

CLASSIFICATION OF ALL 3-MANIFOLDS WITH YAMABE INVARIANT GREATER THAN \mathbb{RP}^3

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ABSTRACT. We show that, for all nonnegative integers k , ℓ , m and n , the Yamabe invariant of

$$\#k(\mathbb{RP}^3)\#\ell(\mathbb{RP}^2 \times S^1)\#m(S^2 \times S^1)\#n(S^2 \tilde{\times} S^1)$$

is equal to the Yamabe Invariant of \mathbb{RP}^3 , provided $k+\ell \geq 1$. We then complete the classification (started by Bray and the second author) of all closed 3-manifolds with Yamabe invariant greater than \mathbb{RP}^3 , by showing that such manifolds are either S^3 or finite connected sums $\#m(S^2 \times S^1)\#n(S^2 \tilde{\times} S^1)$ for $m+n \geq 1$, where $S^2 \tilde{\times} S^1$ is the nonorientable S^2 -bundle over S^1 .

A key ingredient is an Aubin's Lemma [3], which says that if the Yamabe constant is positive, then it is strictly less than the Yamabe constant of any of its non-trivial finite conformal coverings. This lemma, combined with the inverse mean curvature flow technique, and with analysis of the Green's function for the conformal Laplacian on specific finite and infinite Riemannian coverings, will allow us to construct a family of nice test functions on the finite coverings and thus prove the desired result.

1. INTRODUCTION

It is well known that, for a closed smooth manifold M of $\dim M = n \geq 3$, a Riemannian metric is Einstein if and only if it is a critical point of the normalized Einstein-Hilbert functional E on the space $\mathcal{M}(M)$ of all Riemannian metrics on M

$$E : \mathcal{M}(M) \rightarrow \mathbb{R}, \quad g \mapsto E(g) = \frac{\int_M R_g dV_g}{\text{Vol}_g(M)^{(n-2)/n}}.$$

Here, R_g , dV_g and $\text{Vol}_g(M)$ denote respectively the scalar curvature, the volume form of the metric g and the volume of (M, g) . Accompanied by this functional, a natural differential-topological invariant $Y(M)$ of M , called the Yamabe invariant, is defined as follows. Let $C = [g] := \{e^{2f} \cdot g \mid f \in C^\infty(M)\}$ and $\mathcal{C}(M)$ denote respectively the conformal class of a Riemannian metric g and the space of all conformal classes on M . Then, the Yamabe invariant $Y(M)$ is defined by

$$Y(M) := \sup_{C \in \mathcal{C}(M)} \inf_{g \in C} E(g).$$

Contrarily to what was stated in [7], this invariant was introduced independently by O. Kobayashi [15] and Schoen [30]. We note that it is sometimes called the σ -invariant.

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To put this definition in context, first recall that E is neither bounded from above or below for any M . However, to define a natural invariant of M arising from E , the study of the second variation of E done in [16, 31] (cf. [6]) leads to a mini-max method. At a critical metric $h \in \mathcal{M}_1(M) := \{g \in \mathcal{M}(M) \mid \text{Vol}_g(M) = 1\}$ of E , the metric h globally minimizes E in $\mathcal{M}_1(M) \cap [h]$. Moreover, if (M, h) is not conformally equivalent to $(S^n, [g_0])$, E is strictly, globally minimized among variations from $T_h(\mathcal{M}_1(M) \cap [h])$. Here, g_0 is the canonical metric of constant curvature one on the standard n -sphere S^n . On the other hand, if we consider the infinite dimensional subspace \mathbf{V} consisting of infinitesimal deformations orthogonal to $T_h[h]$ and the subspace of those arising from the diffeomorphism group $\text{Diff}(M)$ of M , E is locally maximized among those from \mathbf{V} modulo a finite dimensional subspace (possibly the zero subspace). These observations roughly imply that critical points are always saddle points over the moduli space $\mathcal{M}_1(M)/\text{Diff}(M)$.

The restriction of E to any given conformal class $[g]$ is always bounded from below. Hence, we can consider the following conformal invariant

$$\begin{aligned} Y(M, [g]) &:= \inf\{E(\tilde{g}) \mid \tilde{g} = u^{4/(n-2)} \cdot g \in [g], u > 0 \text{ in } C^\infty(M)\} \\ &= \inf \left\{ Q_g(u) := \frac{\int_M \left(\frac{4(n-1)}{n-2} |\nabla u|^2 + R_g u^2 \right) dV_g}{\left(\int_M u^{2n/(n-2)} dV_g \right)^{(n-2)/n}} \mid u > 0 \text{ in } C^\infty(M) \right\}, \end{aligned}$$

which is called the *Yamabe constant* of $[g]$. A remarkable theorem [37, 4, 5, 29] (cf. [22, 35]) of Yamabe, Trudinger, Aubin, and Schoen asserts that each conformal class $[g]$ contains metrics \check{g} , called *Yamabe metrics*, which realize the minimum

$$Y(M, [g]) = E(\check{g}).$$

These metrics must have constant scalar curvature $R_{\check{g}} = Y(M, [g]) \cdot \text{Vol}_{\check{g}}(M)^{-2/n}$. Aubin [4] (cf. [5]) also proved that

$$Y(M, C) \leq Y(S^n, [g_0]) = n(n-1)\text{Vol}_{g_0}(S^n)^{2/n}$$

for any $C \in \mathcal{C}(M)$. This implies that

$$Y(M) = \sup_{C \in \mathcal{C}(M)} Y(M, C) \leq Y(S^n) = n(n-1)\text{Vol}_{g_0}(S^n)^{2/n}.$$

There is a substantial body of work on Yamabe invariants and we will briefly review some significant results (see for the surveys [1, 17, 21, 36] for example). The Yamabe invariant $Y(M)$ is positive if and only if M carries a metric of positive scalar curvature. Some classification theorems for manifolds with positive Yamabe invariant have been obtained in [8, 9, 32, 33, 36] which show, for instance, that the n -torus T^n has $Y(T^n) = 0$. There are also some useful surgery theorems [15, 25, 26] for Yamabe invariants. LeBrun [20] proved that, for compact complex surfaces, the Yamabe invariant is positive, zero, or negative, if and only if the Kodaira dimension is $-\infty, 0$ or 1 , or 2 , respectively. LeBrun and collaborators [10, 14, 18, 19] also computed the Yamabe invariants for a large class of 4-manifolds, including Kähler surfaces X with $Y(X) < 0$ and $0 < Y(X) < Y(S^4)$. In [21], it is constructed a Kähler surface which admits Einstein metrics but the Yamabe invariant is not achieved by any Yamabe metric.

Regarding 3-manifolds, the following conjecture by Schoen [31] is one of the most fundamental unsolved problems related to Yamabe invariants.

Conjecture. *If M is a closed 3-manifold which admits a constant curvature metric h , then*

$$Y(M) = E(h).$$

Note that, for an Einstein metric h on an n -manifold N ($n \geq 3$), we have that $E(h) = Y(N, [h])$. If $E(h) > 0$, this comes from Obata's Theorem [24], while the nonpositive case comes from the uniqueness (up to a normalizing constant) of Yamabe metrics. Setting

$$Y_1 := Y(S^3) = 6 \cdot \text{Vol}_{g_0}(S^3)^{2/3}, \quad Y_k := \frac{Y_1}{k^{2/3}} \quad \text{for } k \in \mathbb{N},$$

this conjecture implies, in particular, that if S^3/Γ is a smooth quotient by a finite group Γ of order k , then $Y(S^3/\Gamma) = Y_k$. This conjecture is known to be true for any closed manifold uniformized by T^3 and for S^3 . In [7], Bray and the second author showed that it is also true for \mathbb{RP}^3 .

Theorem 1.1 (Bray–Neves). *Let M be a closed 3-manifold satisfying the following conditions:*

- (i) M is not diffeomorphic to S^3 ;
- (ii) M does not contain any non-separating embedded 2-sphere;
- (iii) M does not contain any two-sided real projective plane \mathbb{RP}^2 .

Then,

$$Y(M, [g]) \leq Y_2$$

for any conformal class $[g]$. Moreover, the equality holds if and only if (M, g) is conformally equivalent to \mathbb{RP}^3 with the metric h_0 of constant curvature one. In particular, $Y(M) \leq Y_2$.

Remark 1.2.

- (a) The result concerning the conformal equivalence to \mathbb{RP}^3 was not investigated in [7]. However, it follows easily from the characterization of the equality condition in the *Riemannian Penrose Inequality* [13, Section 8].
- (b) The condition in Theorem 1.1 (iii) is vacuous when M is orientable. A more general condition was stated in [7] and it includes the case where M is $\mathbb{RP}^2 \times S^1$. The necessity of such condition is addressed later.

As a consequence, they obtained the following corollaries.

Corollary 1.3.

$$Y(\mathbb{RP}^3) = Y_2, \quad Y(\mathbb{RP}^2 \times S^1) = Y_2.$$

Corollary 1.4. *The only prime 3-manifolds with Yamabe invariant greater than Y_2 are S^3 , $S^2 \times S^1$ and $S^2 \tilde{\times} S^1$ (the nonorientable S^2 bundle over S^1).*

All these manifolds have Yamabe invariant equal to Y_1 . The fact that $Y(S^2 \times S^1) = Y_1$ was proven independently by Kobayashi [15] and Schoen [31].

We now describe, in an informal way, the idea behind the proof of Theorem 1.1. Most of the definitions can be found in Section 3 but, for a more elaborated exposition, the reader is encouraged to see [7, Section 3 and 4]. All the statements regarding the inverse mean curvature flow are proven in [13].

The general approach was to construct a nice test function, not on (M, g) , but instead on a conformally equivalent scalar-flat, asymptotically flat manifold $(M - \{p\}, g_{AF})$, where p is a point in M . Bray and Neves observed that if this procedure is

done for $(\mathbb{R}\mathbb{P}^3, h_0)$, then the optimal test function u_0 on the corresponding scalar-flat, asymptotically flat manifold (which is the Schwarzschild manifold), has the same level sets as the inverse mean curvature flow starting on the unique outermost minimal surface.

The main idea was then to construct a test function u on the asymptotically flat manifold $M - \{p\}$ with the same level sets as the inverse mean curvature flow $\{\Sigma_t\}_{t \geq 0}$ starting on an outermost minimal surface. For each time t , u assumes on Σ_t exactly the value that u_0 assumes along the inverse mean curvature flow at time t on the Schwarzschild manifold.

They showed next that if the Hawking quasi-local mass is nondecreasing, which happens precisely if $\chi(\Sigma_t) \leq 2$ for all t , then $E(u^4 g_{AF}) \leq Y_2$. The topological assumptions on Theorem 1.1 assure that, if the initial condition has Euler characteristic less or equal than two, then this condition is preserved by the inverse mean curvature (see Theorem 3.4 and Lemma 3.5) and so the result follows.

A question left open in [7] was how to compute, for instance, $Y(\mathbb{R}\mathbb{P}^3 \# (S^2 \times S^1))$ or $Y((\mathbb{R}\mathbb{P}^2 \times S^1) \# (\mathbb{R}\mathbb{P}^2 \times S^1))$. The problem with the above idea is that, in this case, is impossible to guarantee the monotonicity of the Hawking quasi-local mass. The lack of monotonicity is a serious issue. As a matter of fact, we point out that the reason why this idea does not work for $S^2 \times S^1$ (which has the Yamabe invariant Y_1), is precisely because the Hawking quasi-local mass does not remain monotone (for some interval of time, the flow might become a disconnected union of 2-spheres). In this paper, we find a way to overcome this difficulty.

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2. MAIN RESULTS

We now describe the main results of this paper. We denote the connected sum of k copies of a closed manifold M by $\#kM$ and the nonorientable S^2 -bundle over S^1 by $S^2 \tilde{\times} S^1$.

The first result is a generalization of Theorem 1.1.

Theorem 2.1. (1) *Let M be a closed 3-manifold satisfying the following conditions:*

- (i) *M is not diffeomorphic to S^3 ;*
- (ii) *M does not contain any non-separating embedded 2-sphere;*
- (iii) *M does not contain any two-sided real projective plane $\mathbb{R}\mathbb{P}^2$.*

If L is a closed 3-manifold which has a non-separating 2-sphere, then

$$Y(M \# L, [g]) < Y_2$$

for any conformal class $[g]$. In particular, $Y(M \# L) \leq Y_2$.

- (2) *Let M be a closed 3-manifolds with a two-sided $\mathbb{R}\mathbb{P}^2$, then*

$$Y(M, [g]) < Y_2$$

for any conformal class $[g]$. In particular, $Y(M) \leq Y_2$.

The next corollary follows easily from Theorem 2.1.

Corollary 2.2. *For any closed 3-manifold L which is not diffeomorphic to S^3 , then*

$$Y(\mathbb{RP}^3 \# L, [g]) < Y_2 \quad \text{and} \quad Y((\mathbb{RP}^2 \times S^1) \# L, [\tilde{g}]) < Y_2$$

for any conformal classes $[g]$ and $[\tilde{g}]$, respectively. In particular,

$$Y(\mathbb{RP}^3 \# L) \leq Y_2 \quad \text{and} \quad Y((\mathbb{RP}^2 \times S^1) \# L) \leq Y_2.$$

Another immediate corollary is the following classification.

Corollary 2.3. *The only closed 3-manifolds with Yamabe invariant greater than Y_2 are*

$$S^3 \quad \text{and} \quad \#k(S^2 \times S^1) \# \ell(S^2 \tilde{\times} S^1),$$

where k and ℓ are any nonnegative integers with $k + \ell \geq 1$.

Proof. This follows from the Prime Decomposition Theorem, which says that any closed 3-manifold, not diffeomorphic to S^3 , can be decomposed as

$$M \# k(S^2 \times S^1) \# \ell(S^2 \tilde{\times} S^1) \# mQ$$

for some nonnegative integers k, ℓ , and m which is either 0 or 1. Here, M satisfies the conditions (ii) and (iii) of Theorem 2.1, and Q is a connected sum of prime 3-manifolds each of which has a two-sided \mathbb{RP}^2 (see [12]). Theorem 2.1 implies that M is S^3 and that $m = 0$. \square

For any two given manifolds M_1 and M_2 with nonnegative Yamabe invariants, Kobayashi [15] proved the following fundamental inequality for the connected sum $M_1 \# M_2$

$$Y(M_1 \# M_2) \geq \min\{Y(M_1), Y(M_2)\}.$$

Recall that $Y(S^2 \times S^1) = Y_1 = Y(S^2 \tilde{\times} S^1)$. Then, we get

Corollary 2.4. *For any nonnegative integers k, ℓ, m, n with $k + \ell \geq 1$, we have*

$$Y(\#k\mathbb{RP}^3 \# \ell(\mathbb{RP}^2 \times S^1) \# m(S^2 \times S^1) \# n(S^2 \tilde{\times} S^1)) = Y_2.$$

3. TOOL BOX REVIEW

The rest of the paper is devoted to the proof of Theorem 2.1, and so we start by briefly reviewing the indispensable key results. The main references are [7], [13], and [22] for the first two subsections and [3] for the last subsection.

3.1. Yamabe constants of asymptotically flat 3-manifolds. For any closed conformal 3-manifold (M, C) with positive Yamabe constant, there exists a scalar-flat, asymptotically flat metric $g_{AF} \in C$ on $M - \{\text{point}\}$. It turns out that this asymptotically flat manifold is more convenient for studying Yamabe constants. This has played a central role in [29], [7], and it will also be important in our approach.

Definition 3.1. A Riemannian 3-manifold (N, h) is said to be *asymptotically flat* if there exists a compact set $K \subset N$ such that $N - K$ is diffeomorphic to $\mathbb{R}^3 - \{|x| \leq 1\}$ and that, in the coordinate chart defined by this diffeomorphism, the metric $h = \sum_{i,j} h_{ij}(x) dx^i dx^j$ satisfies the following:

$$h_{ij} = \delta_{ij} + O(|x|^{-1}), \quad h_{ij,k} = O(|x|^{-2}), \quad h_{ij,kl} = O(|x|^{-3}).$$

Let (M, g) be a closed 3-manifold with positive scalar curvature. Define the *conformal Laplacian* to be

$$L_g := -8\Delta_g + R_g,$$

and consider the normalized Green's function G_p for L_g with a pole at $p \in M$, that is,

$$L_g G_p = c_0 \cdot \delta_p \quad \text{on } M \quad \text{and} \quad \lim_{q \rightarrow p} \text{dist}(p, q) G_p(q) = 1.$$

Here, $c_0 > 0$ and δ_p stand respectively for a specific universal positive constant and the Dirac δ -function at p . The maximum principle implies that G_p is positive on $M^* := M - \{p\}$, and hence $g_{AF} := G_p^4 g$ defines a scalar-flat, asymptotically flat metric on M^* (see [22, Section 6]).

For the noncompact manifold (M^*, g_{AF}) , the Yamabe constant is defined by

$$\begin{aligned} Y(M^*, [g_{AF}]) &:= \inf \left\{ \frac{\int_{M^*} 8|\nabla u|^2 dV_{g_{AF}}}{\left(\int_{M^*} u^6 dV_{g_{AF}}\right)^{1/3}} \mid u \not\equiv 0 \text{ in } C_c^\infty(M^*) \right\} \\ &= \inf \left\{ \frac{\int_{M^*} 8|\nabla u|^2 dV_{g_{AF}}}{\left(\int_{M^*} u^6 dV_{g_{AF}}\right)^{1/3}} \mid u \not\equiv 0 \text{ in } W^{1,2}(M^*, g_{AF}) \right\}, \end{aligned}$$

as well as that of a general open conformal manifold, cf. [34]. From the Sobolev embedding theorem for asymptotically flat manifolds, we note that

$$W^{1,2}(M^*, g_{AF}) \hookrightarrow L^6(M^*, g_{AF}).$$

It is readily seen that $Y(M^*, [g_{AF}]) = Y(M, [g])$.

3.2. Inverse Mean Curvature Flow. Like in [7], the inverse mean curvature flow developed by Huisken and Ilmanen in [13] will be a crucial tool to prove Theorem 2.1. The main difference this time is its application to a *family* of asymptotically flat manifolds *with boundary*. We now introduce some of their terminology and present the main results of [13] that will be necessary for our purposes.

In this subsection, (N, h) always stands for an asymptotically flat 3-manifold with boundary ∂N consisting of a union of minimal surfaces. For a C^1 surface Σ contained in N , $H \in L^1_{\text{loc}}(\Sigma)$ is called the *weak mean curvature* of Σ provided

$$\int_{\Sigma} \text{div}_N(X) dA_h = \int_{\Sigma} H \langle X, \nu \rangle dA_h$$

for all vector fields X with compact support on N , where ν is the exterior unit normal vector. If Σ is smooth, $-H\nu$ coincides with the usual mean curvature of Σ .

A precompact set E of N is called a *minimizing hull in N* if it minimizes area on the outside, that is,

$$|\partial^* E| \leq |\partial^* F|$$

for any F containing E such that $F - E \subset\subset N$, where $\partial^* F$ is the reduced boundary of F .

For any surface Σ of (N, h) , $|\Sigma| = |\Sigma|_h$ stands for the 2-dimensional Hausdorff measure of Σ . Huisken and Ilmanen [13, Section 3 and 6] proved the following existence result for inverse mean curvature flow.

Theorem 3.2 (Existence). *Let E_0 be an open precompact minimizing hull in N with smooth boundary ∂E_0 . Then, there exists a precompact locally Lipschitz ϕ on N satisfying:*

- (i) For $t \geq 0$, $\Sigma_t := \partial\{\phi < t\}$ defines an increasing family of $C^{1,\alpha}$ surfaces ($0 < \alpha < 1$) such that $\Sigma_0 = \partial E_0$;
- (ii) For almost all $t \geq 0$, the weak mean curvature of Σ_t is $|\nabla\phi|_{\Sigma_t}$;
- (iii) For almost all $t \geq 0$,

$$|\nabla\phi|_{\Sigma_t} \neq 0 \quad \text{for almost all } x \in \Sigma_t$$

(with respect to the surface measure) and

$$|\Sigma_0|e^t \leq |\Sigma_t| \leq (|\Sigma_0| + |\partial N|)e^t \quad \text{for all } t \geq 0.$$

When ϕ is smooth with nonvanishing gradient, (ii) implies that the surfaces $\{\phi = t\}_{t \geq 0}$ solve the inverse mean curvature flow with initial condition ∂E_0 . More precisely, in this situation, the speed of this flow is given by

$$|\nabla\phi|^{-1}\nu,$$

where ν is the exterior unit normal to $\{\phi = t\}$.

Definition 3.3. For a compact C^1 hypersurface Σ with weak mean curvature H in $L^2(\Sigma)$, the *Hawking quasi-local mass* is defined to be

$$m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{(16\pi)^3}} \left(16\pi - \int_{\Sigma} H^2 dA_h \right).$$

For the rest of this subsection, assume that E_0 is an open connected precompact minimizing hull in N , with smooth boundary ∂E_0 , and that $\{\Sigma_t\}_{t \geq 0}$ is a weak solution for the inverse mean curvature flow with initial condition $\Sigma_0 = \partial E_0$.

Under the condition that (N, h) has nonnegative scalar curvature, we have the following remarkable property for the weak inverse mean curvature flow [13].

Theorem 3.4 (Geroch Monotonicity). *Let (N, h) be an asymptotically flat 3-manifold with nonnegative scalar curvature. The Hawking quasi-local mass $m_H(\Sigma_t)$ is a nondecreasing function of t provided $\chi(\Sigma_t) \leq 2$ for all t .*

The monotonicity of the Hawking quasi-local mass will be of major importance in proving Theorem 2.1. The next Lemma gives topological conditions on N that guarantee this monotonicity.

Lemma 3.5. *Assume that besides the assumption in Theorem 3.4, N does not contain any non-separating embedded 2-sphere and any two-sided real projective plane. Then,*

$$\chi(\Sigma_t) \leq 2$$

for all $t \geq 0$, provided that ∂E_0 is connected.

Proof. Let ϕ be the function defining the weak solution $\{\Sigma_t\}_{t \geq 0}$. According to [13, Section 4 and 6], the connectedness of E_0 combined with the assumptions implies that both $\{\phi < t\}$ and $\{\phi > t\}$ remain connected for all t .

It is also proven in [13] that, for each $t > 0$, Σ_t can be approximated by earlier surfaces Σ_s for which $\nabla\phi$ never vanishes. Hence, without loss of generality, we can assume that

$$\Sigma_t = \{\phi = t\} = \partial\{\phi > t\} = \partial\{\phi < t\}.$$

The connectedness of the two sets on the right of the above expression implies that Σ_t is two-sided. We now show that $\chi(\Sigma_t) > 2$ is impossible.

If Σ_t contains a 2-sphere, then this 2-sphere has to bound a connected region, and hence $\chi(\Sigma_t) = 2$ because $\{\phi < t\}$ is connected. If Σ_t contains a real projective plane, then it has to be one-sided. This contradicts that Σ_t is two-sided. \square

3.3. Aubin's Lemma. The next Lemma, due to Aubin [3], is a key tool to prove Theorem 2.1. For the sake of completeness, we give its proof.

Lemma 3.6 (Aubin). *Let (M, g) be a closed n -manifold with positive Yamabe constant and (M_k, g_k) a non-trivial k -fold Riemannian covering (i.e., $k \geq 2$). Then,*

$$Y(M, [g]) < Y(M_k, [g_k]).$$

Proof. Let γ_i ($i = 1, \dots, k$) be the deck transformations of M_k corresponding to the k -fold covering and u a normalized positive function that minimizes the Yamabe quotient Q_{g_k} , that is,

$$Q_{g_k}(u) = E(u^{4/(n-2)}g_k) = Y(M_k, [g_k]) \quad \text{and} \quad \int_{M_k} u^{2n/(n-2)} dV_{g_k} = 1.$$

Setting a positive function v on M_k by

$$v := \sum_{i=1}^k u \circ \gamma_i,$$

we get that

$$L_k v = Y(M_k, [g_k]) \sum_{i=1}^k (u \circ \gamma_i)^{(n+2)/(n-2)},$$

where $L_k := -\frac{4(n-1)}{n-2}\Delta_{g_k} + R_{g_k}$ is the conformal Laplacian with respect to g_k .

Define v_0 to be the function on M whose lift to M_k is v . Then,

$$\begin{aligned} Q_{g_k}(v_0) &= k^{-2/n} Q_{g_k}(v_0) \\ &= k^{-2/n} \frac{\int_{M_k} v(L_k v) dV_{g_k}}{\left(\int_{M_k} v^{2n/(n-2)} dV_{g_k}\right)^{(n-2)/n}} \\ (1) \quad &= k^{-2/n} Y(M_k, [g_k]) \frac{\int_{M_k} v \left(\sum_{i=1}^k (u \circ \gamma_i)^{(n+2)/(n-2)}\right) dV_{g_k}}{\left(\int_{M_k} v^{2n/(n-2)} dV_{g_k}\right)^{(n-2)/n}}. \end{aligned}$$

Using Hölder's inequality twice and the following strict inequality:

$$\left(\sum_{i=1}^k a_i^p\right)^{1/p} < \sum_{i=1}^k a_i \quad \text{for } k \geq 2, p > 1, a_i > 0 (i = 1, \dots, k),$$

we obtain that

$$\begin{aligned}
& \int_{M_k} v \left(\sum_{i=1}^k (u \circ \gamma_i)^{(n+2)/(n-2)} \right) dV_{g_k} \\
& \leq \int_{M_k} v \left(\sum_{i=1}^k (u \circ \gamma_i)^{2n/(n-2)} \right)^{2/n} \left(\sum_{i=1}^k (u \circ \gamma_i)^{n/(n-2)} \right)^{(n-2)/n} dV_{g_k} \\
(2) \quad & < \int_{M_k} v^2 \left(\sum_{i=1}^k (u \circ \gamma_i)^{2n/(n-2)} \right)^{2/n} dV_{g_k} \\
& \leq \left(\int_{M_k} v^{2n/(n-2)} dV_{g_k} \right)^{(n-2)/n} \left(\int_{M_k} \sum_{i=1}^k (u \circ \gamma_i)^{2n/(n-2)} dV_{g_k} \right)^{2/n} \\
& = k^{2/n} \left(\int_{M_k} v^{2n/(n-2)} dV_{g_k} \right)^{(n-2)/n}.
\end{aligned}$$

It then follows from (1) and (2) that

$$Q_g(v_0) < Y(M_k, [g_k]),$$

and hence

$$Y(M, [g]) < Y(M_k, [g_k]).$$

□

4. OVERALL STRATEGY

In this section, we describe our approach to Theorem 2.1-(1) in the simple case that $L = S^2 \times S^1$. Let

$$N := M \# (S^2 \times S^1),$$

where M is a closed 3-manifold satisfying the conditions (i)-(iii) in Theorem 2.1. Without loss of generality, we may assume that $Y(N, [g]) > 0$ and g is a Yamabe metric with $\text{Vol}_g(N) = 1$. Note that $R_g \equiv Y(N, [g]) > 0$.

Consider Riemannian coverings of (N, g) defined in the following way. Take a smooth loop c in $S^2 \times S^1$ whose homotopy class $\langle c \rangle$ generates $\pi_1(S^2 \times S^1)$. We regard the loop c as a loop in N . Then, (N_k, g_k) and (N_∞, g_∞) denote, respectively, the k -fold Riemannian covering and the infinite Riemannian covering associated to $\langle c \rangle$. Note that, topologically,

$$N_k = \#k M \# (S^2 \times S^1), \quad N_\infty = \#_1^\infty M \# (S^2 \times \mathbb{R}).$$

From Aubin's Lemma 3.6, we note that, for all $k \geq 2$,

$$Y(N, [g]) < Y(N_k, [g_k]),$$

and this implies that

$$Y(N, [g]) < \limsup_{k \rightarrow \infty} Y(N_k, [g_k]).$$

The strict inequality comes from the fact that we can choose an increasing subsequence of $\{Y(N_k, [g_k])\}_{k \geq 1}$, for instance, $\{Y(N_{2^m}, [g_{2^m}])\}_{m \geq 1}$. Moreover, modifying the techniques in [15, Section 6] and [2, Sections 5 and 6], we can prove the following

$$\lim_{k \rightarrow \infty} Y(N_k, [g_k]) = Y(N_\infty, [g_\infty]).$$

However, the above strict inequality is sufficient for our proof that $Y(N, [g]) < Y_2$.

Unlike each N_k , the noncompact manifold N_∞ satisfies the conditions (i)-(iii) of Theorem 1.1, and hence, at least heuristically, this suggests that

$$\limsup_{k \rightarrow \infty} Y(N_k, [g_k]) = Y(N_\infty, [g_\infty]) \leq Y_2.$$

It should be also pointed out that the scalar-flat, asymptotically flat manifold conformally equivalent to $(N_\infty - \{\text{point}\}, g_\infty)$, will have two singularities created by the two ends of N_∞ . Therefore, we can not apply the inverse mean curvature flow technique directly to this manifold. Instead, we apply it to the family of asymptotically flat manifolds without singularities, but with boundary, arising from $(N_k^*, g_{k,AF})$ below. Then, we will prove Theorem 4.1 below.

Fix a point $p_\infty \in N_\infty$ and call its projection to each N_k also by p_∞ . Let G_k be the normalized Green's function on N_k for the conformal Laplacian L_{g_k} with the pole at p_∞ . Consider the following scalar-flat, asymptotically flat metrics

$$g_{k,AF} := G_k^4 g_k \quad \text{on} \quad N_k^* := N_k - \{p_\infty\}.$$

Theorem 4.1.

$$\limsup_{k \rightarrow \infty} Y(N_k^*, [g_k]) \leq Y_2.$$

The proof is given in the next two sections. Some extra difficulties arise for the general case because the infinite Riemannian covering will have infinitely many ends. Fortunately, the method developed here and in Sections 5, 6 is robust enough to handle those cases. The proof will be given in Section 7.

5. PROOF OF MAIN THEOREM: SIMPLE CASE

Under the same assumptions of Section 4, consider the normalized minimal positive Green's function G_∞ on (N_∞, g_∞) for the conformal Laplacian L_{g_∞} with the pole at p_∞ (cf. [34, Section 1], [2, Section 6]). This function is defined on $N_\infty^* := N_\infty - \{p_\infty\}$, and it is essential to control the geometry of $(N_k^*, g_{k,AF})$. All the statements in this section regarding these manifolds follow from the good understanding we have for G_∞ .

We start by collecting the properties G_∞ and G_k on Lemma 5.1. After this, we describe the main idea behind the proof of Theorem 4.1 and state four auxiliary lemmas, whose proofs will be given in the next section. We then prove Theorem 4.1.

For reasons of simplicity, we set the following notation for the rest of the paper. Any quantity Q that depends on $g_{k,AF}$ or g_k will be denoted by $Q_{k,AF}$ or Q_k , respectively. If it is obvious from the context which metric is being considered, we will just denote it by Q .

Let γ be the deck transformation of N_∞ corresponding to the homotopy class of $\langle c \rangle \in \pi_1(N)$, that is,

$$N_k = N_\infty / \langle \gamma^k \rangle \quad \text{for} \quad k \geq 1.$$

For simplicity, we often identify a set in N_∞ with its projection to each N_k and a function on N_k with its lift to N_∞ . For instance, G_k denotes both the normalized Green's function on N_k and its lift to N_∞ .

The standard arguments for G_k and G_∞ in [34], [11], [2] imply the following lemma. Here, note that each g_k and g_∞ are metrics of positive constant scalar curvature $Y(N, [g]) > 0$.

Lemma 5.1.

(i) For all $k \geq 1$,

$$G_k = \sum_{j \in \mathbb{Z}} G_\infty \circ \gamma^{jk} \quad \text{on } N_\infty;$$

(ii) For all $k \geq 2$,

$$0 < G_\infty < G_k < G_1,$$

and G_k converges uniformly in C^ℓ (for every $\ell \geq 1$) to G_∞ on every compact subset of N_∞^* ;

(iii) For every compact set K of N_∞^* ,

$$\lim_{j \rightarrow \infty} \sup \{G_\infty(\gamma^j(x)) \mid x \in K\} = 0;$$

(iv) For any open set O containing p_∞ , there exists a constant L independent of k such that, for all $k \geq 1$,

$$|\nabla G_k|_k \leq L G_k \quad \text{on } N_k - O;$$

(v)

$$\int_{N_\infty} G_\infty dV_\infty < \infty.$$

Take a smooth connected embedded 2-sphere S in N_∞ with $p_\infty \in S$ such that the projection from S into N is injective and its image in N and c intersects transversely at one point. Then, consider the embedded 2-spheres $F_k := \gamma^{[k/2]}(S) \sqcup \gamma^{[k/2]-k}(S)$ in N_∞ for each $k \geq 2$. Denote the projection of F_k ($\subset N_\infty$) to each N_k also by F_k . Note that F_k is connected and that

$$\lim_{k \rightarrow \infty} \text{dist}_{g_k}(p_\infty, F_k) = \infty.$$

The topological condition on M implies that all non-separating 2-spheres of N_k^* are cobordant to each other. Such 2-spheres correspond to a “neck” on N_k^* . The properties (ii) and (iii) say that $(N_k^*, g_{k,AF})$ has, for k large enough, a very small “neck”. This is the content of the following Lemma.

Lemma 5.2. *For every k , there exists an area-minimizing, non-separating, connected embedded 2-sphere S_k of $(N_k^*, g_{k,AF})$ such that*

$$\lim_{k \rightarrow \infty} |S_k|_{k,AF} = 0,$$

where $|S_k|_{k,AF}$ denotes the area of S_k with respect to $g_{k,AF}$.

Set X_k to be the metric completion of

$$N_k^* - S_k = N_k - (\{p_\infty\} \cup S_k),$$

and denote the smooth metric extension of $g_{k,AF}$ to X_k by the same symbol. Then, $(X_k, g_{k,AF})$ becomes an asymptotically flat manifold with minimal boundary consisting of two 2-spheres. This corresponds to “cutting out the small neck” of N_k^* . The advantage of working with X_k rather than N_k^* is that, as can be easily seen, any embedded 2-sphere of X_k bound a connected region. Therefore, by Theorem 3.4 and Lemma 3.5, the monotonicity of the Hawking quasi-local mass is preserved for each solution of inverse mean curvature flow on $(X_k, g_{k,AF})$, provided that its initial condition is the boundary of a connected region. Taking into account that the area $|\partial X_k|_{k,AF}$ of ∂X_k goes to zero as k goes to infinity, we expect to construct a nice family of test functions u_k with Yamabe quotient $Q_{g_{k,AF}}(u_k)$ approaching Y_2 .

Using the inverse mean curvature flow techniques in the same spirit as in [7], we have the following result.

Lemma 5.3. *Let (X, h) be an asymptotically flat 3-manifold with minimal boundary ∂X . Assume further there exists a weak solution for the inverse mean curvature flow, with initial condition a minimal surface Σ_0 , that keeps the Hawking quasi-local mass nondecreasing. Then, there exists a function v in $W^{1,2}(X, h)$ such that $0 \leq v \leq 1$ on X and*

$$\frac{\int_X 8|\nabla v|^2 dV_h}{\left(\int_X v^6 dV_h\right)^{1/3}} \leq Y_2 + \theta \left(|\partial X| |\Sigma_0|^{-1} \right),$$

where θ is a continuous function independent of (X, h) and vanishing at zero.

An *outermost minimal surface* of an asymptotically flat 3-manifold (X, h) is a minimal surface (possibly disconnected) which encloses all other compact minimal surfaces. Since each N_k^* is not diffeomorphic to \mathbb{R}^3 , a result due to Meeks, Simon and Yau [23] implies that $(X_k, g_{k,AF})$ has a unique outermost minimal surface Ω which is the disjoint union of a finite number of weakly embedded 2-spheres (i.e., limits of uniformly smooth embedded 2-spheres). Moreover, the region exterior to the outermost minimal surface is \mathbb{R}^3 minus a finite number of closed balls (see [13, Section 4] for details). The next Lemma guarantees that one of the connected components of the outermost minimal surface has area bounded below independently of k .

Lemma 5.4. *The outermost minimal surface of each $(X_k, g_{k,AF})$ has a connected component $\Sigma(k)$ with area bounded from below independently of k .*

Remark 5.5. The surface $\Sigma(k)$ is a minimizing hull because it is contained in the outermost minimal surface. If $\Sigma(k)$ is a 2-sphere, then it is the boundary of a precompact connected open set, and hence it can be used as an initial condition for the inverse mean curvature flow (see Theorem 3.2). If $\Sigma(k)$ is a one-sided $\mathbb{R}P^2$, then it is the limit of uniformly smooth embedded 2-spheres that are minimizing hulls bounding a precompact open connected set. We then take the limit of the weak solutions to inverse mean curvature flow starting at each of these 2-spheres, in order to get a weak solution to the inverse mean curvature flow with initial condition $\Sigma(k)$. This can be done rigorously using the compactness theorem and the a priori bounds proved in [13, Sections 2 and 3].

It can be easily seen that Theorem 4.1 is false when M is either S^3 or $S^2 \times S^1$, in which case N is $S^2 \times S^1$ or $(S^2 \times S^1) \# (S^2 \times S^1)$, respectively. The technical reason why this proof doesn't work for those cases is because Lemma 5.4 doesn't hold. It is worthwhile to point out that, among the auxiliary lemmas we state in this section, Lemma 5.4 is the only where the topological restrictions on M play a role.

The function v given by Lemma 5.3 may not vanish on the boundary. Therefore, if we want to apply this result to our sequence of asymptotically flat manifolds $(X_k, g_{k,AF})$, we need to multiply the resulting functions by some cut-off function vanishing on the boundary ∂X_k . Otherwise, we could not extend it to be a valid test function on N_k^* . This is a delicate issue because, in general, the multiplication by a cut-off function makes the Yamabe quotient increase strictly. Fortunately, using Lemma 5.1 (iv) and (v), we can prove the following result.

Lemma 5.6. *For every $\varepsilon > 0$, there exists a sequence of Lipschitz functions η_k ($0 \leq \eta_k \leq 1$) such that*

$$\eta_k \equiv 0 \quad \text{on } \partial X_k, \quad \eta_k \equiv 1 \quad \text{outside a compact set of } X_k,$$

and for all k sufficiently large

$$\int_{X_k} |\nabla \eta_k|^2 dV_{k,AF} \leq \varepsilon, \quad \text{Vol}_{k,AF}(\{x \in X_k \mid \nabla \eta_k \neq 0\}) \leq \varepsilon.$$

We can now prove Theorem 4.1.

Proof of Theorem 4.1. The proof follows if we find a sequence of test functions u_k vanishing on X_k such that

$$\limsup_{k \rightarrow \infty} \frac{\int_{X_k} 8|\nabla u_k|^2 dV_{k,AF}}{\left(\int_{X_k} u_k^6 dV_{k,AF}\right)^{1/3}} \leq Y_2$$

Consider the minimal surface $\Sigma(k)$ given by Lemma 5.4. There exists a solution to the inverse mean curvature flow with initial condition $\Sigma(k)$ (Remark 5.5), and the topological condition on X_k implies the monotonicity of the Hawking quasi-local mass along the solution. From Lemma 5.3, we have the existence, for all k , of v_k ($0 \leq v_k \leq 1$) in $W^{1,2}(X_k, g_{k,AF})$ such that

$$\frac{\int_{X_k} 8|\nabla v_k|^2 dV_{k,AF}}{\left(\int_{X_k} v_k^6 dV_{k,AF}\right)^{1/3}} \leq Y_2 + \theta \left(|\partial X_k|_{k,AF} |\Sigma(k)|_{k,AF}^{-1} \right).$$

Take any small $\varepsilon > 0$ and let η_k ($0 \leq \eta_k \leq 1$) denote the Lipschitz function given in Lemma 5.6 for all sufficiently large k . Consider the admissible nonnegative function in $W^{1,2}(N_k^*, g_{k,AF})$ given by

$$u_k := \eta_k v_k.$$

We then have

$$\begin{aligned} \int_{X_k} |\nabla u_k|^2 dV_{k,AF} &\leq \int_{X_k} |\nabla v_k|^2 dV_{k,AF} + 2\sqrt{\int_{X_k} |\nabla v_k|^2 dV_{k,AF}} \sqrt{\int_{X_k} |\nabla \eta_k|^2 dV_{k,AF}} \\ &\quad + \int_{X_k} |\nabla \eta_k|^2 dV_{k,AF} \\ &\leq q_k \left(\int_{X_k} v_k^6 dV_{k,AF} \right)^{1/3} + 2\sqrt{\varepsilon q_k} \left(\int_{X_k} v_k^6 dV_{k,AF} \right)^{1/6} + \varepsilon, \end{aligned}$$

where

$$q_k := Y_2 + \theta \left(|\partial X_k|_{k,AF} |\Sigma(k)|_{k,AF}^{-1} \right) > 0.$$

Following the proof of Lemma 5.3 we see that

$$\int_{X_k} v_k^6 dV_{k,AF} \geq |\Sigma(k)|_{k,AF}^{3/2} \left(C_0 + \theta \left(|\partial X_k|_{k,AF} |\Sigma(k)|_{k,AF}^{-1} \right) \right) \geq C > 0$$

for some positive constants C_0, C independent of k , where the second inequality comes from combining Lemma 5.2 with the lower bound on $|\Sigma(k)|_{k,AF}$ (Lemma 5.4), and with the continuity of θ . Hence, using the properties of η_k given by Lemma 5.6, we obtain

$$\begin{aligned} \frac{\int_{X_k} v_k^6 dV_{k,AF}}{\int_{X_k} u_k^6 dV_{k,AF}} &\leq \frac{\int_{X_k} v_k^6 dV_{k,AF}}{\int_{X_k} v_k^6 dV_{k,AF} - \text{Vol}_{k,AF}(\{x \in X_k \mid \nabla \eta_k \neq 0\})} \\ &\leq \frac{C}{C - \varepsilon} \end{aligned}$$

for some positive constant C independent of k , and hence

$$\frac{\int_{X_k} |\nabla u_k|^2 dV_{k,AF}}{\left(\int_{X_k} u_k^6 dV_{k,AF}\right)^{1/3}} \leq q_k \left(\frac{C}{C - \varepsilon}\right)^{1/3} + 2\sqrt{q_k} \frac{\sqrt{\varepsilon} C^{1/6}}{(C - \varepsilon)^{1/3}} + \frac{\varepsilon}{(C - \varepsilon)^{1/3}}.$$

Combining Lemma 5.4 with the fact that $|\partial X_k|_{AF}$ converges to zero (Lemma 5.2), we have

$$\lim_{k \rightarrow \infty} q_k = Y_2.$$

The proof follows by letting ε go to zero. \square

6. PROOFS OF AUXILIARY LEMMAS

We first prove Lemma 5.2, then Lemma 5.3, Lemma 5.4, and finally Lemma 5.6.

Proof of Lemma 5.2. Consider the same smooth connected embedded 2-spheres S in N_∞ and F_k in N_k^* that were constructed right before starting Lemma 5.2. Then

$$|\gamma^j(S)|_{\infty,AF} = \int_S G_\infty^4 \circ \gamma^j dA_\infty.$$

Hence, from Lemma 5.1 (ii) and (iii),

$$\lim_{k \rightarrow \infty} |F_k|_{k,AF} = 0.$$

We minimize area in the isotopy class of F_k . The theory developed in [23] says that the area minimizer exists and is a smooth disjoint union of 2-spheres and one-sided real projective planes. Moreover, according to [23, Section 3], the connected components of the area minimizer can be jointed by arbitrarily thin tubes so that the resulting surface is isotopic to F_k . Hence, the area minimizer has to contain a smooth minimal 2-sphere S_k that is non-separating.

The desired assertion follows from the obvious fact that $|S_k|_{k,AF} \leq |F_k|_{k,AF}$. \square

Proof of Lemma 5.3. We use the same idea of [7] and construct a test function v whose level sets coincide with the weak solution of the inverse mean curvature flow. This time we have to take into account that X has a minimal boundary ∂X .

Let ϕ be the weak solution to the inverse mean curvature flow with initial condition Σ_0 and set $\Sigma_t := \partial\{\phi < t\}$. Like in [7], v is defined to be

$$v(x) := \begin{cases} f(0) & \text{if } \phi(x) \leq 0, \\ f(\phi(x)) & \text{if } \phi(x) > 0, \end{cases}$$

where

$$f(t) = \frac{1}{\sqrt{2e^t - e^{t/2}}}.$$

After a simple computation using the coarea formula [7, Section 6], we get

$$(3) \quad \int_X |\nabla v|^2 dV_h = \int_0^\infty f'(t)^2 \int_{\Sigma_t} H dA_h dt,$$

$$(4) \quad \int_X v^6 dV_h \geq \int_0^\infty f(t)^6 |\Sigma_t|^2 \left(\int_{\Sigma_t} H dA_h \right)^{-1} dt.$$

Next, we find an upper bound for $\int_{\Sigma_t} H dA$.

Lemma 6.1. *For all $t \geq 0$,*

$$\int_{\Sigma_t} H dA_h \leq \sqrt{16\pi|\Sigma_0|(e^t - e^{t/2})} + \sqrt{16\pi|\partial X|e^t}.$$

Proof. Using the monotonicity of the Hawking quasi-local mass, we have

$$\begin{aligned} \sqrt{\frac{|\Sigma_t|}{(16\pi)^3}} \left(16\pi - \int_{\Sigma_t} H^2 dA_h \right) &= m_H(\Sigma_t) \\ &\geq m_H(\Sigma_0) \\ &= \sqrt{\frac{|\Sigma_0|}{16\pi}}. \end{aligned}$$

Hence, it follows from Hölder's inequality that

$$\begin{aligned} \int_{\Sigma_t} H dA_h &\leq \sqrt{|\Sigma_t| \int_{\Sigma_t} H^2 dA_h}, \\ &\leq \sqrt{16\pi \left(|\Sigma_t| - \sqrt{|\Sigma_t||\Sigma_0|} \right)}, \end{aligned}$$

and thus, by Theorem 3.2 (iii),

$$\begin{aligned} \int_{\Sigma_t} H dA_h &\leq \sqrt{16\pi|\Sigma_0|(e^t - e^{t/2})} + \sqrt{16\pi|\partial X|e^t} \\ &\leq \sqrt{16\pi|\Sigma_0|(e^t - e^{t/2})} + \sqrt{16\pi|\partial X|e^t}. \end{aligned}$$

□

Combining Lemma 6.1 first with (3) and then with (4), we get

$$\int_X |\nabla v|^2 dV_h \leq \sqrt{16\pi|\Sigma_0|} \left(\int_0^\infty f'(t)^2 \sqrt{e^t - e^{t/2}} dt + C|\partial X||\Sigma_0|^{-1} \right),$$

for some positive constant C . Then, Theorem 3.2 (iii) implies the following

$$\begin{aligned} \int_X v^6 dV_h &\geq \int_0^\infty \frac{f(t)^6 |\Sigma_t|^2}{\sqrt{16\pi |\Sigma_0| (e^t - e^{t/2})} + \sqrt{16\pi |\partial X| e^t}} dt \\ &\geq |\Sigma_0|^{3/2} \int_0^\infty \frac{f(t)^6 e^{2t}}{\sqrt{16\pi (e^t - e^{t/2})} + \sqrt{16\pi |\partial X| |\Sigma_0|^{-1} e^t}} dt \\ &= |\Sigma_0|^{3/2} \left(\int_0^\infty \frac{f(t)^6 e^{2t}}{\sqrt{16\pi (e^t - e^{t/2})}} dt + \theta \left(|\partial X| |\Sigma_0|^{-1} \right) \right), \end{aligned}$$

where θ is some function independent of the metric. It can be easily seen that θ is continuous and vanishes when $|\partial X| |\Sigma_0|^{-1}$ is zero. According to [7], we know that

$$\frac{(16\pi)^{2/3} \int_0^\infty f'(t)^2 \sqrt{e^t - e^{t/2}} dt}{\left(\int_0^\infty f(t)^6 e^{2t} (e^t - e^{t/2})^{-1/2} dt \right)^{1/3}} = \frac{Y_2}{8},$$

and hence

$$\begin{aligned} \frac{\int_X |\nabla v|^2 dV_h}{\left(\int_X v^6 dV_h \right)^{1/3}} &\leq \frac{(16\pi)^{2/3} \int_0^\infty f'(t)^2 \sqrt{e^t - e^{t/2}} dt}{\left(\int_0^\infty f(t)^6 e^{2t} (e^t - e^{t/2})^{-1/2} dt \right)^{1/3}} + \theta \left(|\partial X| |\Sigma_0|^{-1} \right) \\ &= \frac{Y_2}{8} + \theta \left(|\partial X| |\Sigma_0|^{-1} \right), \end{aligned}$$

where θ is a continuous function vanishing at zero and independent of the metric h . \square

Proof of Lemma 5.4. We argue by contradiction. Suppose that there exists an infinite number of asymptotically flat 3-manifolds $(X_k, g_{k,AF})$ (which we still index by k), such that the area of every connected component of the outermost minimal surface $\Omega_k \subset X_k$ goes to zero when k goes to infinity.

We show the existence of a connected component $\Omega_{k,0}$ of Ω_k for which there exist two subsets A_k, B_k of X_k , both intersecting $\Omega_{k,0}$, and such that the distance between A_k and B_k (with respect to $g_{k,AF}$) is bounded from below independently of k . Thus, $\Omega_{k,0}$ will have, for k large enough, very small area and diameter bounded from below. Moreover, it will be apparent from the construction of A_k and B_k that the complement of both sets in X_k has bounded sectional curvature independently of k . A simple application of the monotonicity formula shows that $\Omega_{k,0}$ cannot have very small area (cf. [27], [28]).

First, we construct A_k . Take a positive constant α such that

$$A_k := G_k^{-1}[\alpha, +\infty)$$

is not simply connected. For all k sufficiently large, this set projects injectively on X_k . Thus, we can identify A_k with its projection to X_k . Since the region exterior to Ω_k is simply connected, we get that one of its connected components $\Omega_{k,0}$ must intersect A_k .

Define

$$B_k := G_k^{-1}[0, \alpha/2].$$

We now show that, for all k sufficiently large, B_k intersects $\Omega_{k,0}$. Consider

$$s_k := \inf \left\{ |\Sigma|_k \mid \Sigma \text{ is a homotopically non-trivial surface of } N_k \right\},$$

and note that $0 < s_1 \leq s_k$ for all k . For any k large enough satisfying

$$|\Omega_{k,0}|_{k,AF} = \int_{\Omega_{k,0}} G_k^4 dA_k \leq (\alpha/2)^4 s_1,$$

then we get

$$\Omega_{k,0} \cap G_k^{-1}[0, \alpha/2] \neq \emptyset.$$

We just need to check that the distance between A_k and B_k is bounded from below independently of k . Let

$$d := \text{dist}_{g_\infty} (G_\infty^{-1}[\alpha, +\infty), G_\infty^{-1}[0, \alpha/2]) > 0.$$

Regarding B_k as its lift to X_∞ , we have, from Lemma 5.1 (ii),

$$B_k \subset G_\infty^{-1}[0, \alpha/2].$$

Therefore, for all k sufficiently large,

$$\text{dist}_{k,AF}(A_k, B_k) \geq \text{dist}_{\infty,AF}(A_k, G_\infty^{-1}[0, \alpha/2]) \geq (\alpha/2)^2 d.$$

□

Proof of Lemma 5.6. For any $\delta > 0$, set

$$\eta_{k,\delta} := \min \left\{ 1, \frac{(G_k - \delta)_+}{\delta} \right\},$$

where $(G_k - \delta)_+$ denotes the positive part of $(G_k - \delta)$. First note that ∂X_k is minimal and its area goes to zero when k goes to infinity. Using a similar arguments to the proof of Lemma 5.4, we obtain that, for all k sufficiently large,

$$\partial X_k \subset G_k^{-1}[0, \delta].$$

It suffices to check that, for any $\varepsilon > 0$, we can find a small $\delta > 0$ so that

$$\int_{X_k} |\nabla \eta_{k,\delta}|^2 dV_{k,AF} \leq \varepsilon \quad \text{and} \quad \text{Vol}_{k,AF}(\{x \in X_k \mid \nabla \eta_k \neq 0\}) \leq \varepsilon$$

for all k sufficiently large.

For any open set O containing p_∞ in N_∞ , Lemma 5.1 (iii) and (v) imply that

$$\int_{N_\infty - O} G_\infty^6 dV_\infty < \infty.$$

Combining this with also Lemma 5.1 (iii), we can choose a small $\delta > 0$ so that

$$\int_{G_\infty^{-1}[\delta, 2\delta]} G_\infty^6 dV_\infty \leq \frac{\varepsilon}{2}.$$

The uniform approximation property of Lemma 5.1 (ii) guarantees that

$$\text{Vol}_{k,AF}(\{x \in X_k \mid \nabla \eta_k \neq 0\}) \leq \int_{G_k^{-1}[\delta, 2\delta]} G_k^6 dV_k < \varepsilon$$

for all k large enough. Likewise, Lemma 5.1 (iv) implies that

$$\int_{X_k} |\nabla \eta_{k,\delta}|_{k,AF}^2 dV_{k,AF} \leq \int_{G_k^{-1}[\delta, 2\delta]} G_k^2 \delta^{-2} |\nabla G_k|^2 dV_k \leq 4L^2 \int_{G_k^{-1}[\delta, 2\delta]} G_k^2 dV_k.$$

Arguing as before, we can find a small $\delta > 0$ so that

$$4L^2 \int_{G_k^{-1}[\delta, 2\delta]} G_k^2 dV_k < \varepsilon.$$

for all k sufficiently large. This completes the proof. \square

7. PROOF OF MAIN THEOREM: GENERAL CASE

In this section, we complete the proof of Theorem 2.1. For the proof of the assertion (1) in Theorem 2.1, set

$$N := M \# L,$$

where M is a closed 3-manifold satisfying the conditions (i)-(iii) in Theorem 2.1, and L is a closed 3-manifold containing a non-separating embedded 2-sphere. For the proof of the assertion (2) in Theorem 2.1, we consider $N = M \# L$ as below provided that $M = S^3$ and that L contains a two-sided $\mathbb{R}P^2$. Without loss of generality, we may assume that $Y(N, [g]) > 0$ and g is a Yamabe metric with $\text{Vol}_g(N) = 1$.

We now describe a suitable infinite Riemannian covering of (N, g) . If L has no two-sided $\mathbb{R}P^2$, define L_∞ to be the universal cover of L . If L has a two-sided $\mathbb{R}P^2$, then $\pi_1(L)$ has an element γ of order two. In this case, define L_∞ to be $\tilde{L}/\langle \gamma \rangle$, where \tilde{L} is the universal cover of L . Finally, let (N_∞, g_∞) be the infinite Riemannian covering of (N, g) associated to the infinite covering L_∞ of L , that is, topologically,

$$N_\infty = \#_1^\infty M \# L_\infty.$$

The important properties of N_∞ are that it is not simply-connected and that every embedded 2-sphere and every two-sided real projective plane bound a connected region.

Fix $p_\infty \in N_\infty$ and consider a sequence of finite Riemannian coverings (N_k, g_k) of (N, g) satisfying $(N_1, g_1) = (N, g)$ and the following: For each $k \geq 2$, there exists a fundamental domain of N_k containing p_∞ such that its closure $\widehat{N}_k (\subset N_\infty)$ satisfies

$$\{x \in N_\infty \mid \text{dist}_{g_\infty}(x, p_\infty) \leq k\} \subsetneq \widehat{N}_k.$$

We can assume, without loss of generality, that N_{k+1} is a finite covering of N_k , $\partial \widehat{N}_k$ is smooth, and $\partial \widehat{N}_{k+1}, \partial \widehat{N}_k$ have the same projection on N_k for all $k \geq 2$. Denote by $\widehat{N} (\subset N_\infty)$ the closure of a fundamental domain of N containing p_∞ such that $\partial \widehat{N}_2$ and $\partial \widehat{N}$ have the same projection on N .

Set

$$W_n := \{\gamma \in \pi_1(N_\infty) \mid \text{dist}_{g_\infty}(\gamma(p_\infty), p_\infty) \geq n\}.$$

The finite coverings N_k have the property that

$$\pi_1(N_k) - \{e\} \subset W_k.$$

We denote the projection of p_∞ to each N_k also by p_∞ . Like in Section 4, let G_k be the normalized Green's function on N_k for the conformal Laplacian L_{g_k} with the pole at p_∞ . Consider the following scalar-flat, asymptotically flat metrics

$$g_{k,AF} := G_k^4 g_k \quad \text{on} \quad N_k^* := N_k - \{p_\infty\}.$$

Using Aubin's Lemma 3.6,

$$Y(N, [g]) < Y(N_k, [g_k]) < Y(N_{k+1}, [g_{k+1}])$$

for all $k \geq 2$. Therefore, it suffices to prove the following theorem.

Theorem 7.1.

$$\lim_{k \rightarrow \infty} Y(N_k^*, [g_{k,AF}]) \leq Y_2.$$

Proof. The strategy for the proof is the same as in Sections 4 and 5.

Consider the normalized minimal positive Green's function G_∞ on N_∞ for L_{g_∞} with the pole at p_∞ . This function is defined on $N_\infty^* := N_\infty - \{p_\infty\}$. Note that Lemma 5.1 still holds, with some modifications, in this new setting. For example, Lemma 5.1 (i) becomes

$$(5) \quad G_k = \sum_{\gamma \in \pi_1(N_k)} G_\infty \circ \gamma \quad \text{on } N_\infty$$

for all $k \geq 1$, and Lemma 5.1 (iii) now says that

$$(6) \quad \lim_{n \rightarrow \infty} \sup \{G_\infty(\gamma(x)) \mid x \in K, \gamma \in W_n\} = 0$$

for every compact set K of N_∞^* .

The main difference between the case we consider in this section and the one considered in Section 5, is that the number of “necks” in N_k^* increases with k . Nevertheless, we show that these “necks” are very small when k is large enough.

Lemma 7.2. *For any k , there exists a family Γ_k of disjoint embedded surfaces in N_k^* such that each of its connected components is either a 2-sphere or a two-sided \mathbb{RP}^2 which minimizes area (with respect to $g_{k,AF}$), and that*

$$\lim_{k \rightarrow \infty} |\Gamma_k|_{k,AF} = 0.$$

Moreover, every embedded 2-sphere and every embedded two-sided \mathbb{RP}^2 of $N_k^ - \Gamma_k$ bound a connected region.*

Proof. Take a subset S of the connected components of $\partial\widehat{N}$ which projects injectively into N^* , such that every embedded 2-sphere and every embedded two-sided \mathbb{RP}^2 of $N - S$ separate. Note that $N - S$ is connected and that S is the disjoint union of connected components $\{S_j\}_{j=1}^{j_0}$, which are either embedded 2-spheres or two-sided real projective planes. From the construction of N_k , there exist a sequence of positive integers $q(k)$ going to infinity and subsets

$$I_{j,k} \subset W_{q(k)} \subset \pi_1(N) \quad \text{for } j = 1, \dots, j_0,$$

such that

$$(I_{1,k}(S_1) \sqcup \dots \sqcup I_{j_0,k}(S_{j_0})) \subset \partial\widehat{N}_k,$$

and every embedded 2-sphere and every two-sided \mathbb{RP}^2 of

$$N_k - (I_{1,k}(S_1) \sqcup \dots \sqcup I_{j_0,k}(S_{j_0}))$$

separate, where

$$I_{j,k}(S_j) := \{\gamma(S_j) \mid \gamma \in I_{j,k}\} \quad \text{for } j = 1, \dots, j_0.$$

Note also that $N_k - (I_{1,k}(S_1) \sqcup \dots \sqcup I_{j_0,k}(S_{j_0}))$ is connected. We now check that

$$\lim_{k \rightarrow \infty} |I_{1,k}(S_1) \sqcup \dots \sqcup I_{j_0,k}(S_{j_0})|_{k,AF} = 0.$$

From (5), we have for $j = 1, \dots, j_0$,

$$\sum_{\gamma \in \pi_1(N)} \int_{S_j} G_\infty \circ \gamma dV_\infty = \int_{S_j} G_1 dV_1 < \infty.$$

Hence,

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{I_{j,k}(S_j)} G_k dV_k &= \lim_{k \rightarrow \infty} \sum_{\gamma \in \pi_1(N_k)} \int_{I_{j,k}(S_j)} G_\infty \circ \gamma dV_\infty \\ &\leq \lim_{k \rightarrow \infty} \sum_{\gamma \in W_{q(k)}} \int_{S_j} G_\infty \circ \gamma dV_\infty \\ &= 0, \end{aligned}$$

provided that $q(k)$ goes to infinity. Using (6), we have that

$$\lim_{k \rightarrow \infty} \sup \{G_\infty(x) \mid x \in I_{i,k}(S_i)\} = 0 \quad \text{for } j = 1, \dots, j_0,$$

and thus this implies

$$\begin{aligned} \lim_{k \rightarrow \infty} |I_{j,k}(S_j)|_{k,AF} &= \lim_{k \rightarrow \infty} \int_{I_{j,k}(S_j)} G_k^4 dV_k \\ &\leq \lim_{k \rightarrow \infty} \int_{I_{j,k}(S_j)} G_k dV_k \\ &= 0. \end{aligned}$$

The proof follows by minimizing area (with respect to $g_{k,AF}$) in the isotopy class of $I_{1,k}(S_1) \sqcup \dots \sqcup I_{j_0,k}(S_{j_0})$. The existence and regularity theory in [23] says that the area minimizer is, in this case, a disjoint union of 2-spheres and real projective planes. Again, according to [23, Section 3], the connected components of the area minimizer can be jointed by arbitrarily thin tubes so that the resulting surface is isotopic to $I_{1,k}(S_1) \sqcup \dots \sqcup I_{j_0,k}(S_{j_0})$. Hence, the area minimizer has to contain a minimal surface $\Gamma_k (\subset N_k^*)$ consisting of 2-spheres or two-sided real projective planes such that every embedded 2-sphere and every two-sided $\mathbb{R}\mathbb{P}^2$ in $N_k^* - \Gamma_k$ separate. \square

Set X_k to be the metric completion of

$$N_k^* - \Gamma_k = N_k - (\{p_\infty\} \cup \Gamma_k),$$

and denote the smooth metric extension of $g_{k,AF}$ to X_k by the same symbol. Then, $(X_k, g_{k,AF})$ becomes an asymptotically flat manifold with minimal boundary consisting of a disjoint union of 2-spheres and projective planes. Note that Theorem 3.4 and Lemma 3.5 imply the monotonicity of the Hawking quasi-local mass for each solution of the inverse mean curvature flow on $(X_k, g_{k,AF})$, provided that its initial surface is the boundary of a connected region.

Since $\pi_1(N_\infty^*)$ is not trivial, there exists a positive constant α such that $G_\infty^{-1}[\alpha, +\infty)$ is not simply connected. The proof of Lemma 5.4 can then be easily modified to give the following.

Lemma 7.3. *The outermost minimal surface of each $(X_k, g_{k,AF})$ has a connected component $\Sigma(k)$ with area bounded from below independently of k .*

The argument in the proof of Lemma 5.6 is still applicable to this new setting. The same arguments as in the proof of Theorem 4.1 in Section 4 completes the proof of Theorem 7.1. \square

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