1} |\phi_1| \leq C|\phi|^2_{C^{2,\alpha}} + o(\exp(-\underline{r}))|\phi|_{C^{0,\alpha}}.$$

Therefore,

$$|\phi_1|_{C^{2,\alpha}} \leq C|\phi_0|^2_{C^{2,\alpha}} + o(\exp(-\underline{r}))|\phi_0|_{C^{0,\alpha}},$$

where C is some uniform constant. Because the $C^{2,\alpha}$ norm of ϕ_0 is bounded by its $C^{0,\alpha}$ norm, we obtain from (27) that

$$\begin{aligned} |\phi_0|_{C^{2,\alpha}} &\leq O(\exp(\underline{r}))|\phi|^2_{C^{2,\alpha}} + o(1)|\phi|_{C^{0,\alpha}} \\ &\leq O(\exp(\underline{r}))|\phi_0|^2_{C^{2,\alpha}} + o(1)|\phi_0|_{C^{0,\alpha}}. \end{aligned}$$

From Theorem 7.1 we have that $|\phi|_{C^{2,\alpha}} = o(\exp(-\underline{r}))$ and thus

$$|\phi_0|_{C^{0,\alpha}} \leq o(1)|\phi_0|_{C^{0,\alpha}}.$$

Consequently, for t sufficiently large, $\phi_0 = 0$ and this implies that $\phi_1 = 0$. Hence $w^1_t = w^2_t$ for all t sufficiently large and this is the same as $\Sigma^1_t = \Sigma^2_t$. □

8.2. **Existence.** We show existence of a foliation by stable spheres with constant mean curvature when the mass of the asymptotically Anti de Sitter-Schwarzschild metric is positive. This result was previously shown by Rigger [15] using a modified mean curvature flow approach. The argument we use is a straightforward adaptation of the arguments used by Rugang Ye in [18], where he showed a similar theorem in the context of asymptotically flat manifolds. We include the proof of existence of a foliation for the sake of completeness.

Theorem 8.3. *If the mass of the asymptotically Anti de Sitter- Schwarzschild metric is nonzero, the manifold M admits, outside a compact set, a foliation by spheres with constant mean curvature. Moreover, if the mass is positive, then the spheres are stable.*

Proof. Given $\phi \in C^\infty(S^2)$ and r sufficiently large, let

$$\Sigma_r(\phi) \equiv \{(r + \phi(x), x) \,|\, x \in S^2\}$$

and set $F(r, \phi) \equiv H(\Sigma_r(\phi))$. Because the metric g is a C^3 perturbation of g_m, we have from Lemma 3.1 that

$$F(r, 0) = 2\cosh r/\sinh r - m/\sinh^3 r + O(\exp(-5r))$$

and

$$(|A|^2 + R(\nu, \nu))_{\Sigma_r(0)} = 8\pi/|\Sigma_r(0)| - 3m/\sinh^3 r + O(\exp(-5r)).$$

For every r sufficiently large, we want to find ϕ so that

$$F(r,\phi) = 2\cosh r/\sinh r - m/\sinh^3 r.$$

Using Taylor's formula like in the previous subsection, the above equation is equivalent to solve on S^2

$$\Delta_0\phi + (2 - 3m/\sinh r)\phi = P(\phi) + Q(\phi) + N,$$

where

$$N \equiv |\Sigma_r(0)|(F(r,0) - 2\cosh r/\sinh r + m/\sinh^3 r)/(4\pi)$$

satisfies

$$|N|_{C^{0,\alpha}} = O(\exp(-3r))$$

and P and Q are such that, for some uniform constant C,

$$|P(\phi)|_{C^{0,\alpha}} \le C|\phi|^2_{C^{2,\alpha}},$$
$$|P(\phi) - P(\psi)|_{C^{0,\alpha}} \le C(|\phi|_{C^{2,\alpha}} + |\psi|_{C^{2,\alpha}})|\phi - \psi|_{C^{2,\alpha}},$$
$$|Q(\phi)|_{C^{0,\alpha}} \le o(\exp(-r))|\phi|_{C^{0,\alpha}},$$

and

$$|Q(\phi) - Q(\psi)|_{C^{0,\alpha}} \le o(\exp(-r))|\phi - \psi|_{C^{0,\alpha}}.$$

Consider the map

$$T : C^{2,\alpha}(S^2) \longrightarrow C^{2,\alpha}(S^2)$$

such that

$$\Delta_0(T(u)) + (2 - 3m/\sinh r)T(u) = P(u) + Q(u) + N.$$

The map is well defined because the operator on the left hand side is invertible. The existence of a constant mean curvature foliation follows from

Lemma 8.4. *For all r sufficiently large, the map T is a contraction of*

$$\{u \in C^{2,\alpha}(S^2) \,|\, |u|_{C^{2,\alpha}} \le \exp(-r)/r\}$$

onto itself.

Proof. Set $\phi \equiv T(u)$ and consider the decomposition

$$\phi = \phi_0 + \phi_1, \quad P(u) = P_0 + P_1, \quad Q(u) = Q_0 + Q_1, \quad \text{and } N = N_0 + N_1,$$

such that ϕ_0, P_0, Q_0, N_0 belong to the kernel of $\Delta + 2$ and ϕ_1, P_1, Q_1, N_1 are orthogonal to ϕ_0, P_0, Q_0, N_0 respectively. Thus,

$$-3m\phi_0/\sinh r = P_0 + Q_0 + N_0$$

and so

$$|\phi_0|_{C^{2,\alpha}} \le O(\exp(r))|u|^2_{C^{2,\alpha}} + o(1)|u|_{C^{2,\alpha}} + O(\exp(-2r))$$
$$\le o(1)\exp(-r)/r.$$

Furthermore,

$$\Delta_0\phi_1 + (2 - 3m/\sinh r)\phi_1 = P_1(u) + Q_1(u) + N_1$$

and hence, we can argue in the same way as in the proof of Theorem 5.1 and use the orthogonality condition satisfied by ϕ_1 in order to show that

$$|\phi_1|_{C^{2,\alpha}} \leq C|u|^2_{C^{2,\alpha}} + o(\exp(-r))|u|_{C^{2,\alpha}} + O(\exp(-3r)) \\ \leq o(1)\exp(-r)/r$$

for some uniform constant C. Therefore, we have for all r sufficiently large that $|\phi|_{C^{2,\alpha}} \leq \exp(-r)/r$. Finally, we can argue in the same way and check that

$$|T(u) - T(v)|_{C^{2,\alpha}} \leq o(1)|u - v|_{C^{2,\alpha}}.$$

□

Denote by Σ_r the constant mean curvature sphere that is the graph over $\{|x| = r\}$ of a function w_r with $|w_r|_{C^{2,\alpha}} \leq \exp(-r)/r$, where the norm is computed with respect to the standard round metric g_0. We need to show that Σ_r is stable for all r sufficiently large, i.e., we need to show that second variation operator

$$Lf \equiv -\Delta f - \left(|A|^2 + R(\nu,\nu)\right) f$$

has only nonnegative eigenvalues when restricted to functions with zero mean value.

Because Σ_r is the graph of a function with $|w_r|_{C^{2,\alpha}} \leq o(\exp(-r))$, we have that

$$\sup_{\Sigma_r} |A|^2 = o(\exp(-4r)),$$

where this norm is computed with respect to the metric g_r induced by the ambient metric. Moreover, we also have that

$$|A|^2 + R(\nu,\nu) = 8\pi/|\Sigma_r| - 3m/\sinh^3 r + O(\exp(-4r)).$$

In this setting, Theorem 5.1 and Lemma 8.2 apply and so, after applying a suitable diffeomorphism, the normalized metric $\hat{g}_r \equiv 4\pi/|\Sigma_r| g_r$ can be written as $\exp(2\beta_r)g_0$, where $|\beta_r|_{C^{2,\alpha}} \leq o(\exp(-r))$. Hence, in terms of the metric g_0, the positivity of the operator L is equivalent to the positivity of the following operator defined on S^2

$$L_0\phi \equiv -\Delta_0\phi - (2 - 3m/\sinh r + o(\exp(-r))\phi.$$

This is true for all r sufficiently large whenever m is positive. □

APPENDIX A. PROOF OF LEMMA 6.5

Given a smooth function u on S^2, recall that

$$S[u] \equiv \int_{S^2} |\nabla_0 u|^2 d\mu_0 - 2\int_{S^2} u d\mu_0.$$

We want to show that $S[w_t]$ is bounded independently of t. We essentially follow, with some necessary modifications, the proof of [4, Proposition 4.1]. The main idea consists in exploiting the invariance of S under conformal transformations. More precisely, we will find conformal diffeomorphisms F_t for which the family of functions u_t defined by

$$\exp(-2u_t)g_0 = F_t^*(\exp(-2w_t)g_0)$$

is such that

$$\int_{S^2} |\nabla u_t|^2 d\mu_0 \quad \text{and} \quad \int_{S^2} u_t d\mu_0$$

are uniformly bounded. The desired result follows because $S[u_t] = S[w_t]$.

A standard application of Brower's fixed point Theorem (see for instance [2, Lecture 3, Lemma 2]) implies the existence of a conformal diffeomorphism T such that, for $i = 1, 2, 3$,

$$\int_{S^2} x_i \exp(-4w_t \circ F_t + 4\gamma_t)d\mu_0 = 0, \quad \text{where} \quad F_t^*(g_0) = \exp(2\gamma_t)g_0. \tag{28}$$

Moreover, γ_t has an upper bound given by

Lemma A.1. *There is a universal constant C_0 so that*

$$\sup|\gamma_t| \leq C_0 + 2/3(\overline{r}_t - \underline{r}_t).$$

Proof. Assume that the maximum of α_t is attained at the north pole p in S^2. We know that

$$\int_{\{x_3 \geq 0\}} x_3 \exp(-4w_t \circ T + 4\gamma_t)d\mu_0 = \int_{\{x_3 \leq 0\}} (-x_3) \exp(-4w_t \circ T + 4\gamma_t)d\mu_0$$

and so

$$1/2 \int_{\{x_3 \geq 1/2\}} \exp(-4 \sup w_t + 4\gamma_t)d\mu_0 \leq \int_{\{x_3 \leq 0\}} \exp(-4 \inf w_t + 4\gamma_t)d\mu_0. \tag{29}$$

In stereographic coordinates, γ_t can be written as

$$\gamma_t(x) = \log(\lambda(1 + |x|^2)/(\lambda^2 + |x|^2))$$

for some $\lambda \leq 1$. Note that $\sup \gamma_t = -\inf \gamma_t = -\log \lambda$. An explicit computation shows the existence of some r_0 so that, if

$$f(r) \equiv 8/3\lambda^4(\lambda^4 + \lambda^2 + 3r^4 + 3(\lambda^2 + 1)r^2 + 1)/(\lambda^2 + r^2)^3,$$

then

$$\int_{\{x_3 \geq 1/2\}} \exp(4\gamma_t)d\mu_0 = f(0) - f(r_0)$$

and

$$\int_{\{x_3 \leq 0\}} \exp(4\gamma_t)d\mu_0 = f(1).$$

Hence, there are universal constants C_1, C_2, and C_3 so that, for all $\lambda \leq 1$,

$$\int_{\{x_3 \geq 1/2\}} \exp(4\gamma_t)d\mu_0 \geq C_1\lambda^{-2} - C_2\lambda^4$$

and

$$\int_{\{x_3\leq 0\}} \exp(4\gamma_t)d\mu_0 \leq C_3\lambda^4.$$

Therefore, (29) implies that, for some universal constants C_5

$$\lambda^{-6} \leq C_5 \exp(4\sup w_t - 4\inf w_t) + C_5.$$

The result follows because $\sup w_t - \inf w_t = \overline{r}_t - \underline{r}_t$. □

Let

$$u_t \equiv w_t \circ T - \gamma_t + \log \delta_t/2, \quad \text{where} \quad \delta_t \equiv \fint_{S^2} \exp(-2w_t)d\mu_0.$$

The effect of δ_t is to ensure that $\exp(-2u_t)g_0$ has volume 4π and we observe that, combining Theorem 5.1 with Proposition 4.2, we obtain

$$\delta_t = 1 + O(\exp(-\underline{r}_t)) + O(\exp(2\overline{r}_t - 3\underline{r}_t)).$$

Because

$$S[u_t] = S[w_t] - \log \delta_t,$$

it suffices to show that $S[u_t]$ is uniformly bounded.

Because

$$\Delta_0\gamma_t = 1 - \exp(2\gamma_t),$$

we have (14) that

$$\Delta_0 u_t = \exp(-2u_t) - 1 + Q_t,$$

where

$$Q_t = \exp(2\gamma_t)P_t \circ T + \exp(-2u_t)(\delta_t - 1).$$

Integration by parts yields

$$\begin{aligned} 2\int_{S^2} |\nabla_0 u_t|^2 d\mu_0 - 2\int_{S^2} u_t d\mu_0 & \\ &= \int_{S^2} -2u_t \exp(-2u_t)d\mu_0 - \int_{S^2} 2u_t Q_t d\mu_0. \end{aligned}$$

We argue that the last term on the right-hand side is bounded independently of t. Note that Lemma 6.2 and Lemma A.1 imply that

$$\begin{aligned} \int_{S^2} |u_t Q_t| d\mu_0 &\leq \int_{S^2} |u_t \circ T^{-1} P_t| d\mu_0 + |\delta_t - 1| \int_{S^2} |u_t \circ T^{-1}| \exp(-2w_t)d\mu_0 \\ &\leq (|\overline{r}_t - \underline{r}_t| + C_0)\,(O(-\underline{r}_t) + O(\exp(2\overline{r}_t - 3\underline{r}_t))) \end{aligned}$$

and so, because we are assuming condition (1), we obtain

$$\lim_{t\to\infty} \int_{S^2} |u_t Q_t| d\mu_0 = 0.$$

Thus, using Jensen's inequality we obtain that, for all t sufficiently large,

$$2\int_{S^2} |\nabla_0 u_t|^2 d\mu_0 - 2\int_{S^2} u_t d\mu_0 \leq 4\pi \log \fint_{S^2} \exp(-4u_t)d\mu_0 + 1. \tag{30}$$

Lemma A.2.
$$\int_{S^2}|\nabla_0 u_t|d\mu_0$$
is uniformly bounded independently of t.

Proof. Denoting by $G(x,y)$ the Green's function for the Laplacian on S^2 we have
$$u_t(x)=\int_{S^2}u_t d\mu_0-\int_{S^2}\Delta_0 u_t(y)\,G(x,y)d\mu_0(y)$$
and so, because $\nabla G(\cdot,y)$ is in L^1 for all y,
$$\int_{S^2}|\nabla_0 u_t|d\mu_0\leq\int_{S^2}|\Delta_0 u_t|d\mu_0\leq C$$
for some constant C independent of t. □

This lemma and identity (28) allow us to use an improvement of Aubin's inequality [1], due to Alice Chang and Paul Yang [4, Lemma 4.2], which says that under these conditions

$$4\pi\log\fint_{S^2}\exp(-4u_t)d\mu_0\leq 2\int_{S^2}|\nabla_0 u_t|^2 d\mu_0-4\int_{S^2}u_t d\mu_0+C$$
where C is a constant independent of t. Combining this with (30) we get that
$$\hat{u}_t\equiv\int_{S^2}u_t\,d\mu_0$$
is uniformly bounded from above. A uniform bound below follows trivially from $\exp(-2u_t)g_0$ having volume 4π. In order to bound the gradient term we integrate by parts again so that, for all t sufficiently large,
$$\begin{aligned}2\int_{S^2}|\nabla_0 u_t|^2 d\mu_0&=2\int_{S^2}\exp(-2u_t)(\hat{u}_t-u_t)d\mu_0+\int_{S^2}2(\hat{u}_t-u_t)Q_t d\mu_0\\&\leq 2\int_{S^2}(\exp(-2u_t)-2\pi)(\hat{u}_t-u_t)d\mu_0+1\\&\leq 2\pi\log\left((2\pi)^{-1}\int_{S^2}(\exp(-2u_t)-2\pi)\exp(2(\hat{u}_t-u_t))d\mu_0\right)\\&\quad+1\\&\leq 2\pi\log\int_{S^2}\exp(-4u_t)d\mu_0+C\\&\leq\int_{S^2}|\nabla_0 u_t|^2 d\mu_0+C,\end{aligned}$$
where C is a constant independent of t.

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