SINGULARITIES OF LAGRANGIAN MEAN CURVATURE FLOW: ZERO-MASLOV CLASS CASE

ANDRÉ NEVES

Abstract. We study singularities of Lagrangian mean curvature flow in $\mathbb{C}^n$ when the initial condition is a zero-Maslov class Lagrangian. We start by showing that, in this setting, singularities are unavoidable. More precisely, we construct Lagrangians with arbitrarily small Lagrangian angle and Lagrangians which are Hamiltonian isotopic to a plane that, nevertheless, develop finite time singularities under mean curvature flow.

We then prove two theorems regarding the tangent flow at a singularity when the initial condition is a zero-Maslov class Lagrangian. The first one (Theorem A) states that that the rescaled flow at a singularity converges weakly to a finite union of area-minimizing Lagrangian cones. The second theorem (Theorem B) states that, under the additional assumptions that the initial condition is an almost-calibrated and rational Lagrangian, *connected* components of the rescaled flow converges to a *single* area-minimizing Lagrangian cone, as opposed to a possible non-area-minimizing union of area-minimizing Lagrangian cones. The latter condition is dense for Lagrangians with finitely generated $H_1(L,\mathbb{Z})$.

1. Introduction

In the last few years, mean curvature flow of higher codimension submanifolds has attracted some attention. Most of the work done has focused on finding initial conditions that assure the flow will exist for all time. For instance, under some natural convexity assumptions on the image of the Gauss map, long time existence and convergence results have been proved by J. Chen, J. Li, and Tian [4], Smoczyk [14, 15], Smoczyk and M.-T. Wang [16], M.-P. Tsui and M.-T. Wang [17], and M.-T. Wang [18, 19, 20]. On the other hand, finite time singularities for mean curvature flow in the higher codimension case are not so well understood and, reasoning in analogy with minimal surfaces, they are expected to exhibit a far more complicated behavior than in the codimension one case.

There is, therefore, interest in identifying initial conditions for the flow that are broad enough to admit singularities, but restrictive enough so that the singularities are, so to speak, "well-behaved". A natural candidate for such an initial condition is *Lagrangian* because, when the ambient manifold is Kähler-Einstein, the Lagrangian condition is preserved by mean curvature flow (see [12]). Mu-Tao Wang observed in [18] that, when the ambient manifold is Calabi-Yau, *almost-calibrated* Lagrangians (see next section for the definition) cannot develop type I singularities, i.e., no sequence of rescaled flows at a singularity can converge strongly to a homothetically shrinking solution. Later, Jingyi Chen and Jiayu Li [3] showed that in this setting the sequence of rescaled flows converges weakly to an integer rectifiable stationary Lagrangian varifold which is also a cone.

In this paper we study finite time singularities for zero-Maslov class Lagrangians in $\mathbb{C}^n$, a more general condition than being almost-calibrated. The first result, Theorem A, states that the tangent flow at a singularity can be decomposed into a finite union of area-minimizing Lagrangian cones. Theorem B is a more interesting result because, assuming the initial condition is an almost-calibrated and *rational* Lagrangian, it states that the Lagrangian angle converges to a *single* constant on each *connected* component of the rescaled flow. In particular, this implies that connected components of the rescaled flow converge weakly to a single area-minimizing Lagrangian cone, instead of a possible non-area-minimizing union of area-minimizing Lagrangian cones. Heuristically speaking, such property qualifies the formation of singularities as being, so to speak, "well behaved". Without such behavior, it would be hopeless to expect Lagrangian mean curvature flow to be more tractable than general higher codimension mean curvature flow. We remark that any Lagrangian M with $H_1(M,\mathbb{Z})$ finitely generated can always be perturbed in order to become rational.

Assuming some rotational symmetry, we also construct zero-Maslov class exact Lagrangians that develop finite time singularities under Lagrangian mean curvature flow. These examples include Lagrangians with arbitrarily small oscillation of the Lagrangian angle and Lagrangians which are Hamiltonian isotopic to a plane.

The paper is organized as follows. In Section 2 we recall some standard definitions and results that will be useful throughout the rest of the paper. The main two results are discussed and stated in Section 3. Examples of finite time singularities for Lagrangian mean curvature flow are given in Section 4. The first result, Theorem A, is proven in Section 5. In Section 6 we derive evolution equations of some geometric quantities that will be needed in Section 7. In this section we prove Theorem B.

The author would like to express his gratitude to Richard Schoen for all of his guidance and insight. He would also like to thank Leon Simon and Brian White for enlightening discussions and constant availability. Finally, the author is very grateful to the referees for their kind corrections and suggestions that improved this paper.

2. Preliminaries

Let J and ω denote, respectively, the standard complex structure on $\mathbb{C}^n$ and the standard symplectic form on $\mathbb{C}^n$. We consider also the closed complex-valued n-form given by

$$\Omega \equiv dz_1 \wedge \ldots \wedge dz_n$$

and the Liouville form given by

$$\lambda \equiv \sum_{i=1}^{n} x_i dy_i - y_i dx_i, \quad d\lambda = 2\omega,$$

where $z_j = x_j + iy_j$ are complex coordinates of $\mathbb{C}^n$.

A smooth n-dimensional submanifold L in $\mathbb{C}^n$ is said to be *Lagrangian* if $\omega_L = 0$ and this implies that (see [7])

$$\Omega_L = e^{i\theta}\mathrm{vol}_L,$$

where vol_L denotes the volume form of L and θ is some multivalued function called the *Lagrangian angle*. When the Lagrangian angle is a single valued function the Lagrangian is called *zero-Maslov class* and if

$$\cos\theta \geq \varepsilon$$

for some positive ε, then L is said to be *almost-calibrated*. Furthermore, if $\theta \equiv \theta_0$, then L is calibrated by

$$\mathrm{Re}\left(e^{-i\theta_0}\Omega\right)$$

and hence area-minimizing. In this case, L is referred as being *Special Lagrangian*.

Likewise, we define an integral n-varifold L_1 and an integral n-current L_2 to be *Lagrangian* if

$$\int_{L_1} \phi|\omega \wedge \eta|\, d\mathcal{H}^n = 0 \quad \text{for all } n-2 \text{ form } \eta \text{ and all smooth } \phi \in C^\infty_C(\mathbb{C}^n)$$

and

$$\int_{L_2} \phi\, \omega \wedge \eta\, d\mathcal{H}^n = 0 \quad \text{for all } n-2 \text{ form } \eta \text{ and all } \phi \in C^\infty_C(\mathbb{C}^n)$$

respectively. The concept of being Special Lagrangian can be easily extended to the case when L is an integral current

For a smooth Lagrangian, the relation between the Lagrangian angle and the mean curvature is given by the following remarkable property (see for instance [11])

$$H = J\nabla\theta.$$

Let L_0 be a smooth Lagrangian in $\mathbb{C}^n$ such that, for some constant C_0, we have

$$\mathcal{H}^n\big(L_0 \cap B_R(0)\big) \leq C_0 R^n$$

for all R sufficiently large and assume that we have a solution $(L_t)_{0\leq t<T}$ to mean curvature flow for which the second fundamental form of L_t is bounded by a time dependent constant. The same argument used in [12] and the maximum principle for noncompact manifolds proved by Ecker and Huisken in [6] imply that the Lagrangian condition is preserved. In this case, we say that we have a solution to *Lagrangian mean curvature flow*. Moreover, if L_0 is also zero-Maslov class, then this condition is preserved by

the flow and, according to [13], the Lagrangian angles θ_t can be chosen so that

$$\frac{d\theta_t}{dt} = \Delta\theta_t.$$

An immediate application of the parabolic maximum principle shows that the almost-calibrated condition is preserved by Lagrangian mean curvature flow.

A Lagrangian L_0 is said to be *rational* if for some real number a

$$\lambda\left(H_1(L_0,\mathbb{Z})\right) = \{a2k\pi \mid k \in \mathbb{Z}\}.$$

Any Lagrangian having $H_1(L_0,\mathbb{Z})$ finitely generated can be perturbed in order to become rational. When $a = 0$ the Lagrangian is called *exact*. Furthermore, if L_0 is also zero-Maslov class, we will see in Section 6 that the rational condition is preserved by Lagrangian mean curvature flow, i.e.,

$$\lambda\left(H_1(L_t,\mathbb{Z})\right) = \{a2k\pi \mid k \in \mathbb{Z}\}$$

while the solution exists smoothly.

Assume now that the solution to mean curvature flow develops a singularity at the point (x_0,T) in space-time. Then

$$L_s^\sigma := \sigma(L_{T+s/\sigma^2} - x_0) \quad \text{for } -\sigma^2T < s < 0$$

is also a solution to Lagrangian mean curvature flow and it is called a *rescaled flow*. It follows from [9, Lemma 8] that for every sequence (σ_i) going to infinity there is a a subsequence for which the mean curvature flow

$$(L_s^{\sigma_i})_{-\sigma_i^2T<s<0}$$

converges weakly to a homothetically shrinking weak solution of mean curvature flow (Brakke flow). This solution is called *tangent flow* and depends on the sequence (σ_i) taken.

3. Statement of results

Let $(L_t)_{0\leq t<T}$ be a smooth solution to Lagrangian mean curvature flow in $\mathbb{C}^n$ satisfying, for some constant C_0, the area bounds

$$\mathcal{H}^n\big(L_0 \cap B_R(0)\big) \leq C_0R^n$$

for all R sufficiently large. Furthermore, assume that the flow develops a finite time singularity at time T and that L_0 is zero-Maslov class with bounded Lagrangian angle. We denote the Lagrangian angle of a rescaled flow $(L_s^i)_{s<0}$ by $\theta_{i,s}$. Arguing informally, the following theorem states that a sequence of rescaled flows at a singularity converges weakly to a finite union of integral Special Lagrangian cones.

Theorem A. *If L_0 is zero-Maslov class with bounded Lagrangian angle, then for any sequence of rescaled flows $(L_s^i)_{s<0}$ at a singularity there exist a finite set $\{\bar\theta_1,\ldots,\bar\theta_N\}$ and integral Special Lagrangian cones*

$$L_1,\ldots,L_N$$

such that, after passing to a subsequence, we have for every smooth function ϕ compactly supported, every f in $C^2(\mathbb{R})$, and every $s<0$

$$\lim_{i\to\infty}\int_{L^i_s} f(\theta_{i,s})\phi\, d\mathcal{H}^n=\sum_{j=1}^N m_j f(\bar\theta_j)\mu_j(\phi),$$

where μ_j and m_j denote the Radon measure of the support of L_j and its multiplicity respectively.

Furthermore, the set $\{\bar\theta_1,\dots,\bar\theta_N\}$ does not depend on the sequence of rescalings chosen.

Remark 3.1.

1) It is possible and expected that, for instance,
$$\{\bar\theta_1,\bar\theta_2,\bar\theta_3\}=\{0,\pi,2\pi\}$$
but the supports of L_1, L_2, and L_3 are all the same.
2) In case $n=2$, it is well known that the support of area-minimizing cones are planes intersecting transversely.

Theorem A follows from combining standard ideas from geometric measure theory with the evolution equation

$$\frac{d\theta^2_{i,s}}{dt}=\Delta\theta^2_{i,s}-2|H|^2.$$

More precisely, we will show that after using Huisken monotonicity formula [8] such equation implies that for all $t<0$ and all positive R

$$\lim_{i\to\infty}\int_{-1}^t\int_{L^i_s\cap B_R(0)}|H|^2+|\mathbf{x}^\perp|^2\, d\mathcal{H}^n ds=0, \tag{1}$$

where $\mathbf{x}$ denotes the vector determined by the point x in $\mathbb{C}^n$ and $\mathbf{x}^\perp$ denotes the projection of the vector $\mathbf{x}$ onto the orthogonal complement of $T_xL^i_s$. Hence, for almost all $s<0$ we get that for all positive R

$$\lim_{i\to\infty}\int_{L^i_s\cap B_R(0)}|H|^2+|\mathbf{x}^\perp|^2\, d\mathcal{H}^n=0$$

and this implies that, after passing to a subsequence, L^i_s converges weakly to a stationary integral varifold L which is also a cone. Note that so far L could be a union of three Lagrangian half-planes meeting at angles of $2\pi/3$ along a common boundary. We now sketch briefly why such configuration cannot occur because the proof of Theorem A consists essentially in exploiting this argument. Denote the Lagrangian angles of these half-planes by θ_1,θ_2, and θ_3. Then, for all sufficiently small ε, $\{|\theta_{i,s}-\theta_1|\le\varepsilon\}$ converges to a half-plane and so

$$\lim_{i\to\infty}\mathcal{H}^{n-1}(\{\theta_{i,s}=\theta_1+\varepsilon\}\cap B_R(0))>0$$

This is impossible because, using the coarea formula and Hölder's inequality, we have

$$\lim_{i\to\infty}\int_{-\infty}^{\infty}\mathcal{H}^{n-1}\left(\{\theta_{i,s}=u\}\cap B_R(0)\right)du \;=\; \lim_{i\to\infty}\int_{L^i_s\cap B_R(0)}|H|\,d\mathcal{H}^n \;=\; 0.$$

Theorem A raises the following question: Given Σ^i a sequence of connected components of $L^i_s \cap B_R(0)$ that converges weakly to Σ, does Σ need to be a Special Lagrangian cone? In other words, does $\theta_{i,s}$ need to converge to a constant? According to Theorem A we only know that Σ is a finite union of Special Lagrangian cones which might have different Lagrangian angles and hence not necessarily area-minimizing. An affirmative answer to this question is necessary if one wants to make reasonable the possibility of developing a regularity theory for the flow.

Technically, the difficulty comes from the fact that because the sequence of smooth manifolds L^i_s are becoming singular when i goes to infinity, no Poincaré inequality holds with a constant independent of i and therefore we cannot conclude that, on each connected component of L^i_s, the Lagrangian angles $\theta_{i,s}$ converge to a constant. As a matter of fact, for the sequence of smooth surfaces

$$L_\varepsilon \equiv \{(z,w)\in\mathbb{C}^2 \,|\, zw=\varepsilon\},$$

one can easily construct bounded functions f_ε for which the L^2 norm of its gradient goes to zero when ε goes to zero but nevertheless f_ε converges to a distinct constant on each complex plane. The question raised in the previous paragraph was addressed in [3, Theorem 5.1] but unfortunately this technical aspect was overlooked.

In order to deal with this difficulty, we require L_0 to satisfy two additional conditions, namely that it is an almost-calibrated and rational Lagrangian (see Section 2 for the definitions). We argued in Section 2 that these conditions are preserved by Lagrangian mean curvature flow.

Theorem B. *If L_0 is almost-calibrated and rational, then after passing to a subsequence of $(L^i_s)_{s<0}$ the following property holds for all $R>0$ and almost all $s<0$.*

For any convergent subsequence (in the Radon measure sense) Σ^i of connected components of $B_{4R}(0)\cap L^i_s$ intersecting $B_R(0)$ there exists a Special Lagrangian cone L in $B_{2R}(0)$ with Lagrangian angle $\bar\theta$ such that

$$\lim_{i\to\infty}\int_{\Sigma^i} f(\theta_{i,s})\phi\,d\mathcal{H}^n = m f(\bar\theta)\mu(\phi)$$

for every f in $C(\mathbb{R})$ and every smooth ϕ compactly supported in $B_{2R}(0)$, where μ and m denote the Radon measure of the support of L and its multiplicity respectively.

Next, we give a heuristic argument explaining why the rational condition should play a role. From the pioneering work of Richard Hamilton both on Ricci flow and on mean curvature flow we know that it is helpful to find quantities that are constant on self-similar solutions. For that matter, let us consider

$$L_s \equiv \sqrt{s}L_1$$

to be a solution to Lagrangian mean curvature flow where L_0 is zero-Maslov class. A simple computation reveals that for all $s > 0$

$$H(L_s) = \mathbf{x}^\perp/(2s) \iff 2s\nabla\theta_s = -(J\mathbf{x})^\top \iff 2s\, d\theta_s + \lambda = 0.$$

Thus, we conclude that L_s is exact and that, if we denote by β_s the primitive for the Liouville form λ, then $\beta_s+2s\theta_s$ is constant in space for all s. Arguing informally, this suggests that showing convergence of the Lagrangian angle to a single constant should be equivalent to showing that the primitive for the Liouville form converges to a single constant. The advantage of doing so is that the gradient of β_s is a first order quantity and thus easier to control than the gradient of θ_s which is a second order quantity.

We now sketch the main idea behind the proof of Theorem B. Assume, for the sake of simplicity, that L_0 is exact which implies that for each i there is a family of smooth functions $\beta_{i,s}$ defined on L^i_s such that $d\beta_{i,s} = \lambda$, or equivalently,

$$J\nabla\beta_{i,s}(x) = -\mathbf{x}^\perp \quad \text{for all } x \in M_{i,s}.$$

Moreover, as it will be shown in Section 6, the functions $\beta_{i,s}$ can be chosen so that

$$\frac{d}{ds}(\beta_{i,s} + 2s\theta_{i,s}) = \Delta(\beta_{i,s} + 2s\theta_{i,s}).$$

This evolution equation combined with identity (1) implies that, after passing to a subsequence, $\beta_{i,s} + 2s\theta_{i,s}$ has a limit which is independent of s and so it must converge to some constant $\bar\beta_j$ on each Special Lagrangian cone L_j, with $j = 1, \ldots, N$. Hence, we obtain from Theorem A that $\beta_{i,s}$ converges to $\bar\beta_j - 2s\bar\theta_j$ on each L_j. Moreover, we can assume without loss of generality that the set

$$\Lambda \equiv \{\bar\beta_1 - 2s\bar\theta_1, \ldots, \bar\beta_N - 2s\bar\theta_N\}$$

has N distinct values.

Let Σ_i be a convergent sequence of connected components of $L^i_s \cap B_R(0)$. Because the gradient of $\beta_{i,s}$ is *pointwise bounded* and its L^2-norm converges to zero, we can show that the sequence of functions $\beta_{i,s}$ converges to a single constant when restricted to Σ_i (see Proposition A.1). Thus, the Lagrangian angle of Σ_i must converge to a constant because otherwise two numbers in the set Λ would be equal.

4. Examples of Finite Time Singularities

We construct examples of finite time singularities for mean curvature flow where the initial condition is a zero-Maslov class and exact Lagrangian.

For simplicity, we restrict ourselves to $\mathbb{C}^2$ but we note that the phenomena observed also occur in $\mathbb{C}^n$. Given a curve γ in the complex plane, it is easy to see that

$$L = \{(\gamma\cos\alpha, \gamma\sin\alpha) \mid \alpha \in \mathbb{R}/2\pi\mathbb{Z}\}$$

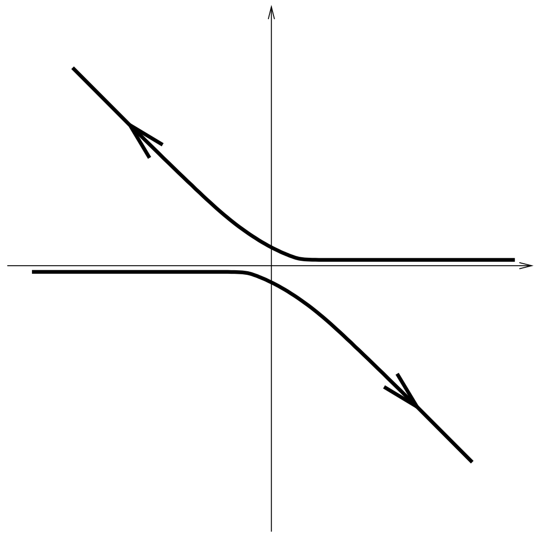

FIGURE 1. Lagrangian surface L_0.

is a Lagrangian surface in $\mathbb{C}^2$. A choice of orientation for the curve γ induces an orientation on L and if $\gamma(s)$ denotes a parametrization of γ, then

$$\Omega_L = \frac{\gamma}{|\gamma|}\frac{\gamma'}{|\gamma'|}\mathrm{vol}_L \quad \text{and} \quad \lambda_L = \langle i\gamma, \gamma'\rangle ds.$$

Hence, we get that L is exact and zero-Maslov class whenever γ is diffeomorphic to a line.

If we evolve L by mean curvature flow, the rotational symmetries are preserved and the corresponding γ_t evolve according to

$$\frac{dz}{dt} = \mathbf{k} - \mathbf{z}^{\perp}/|z|^2, \tag{2}$$

where $\mathbf{k}$ is the curvature of γ and $\mathbf{z}^{\perp}$ denotes the projection of the position vector $\mathbf{z}$ on the orthogonal complement of $T_x\gamma$.

For any $0 < \beta \leq \pi$, consider the following initial condition for the *equivariant mean curvature flow* (2)

$$\gamma_0(s) = (\sin(\pi s/\beta))^{-\beta/\pi}e^{is} \equiv r_0(s)e^{is}, \quad 0 < s < \beta.$$

The corresponding Lagrangian surface L_0 is asymptotic to two oriented planes with Lagrangian angles π and 2β and, when $\beta > \pi/2$, its intersection with $\mathbb{C} \times \{0\}$ can be seen in Figure 1. In order to compute the Lagrangian angle of L_0 we use the formula

$$\theta_0(s) = \arg(\gamma_t\gamma_t') = 2s + \arg(r_0' + ir_0)$$

and obtain that

$$\theta_0(s) = (2 - \pi/\beta)s + \pi, \quad 0 < s < \beta.$$

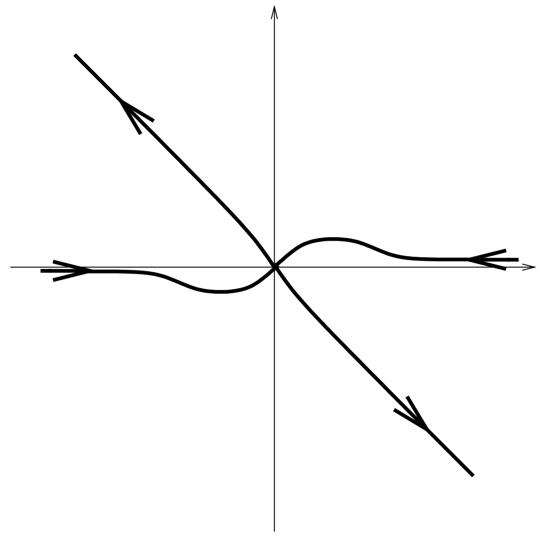

FIGURE 2. Finite time singularity at the origin.

Note that the oscillation of the Lagrangian angle can be made arbitrarily small by choosing β close to $\pi/2$.

We now sketch briefly three distinct behaviors for the equivariant mean curvature flow. When $0 < \beta < \pi/2$, the curve will expand indefinitely because the curvature term on the right hand side of (2) points outward and dominates the first-order term that points inward. As a matter of fact, Anciaux [1] found a self-expander with the same asymptotics at infinity as γ_0. When $\beta = \pi/2$, the Lagrangian surface is one of the Special Lagrangians studied in [7]. Thus, the curvature term equals the first order term on (2) because the curve is a fixed point for the flow. Finally, when $\pi/2 < \beta \leq \pi$, the first order term will be pointing inward and bigger than the curvature term, thus forcing the solution to have a finite time singularity at the origin (see Figure 2). This is the content of the next theorem.

Theorem 4.1. *When $\pi/2 < \beta \leq \pi$, the Lagrangian mean curvature flow starting at L_0 develops a finite time singularity at the origin. The tangent flow is a union of two planes intersecting at a single point, both with Lagrangian angle $\beta/2$.*

Proof. We start by proving short-time existence for the equivariant mean curvature flow. The procedure is well-know among the specialists but we include it here for the sake of completeness.

After rotating the coordinate axis by $(\pi - \beta)/2$, the curve γ_0 can be written as the graph of a function u_0 over the real axis. A straightforward computation shows the existence of some constant C such that

$$|u_0'|_{C^2} + |xu_0' - u_0|_{C^0} \leq C. \tag{3}$$

For each fixed $n \in \mathbb{N}$, consider graphical solutions $\gamma_t^n \equiv (x, u_t^n(x))$ for the equivariant mean curvature flow with boundary conditions

$$u_0^n(x) = u_0(x) \quad \text{for } |x| \leq n, \quad u_t^n(\pm n) = u_0(\pm n) = u_0(n).$$

We will show uniform apriori $C^{2,\alpha}$-estimates for the sequence of functions (u_t^n).

A simple computation reveals that u_t^n solves the quasilinear equation

$$\frac{du}{dt} = \frac{u''}{1+(u')^2} + \frac{xu' - u}{x^2 + u^2}. \tag{4}$$

Lemma 4.2. *There exists positive s_0 and ε so that*

$$u_t^n(0) \geq \varepsilon$$

for all $t \leq s_0$ and all $n \in \mathbb{N}$. Moreover, we have for all $t \leq s_0$ that

$$u_0(n)|x|/n \leq u_t^n(x) \leq u_0(x).$$

Proof. Consider a solution $(C_t)_{t\geq 0}$ to (2) having initial condition a circle of small radius centered at the origin that does not intersect γ_0. The maximum principle implies that the graph of u_t^n cannot intersect C_t and so the first assertion follows. The second assertion also follows from the maximum principle because

$$v^n(x) \equiv u_0(n)|x|/n$$

and u_0 are a subsolution and supersolution for (4) respectively. □

The function

$$v_t^n \equiv u_t^n - u_0$$

satisfies the equation

$$\frac{dv}{dt} = \frac{v''}{1+(u' + u_0')^2} + \frac{xv' - v}{x^2 + (v+u_0)^2} + F_t$$

where, due to (3),

$$F_t \equiv \frac{u_0''}{1+(u' + u_0')^2} + \frac{xu_0' - u_0}{x^2 + (v+u_0)^2}$$

is pointwise bounded. Hence, the maximum principle implies that v_t^n is uniformly bounded for all $t \leq s_0$. Moreover, we obtain from Lemma 4.2 that

$$-u_0(n)/n \leq u_t^{n\prime}(-n) \leq u_0'(-n) \quad \text{and} \quad u_0'(n) \leq u_t^{n\prime}(n) \leq u_0(n)/n$$

and so, it follows from (3) that $v_t^{n\prime}(\pm n)$ converges to zero as n goes to infinity. Because $\phi_t^n \equiv v_t^{n\prime}$ satisfies an evolution equation of the form

$$\frac{d\phi}{dt} = a(x, \phi')\phi'' + b(x, \phi, \phi')\phi' + c(x, \phi, \phi')\phi + G_t,$$

where $a > 0$ and c, G_t are uniformly bounded functions, we obtain from the maximum principle that $v_t^{n\prime}$ is uniformly bounded. Standard theory for quasilinear parabolic equations implies the existence of some constant M for which $|u_t^n - u_0|_{C^{2,\alpha}} < M$ for all $t \leq s_0$. Therefore, we can let n go to infinity and obtain a solution $\gamma_t(x) \equiv (x, u_t(x))$ for the equivariant mean curvature flow.

Next, we argue that the flow (γ_t) develops a finite time singularity. We need the following lemma.

Lemma 4.3. *While the solution exists smoothly, the curve γ_t can be parametrized by*

$$\gamma_t(s) = r_t(s)e^{is} \quad \textit{with } r_t(s) > 0, \quad 0 < s < \beta.$$

Proof. For any $0 < \alpha < \beta$, denote by C_α the line

$$C_\alpha = \left\{ re^{i\alpha} \,|\, r \in \mathbb{R} \right\}.$$

Initially, we have that C_α and γ_0 intersect only once. Furthermore, it follows from the short-time existence estimates that γ_t remains in the region below γ_0 and above the x-axis. Hence, the Sturmian Theorem proved by Angenent [2, Proposition 1.2.] implies that C_α and γ_t must intersect exactly once while the solution exists smoothly. □

For the rest of this proof we parameterize the curves γ_t as described in the previous lemma. The equation satisfied by r_t becomes

Lemma 4.4.

$$\frac{dr}{dt} = -\frac{\theta_t'}{r} = \frac{rr'' - 2r^2 - 3(r')^2}{r(r')^2 + r^3},$$

Proof. Denote by ∂_s the tangent vector

$$\partial_s = r'e^{is} + ire^{is}.$$

Then

$$\langle d(re^{is})/dt, i\partial_s \rangle = dr/dt \langle e^{is}, i\partial_s \rangle = -rdr/dt.$$

On the other hand,

$$\langle d(re^{is})/ds, i\partial_s \rangle = \langle H, i\partial_s \rangle = \langle \nabla\theta_t, \partial_s \rangle = \theta_t'$$

and so the first identity follows. The second identity can be checked using

$$\theta_t(s) = 2s + \arg(r_t' + ir_t).$$

□

Let $A_t(\varepsilon)$ denote the area of the triangular-shaped region

$$\{ue^{is} \,|\, \varepsilon \leq s \leq \beta - \varepsilon,\, 0 \leq u \leq r_t(s)\}.$$

Note that

$$2A_t(\varepsilon) = \int_\varepsilon^{\beta-\varepsilon} r_t^2(s)\, ds$$

and that

$$2s < \theta_t(s) < 2s + \pi$$

because $\theta_t(s) = 2s + \arg(r_t' + ir_t)$. Therefore,

$$\frac{d}{dt}A_t(\varepsilon) = -\int_{\varepsilon}^{\beta-\varepsilon} \theta_t'(s)\,ds = (\theta_t(\varepsilon) - \theta_t(\beta-\varepsilon)) < \pi + 2\varepsilon - 2\beta.$$

Because ε can be chosen arbitrarily small, the flow must develop a finite time singularity if $\pi/2 < \beta \leq \pi$.

Denote by T the instant of the first time singularity. We need to show that the singularity occurs at the origin. The key idea consists in showing that if that is not the case, then the tangent flow cannot be a union of Lagrangian planes, which is a contradiction to Theorem A. In order to do so, we need some preliminary lemmas.

Lemma 4.5. *For all $t < T$*

$$\lim_{s\to 0}\theta_0(s) = \pi \quad \textit{and} \quad \lim_{s\to\beta}\theta_t(s) = 2\beta.$$

Proof. The maximum principle applied to θ_t implies that $\pi \leq \theta_t \leq 2\beta$ for all $t < T$. Suppose that there is $t_1 < T$, a sequence (s_i) converging to zero, and $\varepsilon > 0$ for which

$$\lim_{i\to\infty}\theta_{t_1}(s_i) = \pi + 4\varepsilon.$$

Recall that L_t denotes the Lagrangian surfaces corresponding to γ_t and consider the function

$$\phi_{t,\varepsilon} \equiv (\theta_t - \pi - \varepsilon)_+^3$$

which is supported on $\{p \in L_t \mid \theta_t \geq \pi + \varepsilon\}$. Furthermore,

$$\frac{d\phi_{t,\varepsilon}}{dt} \leq \Delta\phi_{t,\varepsilon}.$$

Huisken's monotonicity formula [8] implies that for all i sufficiently large

$$8\varepsilon^3 \leq \int_{L_0}\phi_{0,\varepsilon}\frac{\exp(-|x-x_i|^2/4t_1)}{4\pi t_1}d\mathcal{H}^2$$
$$= \int_{\{\theta_0\geq\pi+\varepsilon\}}\phi_{0,\varepsilon}\frac{\exp(-|x-x_i|^2/4t_1)}{4\pi t_1}d\mathcal{H}^2,$$

where x_i is the point $(\gamma_{t_1}(s_i), 0)$ in $\mathbb{C}^2$. For every $R > 0$, we have for all i sufficiently large that

$$\{\theta_0 \geq \pi + \varepsilon\} \cap B_R(x_i) = \emptyset.$$

Thus

$$\lim_{i\to\infty}\int_{\{\theta_0\geq\pi+\varepsilon\}}\phi_{0,\varepsilon}\frac{\exp(-|x-x_i|^2/4t_1)}{4\pi t_1}d\mathcal{H}^2 = 0$$

and this gives us a contradiction. □

This lemma is used to prove

Lemma 4.6. *For all $t < T$*

$$\frac{dr}{dt} \leq 0.$$

Proof. Taking into account that the parameterization described in Lemma 4.3 creates a tangential component on the deformation vector, we get that

$$\frac{d\theta}{dt} = \Delta_{L_t}\theta + \left\langle \frac{dx}{dt}, \nabla\theta_t \right\rangle = \frac{\theta''}{|\gamma'|^2} + \theta'\left(\frac{1}{r|\gamma'|}\left(\frac{r}{|\gamma'|}\right)' + \frac{dr}{dt}\frac{r'}{|\gamma'|^2}\right).$$

While the solution exists smoothly, we have that

$$\lim_{s\to 0}\theta_t(s) = \pi \quad \text{and} \quad \lim_{s\to\beta}\theta_t(s) = 2\beta$$

and thus, the Sturmian property [2, Proposition 1.2.] implies that the cardinality

$$\#\{s \,|\, \theta_t(s) = y\}$$

is one if $\pi < y < 2\beta$ and zero if $y < \pi$ or $y > 2\beta$. Hence

$$\frac{dr}{dt} = -\frac{\theta_t'}{r} \leq 0$$

for all $t < T$. □

The curves γ_t are symmetric under refection over a line with slope $\tan(\beta/2)$ and so

$$r_t(\beta/2 + s) = r_t(\beta/2 - s) \tag{5}$$

for all $t < T$. This implies that

$$r_t'(\beta/2) = 0 \quad \text{for all } t < T.$$

Lemma 4.7. *For any $t < T$, $r_t(s)$ is decreasing when $s < \beta/2$ and increasing when $s > \beta/2$.*

Proof. Direct computation shows that $\beta/2$ is the only critical point of r_0 and that, denoting r_t' by u_t,

$$\frac{du_t}{dt} = \frac{u_t''}{(r')^2 + r^2} + u_t' b(r_t, u_t, u_t') + u_t c(r_t, u_t, u_t'),$$

where the functions b and c are bounded for each $t < T$. Moreover,

$$\lim_{s\to 0} u_t(s) = \infty \quad \text{and} \quad \lim_{s\to\beta} u_t(s) = -\infty$$

and thus, the Sturmian property [2, Proposition 1.2.] implies that $\beta/2$ is the only critical point of r_t. □

Suppose now that the singularity happens at a point $x_0 \equiv ae^{i\alpha}$, with $0 < a \leq r_0(\alpha)$ and $0 < \alpha < \beta$. From Theorem A, we know that the tangent flow at the singularity is a union of planes and so, by White's regularity Theorem [22],

$$\limsup_{\delta\to 0} \frac{\mathcal{H}^1\left(\gamma_{T-\delta^2} \cap B_\delta(x_0)\right)}{2\delta} \geq 2.$$

We show next that this is impossible because for all δ sufficiently small and all $t < T$

$$\frac{\mathcal{H}^1\left(\gamma_t \cap B_\delta(x_0)\right)}{2\delta} \leq 3/2.$$

Without loss of generality we assume that $\alpha=\beta/2$ (the cases $\alpha>\beta/2$ and $\alpha<\beta/2$ are treated similarly). For any $\delta<a$, Lemma 4.6 and Lemma 4.7 imply that

$$\gamma_t\cap B_\delta(x_0)$$

is either empty or a connected curve. If the latter occurs, there is $\varepsilon(t)<\arcsin(\delta/a)$ for which

$$\gamma_t\cap B_\delta(x_0)=\{\gamma_t(s)\mid |s-\beta/2|<\varepsilon\}.$$

Note that

$$(r_t(\beta/2+\varepsilon)\cos(\varepsilon)-a)^2+(r_t(\beta/2+\varepsilon)\sin(\varepsilon))^2=\delta^2$$

and so

$$|r_t(\beta/2+\varepsilon)\cos(\varepsilon)-a|+|r_t(\beta/2+\varepsilon)\sin(\varepsilon)|\leq\sqrt{2}\delta<3/2\delta.$$

Combining this inequality with Lemma 4.6, Lemma 4.7, and (5), we obtain

$$\begin{aligned}\frac{\mathcal{H}^1\left(\gamma_t\cap B_\delta(x_0)\right)}{2\delta}&=\frac{1}{\delta}\int_{\beta/2}^{\beta/2+\varepsilon}((r_t')^2+r_t^2)^{1/2}ds\\&\leq\frac{r_t(\beta/2+\varepsilon)-r_t(\beta/2)}{\delta}+\varepsilon\frac{r_t(\beta/2+\varepsilon)}{\delta}\\&\leq\frac{r_t(\beta/2+\varepsilon)-a}{\delta}+\varepsilon\frac{r_t(\beta/2+\varepsilon)}{\delta}\\&\leq 3/2\end{aligned}$$

for all δ sufficiently small.

Finally, we argue next that the tangent flow at the singularity is a union of two planes with Lagrangian angle $\pi/2+\beta$. From (5) it follows that

$$\theta_t'(\beta/2+s)=\theta_t'(\beta/2-s)$$

and therefore, because the solution remains asymptotic to two planes with Lagrangian angles π and 2β, we obtain after integration that $\theta_t(\beta/2)=\pi/2+\beta$. From Lemma 4.7 we know that $\gamma_t(\beta/2)$ is the closest point of γ_t to the origin and so Theorem B implies the desired result. □

We can now use Theorem 4.1 to construct an exact and zero-Maslov Lagrangian class which is Hamiltonian isotopic to a Lagrangian plane that, nevertheless, develops a finite time singularity. Denote by L_0 the compact perturbation of a Lagrangian plane which is associated with the curve described in Figure 3. The dashed noncompact curve represents one of the curves described in Theorem 4.1 (slightly rotated so that it is not asymptotic to L_0) which has a finite time singularity at the origin at time T. The dashed circles shown in Figure 3 correspond to a Lagrangian torus, which will have a finite time singularity at time T_1. All these curves can be arranged so that $T<T_1$ and an explicit expression for such curves could be easily found. The short-time existence for the flow with initial condition L_0 follows from the same arguments used in the proof of Theorem 4.1. Because the two noncompact solutions we consider have different asymptotics, the

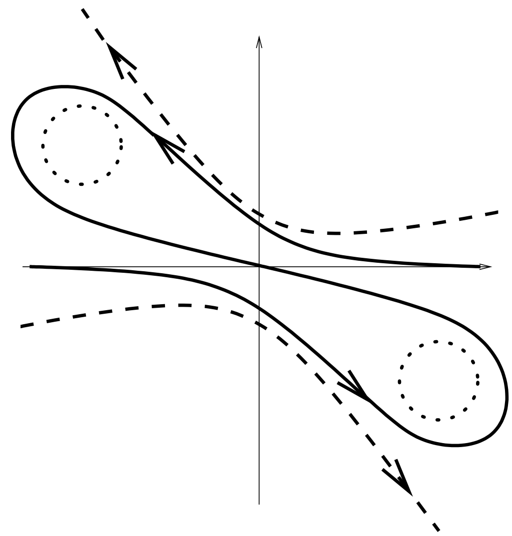

Figure 3. Lagrangian Hamiltonian isotopic to a plane developing a finite time singularity.

maximum principle implies that they can never intersect. Hence, the flow $(L_t)_{t\geq 0}$ must develop a finite-time singularity.

We end this section with a brief heuristic discussion of how could the flow $(L_t)_{t\geq 0}$ be continued after its finite-time singularity. It is expected that in the setting described above, the singularity occurs at the origin. In this situation, the Lagrangian surface at the time of the singularity decomposes into a union of an immersed 2-sphere (the immersion point being at the origin) and a Lagrangian surface diffeomorphic to the Lagrangian plane. As it was pointed out by Tom Ilmanen, there are two possible different evolutions for the Lagrangian surface after the singularity occurs: the immersed 2-sphere that has formed can evolve as an immersed 2-sphere or it can become an embedded torus which then evolves smoothly by mean curvature flow. In either case, the other connected piece will evolve smoothly to a Lagrangian plane.

5. Proof of Compactness Theorem A

The next proposition will be essential to prove Theorem A. As a mean of motivation, it could be easier to read first the proof of Theorem A and come back to the proposition when necessary.

Proposition 5.1. *Let (L^i) be a sequence of smooth zero-Maslov class Lagrangians in $\mathbb{C}^n$ such that, for some fixed $R>0$, the following properties hold:*

(a) *There exists a constant D_0 for which*

$$\mathcal{H}^n(L^i\cap B_{2R}(0))\leq D_0R^n \quad \textit{and} \quad \sup_{L^i\cap B_{2R}(0)}|\theta_i|\leq D_0$$

for all $i \in \mathbb{N}$.

(b)

$$\lim_{i\to\infty} \mathcal{H}^{n-1}(\partial L^i \cap B_{2R}(0)) = 0$$

and

$$\lim_{i\to\infty} \int_{L^i \cap B_{2R}(0)} |H|^2 \, d\mathcal{H}^n = 0.$$

Then there exist a finite set $\{\bar\theta_1, \ldots, \bar\theta_N\}$ *and integral Special Lagrangians*

$$L_1, \ldots, L_N$$

such that, after passing to a subsequence, we have for every smooth function ϕ *compactly supported in* $B_R(0)$ *and every* f *in* $C(\mathbb{R})$

$$\lim_{i\to\infty} \int_{L^i} f(\theta_i)\phi \, d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\bar\theta_j)\mu_j(\phi),$$

where μ_j *and* m_j *denote, respectively, the Radon measure of the support of* L_j *and its multiplicity.*

Proof. From Allard compactness theorem for varifolds [10, Theorem 42.7] we obtain the existence of a subsequence, still denoted by (L^i), converging in $B_{2R}(0)$ to a stationary integer rectifiable varifold L. Moreover,

$$\int_L \phi |\omega \wedge \eta| \, d\mathcal{H}^n = 0$$

for every $n-2$ form η and all smooth $\phi \in C^\infty_C(B_{2R}(0))$, and this implies that L is Lagrangian. It suffices to find integral Special Lagrangians

$$L_1, \ldots, L_N,$$

a finite set $\{\bar\theta_1, \ldots, \bar\theta_N\}$, and some positive ε_0 such that, after passing to a subsequence of (L^i), we have for all smooth ϕ compactly supported in $B_R(0)$, all $0 < \varepsilon < \varepsilon_0$, and all $j = 1, \ldots, N$,

$$\lim_{i\to\infty} \int_{\{|\theta_i - \bar\theta_j| \le \varepsilon\}} \phi \, d\mathcal{H}^n = m_j \mu_j(\phi)$$

and

$$\mu_L(\phi) = \sum_{j=1}^{N} m_j \mu_j(\phi),$$

where μ_L and μ_j denote the Radon measure of L and of the support of L_j respectively, and m_j denotes the multiplicity of L_j.

The idea for the proof is as follows. The regular points of L form a dense open set and therefore we can pick p in $L \cap B_R(0)$ such that, for some positive ρ, $B_\rho(p)$ is contained in $B_R(0)$ and the support of $L \cap B_\rho(p)$ is a smooth Special Lagrangian with angle $\bar\theta_1$. After adding some multiple of π to $\bar\theta_1$ if necessary, we will show the existence of an integral Special Lagrangian L_1

and of $\varepsilon_1 > 0$ such that, for all smooth ϕ with compact support in $B_R(0)$ and all $0 < \varepsilon \leq \varepsilon_1$, we have

$$\lim_{i\to\infty} \int_{\{|\theta_i - \bar{\theta}_1| \leq \varepsilon\}} \phi \, d\mathcal{H}^n = m_1 \mu_1(\phi), \tag{6}$$

where μ_1 is the Radon measure of the support of L_1 and m_1 its multiplicity. Because the support of L_1 is stationary, the monotonicity formula implies that

$$\mu_1(B_{2R}(0))R^{-n} \geq \mu_1(B_R(p))R^{-n} \geq \mu_1(B_\rho(p))\rho^{-n} \geq \gamma_n \tag{7}$$

for some universal constant γ_n.

In order to find $\bar{\theta}_2$ and the integral Special Lagrangian L_2, we repeat this process but this time applied to the sequence

$$P_i \equiv \{|\theta_i - \bar{\theta}_1| \geq \varepsilon_1\},$$

where the boundary will cause no difficulty because, as it will be seen in the proof of Lemma 5.2, we can assume that

$$\lim_{i\to\infty} \mathcal{H}^{n-1}(\{\theta_i = \bar{\theta}_1 \pm \varepsilon_1\} \cap B_{2R}(0)) = 0$$

and hence,

$$\begin{aligned} &\lim_{i\to\infty} \mathcal{H}^{n-1}(\partial P_i \cap B_{2R}(0)) \\ &\quad \leq \lim_{i\to\infty} \big(\mathcal{H}^{n-1}(\partial L_i \cap B_{2R}(0)) + \mathcal{H}^{n-1}(\{\theta_i = \bar{\theta}_1 \pm \varepsilon_1\} \cap B_{2R}(0))\big) = 0. \end{aligned}$$

Condition (a) and the inequality in (7) ensure that this will be done only finitely many times and hence the proposition will be proven as soon as we show (6).

The next lemma will be quite useful throughout the rest of the proof.

Lemma 5.2. *For almost all endpoints a and b, the sequence*

$$N^i \equiv \{a \leq \theta_i \leq b\}$$

contains a subsequence converging, in $B_{2R}(0)$, to a stationary integer rectifiable varifold N in the varifold sense and to an integral current $\widehat{N}$ with $\partial\widehat{N} = 0$ in the current sense.

Proof. For almost all endpoints a and b we have

$$\lim_{i\to\infty} \mathcal{H}^{n-1}\big((\{\theta_i = a\} \cup \{\theta_i = b\}) \cap B_{2R}(0)\big) = 0$$

because, by the coarea formula,

$$\begin{aligned} \int_{-\infty}^{\infty} \mathcal{H}^{n-1}(\{\theta_i = s\} \cap B_{2R}(0)) ds &= \int_{L^i \cap B_{2R}(0)} |H| \, d\mathcal{H}^n \\ &\leq \sqrt{D_0 R^n} \left(\int_{L^i \cap B_{2R}(0)} |H|^2 \, d\mathcal{H}^n \right)^{1/2}. \end{aligned}$$

The first variation formula yields for any vector field Y supported in $B_{2R}(0)$

$$\delta N^i(Y) = -\int_{N^i\cap B_{2R}(0)} \langle H, Y\rangle \, d\mathcal{H}^n + \oint_{\partial N^i \cap B_{2R}(0)} \langle Y, \nu\rangle \, d\mathcal{H}^{n-1},$$

where ν denotes the exterior unit normal. Hence, whenever the sup norm of Y satisfies $|Y|_\infty \leq 1$, we get

$$\begin{aligned} |\delta N^i(Y)| \leq \sqrt{C_0 R^n} \left(\int_{N^i\cap B_{2R}(0)} |H|^2 \, d\mathcal{H}^n \right)^{1/2} \\ + \mathcal{H}^{n-1}((\{\theta_i = a\} \cup \{\theta_i = b\}) \cap B_{2R}(0)) \\ + \mathcal{H}^{n-1}(\partial L^i \cap B_{2R}(0)) \end{aligned}$$

Furthermore, if ϑ is any $n-1$ form compactly supported in $B_{2R}(0)$ with $|\vartheta| \leq 1$, then

$$\begin{aligned} |\partial N^i(\vartheta)| \leq & \mathcal{H}^{n-1}((\{\theta_i = a\} \cup \{\theta_i = b\}) \cap B_{2R}(0)) \\ & + \mathcal{H}^{n-1}(\partial L^i \cap B_{2R}(0)). \end{aligned}$$

We can now apply Allard compactness theorem for varifolds and Federer and Fleming compactness theorem for currents (see [10, Theorem 27.3]) in order to complete the proof of the lemma. □

Condition (a) implies the existence of a finite set $F \subset \mathbb{N}$ such that, whenever $l \notin F$, we have for all $i \in \mathbb{N}$

$$\{|\theta_i - (\bar\theta_1 + l\pi)| \leq \pi\} \cap B_{2R}(0) = \emptyset.$$

Lemma 5.3. *There is a universal constant γ_n so that, for all $\varepsilon < \pi/2$,*

$$\lim_{i\to\infty} \sum_{l\in F} \mathcal{H}^n(\{|\theta_i - (\bar\theta_1 + l\pi)| \leq \varepsilon\} \cap B_\rho(p)) = \mathcal{H}^n(L \cap B_\rho(p)) \geq \gamma_n \rho^n.$$

Proof. The first equality is true because for almost all intervals $[a, b]$ such that

$$[a, b] \cap \{\bar\theta_1 + l\pi \,|\, l \in \mathbb{Z}\} = \emptyset,$$

we have

$$\limsup_{i\to\infty} \mathcal{H}^n\big(\{a \leq \theta_i \leq b\} \cap B_\rho(p)\big) = 0.$$

Otherwise we could, by Lemma 5.2, extract a subsequence converging to a integer rectifiable varifold N with support in L and such that

$$\mu(B_\rho(p)) > 0,$$

where μ is the Radon measure associated to N. This is impossible because for some positive δ we have

$$\sup_{\{a\leq\theta_i\leq b\}} |\cos(\theta_i - \bar\theta_1)| \leq 1 - \delta,$$

and so varifold convergence implies that

$$(1-\delta)\mu(B_\rho(p)) \geq \lim_{i\to\infty}\int_{\{a\leq\theta_i\leq b\}\cap B_\rho(p)}\left|\mathrm{Re}\left(e^{-i\bar{\theta}_1}\Omega\right)\right| d\mathcal{H}^n = \mu(B_\rho(p)).$$

□

Renaming $\bar{\theta}_1$ to be $\bar{\theta}_1 + l\pi$ for some l in F, we can find a sequence (ε_k) converging to zero and a constant $K = K(D_0)$ such that

$$\limsup_{i\to\infty}\mathcal{H}^n\big(\{|\theta_i-\bar{\theta}_1|\leq\varepsilon_k\}\cap B_\rho(p)\big)\geq K\rho^n \tag{8}$$

for all $k\in\mathbb{N}$.

Applying Lemma 5.2 to

$$N^{i,k}\equiv\{|\theta_i-\bar{\theta}_1|\leq\varepsilon_k\},$$

we obtain two sequences (N^k) and $(\widehat{N}^k)$ of stationary integer rectifiable varifolds and integral currents with no boundary respectively. Its Radon measures are denoted by μ_k and $\widehat{\mu}_k$ respectively. Federer and Fleming compactness Theorem implies that $(\widehat{N}^k)$ has a subsequence that converges in $B_{2R}(0)$ to an integral Lagrangian current L_1 with no boundary. Moreover, L_1 is an integral Special Lagrangian because it is calibrated by

$$\vartheta\equiv\mathrm{Re}\left(e^{-i\bar{\theta}_1}\Omega\right)$$

and it is nonempty because, using (8), we obtain that for every nonnegative smooth ϕ compactly supported in $B_{2R}(0)$

$$\begin{aligned}\int_{L_1}\phi\, d\mathcal{H}^n &\geq \lim_{k\to\infty}\widehat{\mu}_k(\vartheta\phi)=\lim_{k\to\infty}\lim_{i\to\infty}\int_{N^{i,k}}\vartheta\phi\\ &\geq\lim_{k\to\infty}\lim_{i\to\infty}\int_{N^{i,k}}\cos\varepsilon_k\phi\, d\mathcal{H}^n=\lim_{k\to\infty}\cos\varepsilon_k\mu_k(\phi)\geq K\rho^n.\end{aligned}$$

Furthermore, the support of each integral current $\widehat{N}^k$ is a stationary rectifiable varifold which, combined with the fact that

$$\widehat{\mu}_{k+1}(\phi)\leq\widehat{\mu}_k(\phi)$$

for every nonnegative ϕ compactly supported in $B_{2R}(0)$ and every $k\in\mathbb{N}$, implies that, for all k sufficiently large, $\widehat{N}^k$ must coincide with L_1 in $B_R(0)$.

□

Before proving Theorem A, we recall the monotonicity formula, found by Huisken in [8], valid for any smooth family of k-dimensional submanifolds $(N_t)_{t\geq 0}$ moving by mean curvature flow in $\mathbb{R}^m$. Consider the backward heat kernel

$$\Phi_{x_0,T}(x,t)=\frac{1}{(4\pi(T-t))^{k/2}}e^{-\frac{|x-x_0|^2}{4(T-t)}}.$$

When $(x_0,T)=(0,0)$, we denote it simply by Φ. The following formula holds

$$\frac{d}{dt}\int_{N_t} f_t\Phi_{x_0,T}\,d\mathcal{H}^n=\int_{N_t}\left(\frac{d}{dt}f_t-\Delta f_t-\left|H+\frac{(\mathbf{x}-\mathbf{x}_0)^{\perp}}{2(T-t)}\right|^2 f_t\right)\Phi_{x_0,T}\,d\mathcal{H}^n,$$

where f_t is a smooth function with polynomial growth at infinity and $(\mathbf{x}-\mathbf{x}_0)^{\perp}$ denotes the orthogonal projection on $(T_xN)^{\perp}$ of the vector determined by the point $(x-x_0)$ in $\mathbb{R}^m$.

Let $(L_t)_{0\le t<T}$ be a solution to Lagrangian mean curvature flow with a singularity at time T.

Theorem A. *If L_0 is zero-Maslov class with bounded Lagrangian angle, then for any sequence of rescaled flows $(L^i_s)_{s<0}$ at a singularity there exist a finite set $\{\bar\theta_1,\ldots,\bar\theta_N\}$ and integral Special Lagrangian cones*

$$L_1,\ldots,L_N$$

such that, after passing to a subsequence, we have for every smooth function ϕ compactly supported, every f in $C^2(\mathbb{R})$, and every $s<0$

$$\lim_{i\to\infty}\int_{L^i_s} f(\theta_{i,s})\phi\,d\mathcal{H}^n=\sum_{j=1}^N m_j f(\bar\theta_j)\mu_j(\phi),$$

where μ_j and m_j denote the Radon measure of the support of L_j and its multiplicity respectively.

Furthermore, the set $\{\bar\theta_1,\ldots,\bar\theta_N\}$ does not depend on the sequence of rescalings chosen.

Proof. We start with the following lemma

Lemma 5.4. *For any $a<b<0$ and any $R>0$, we have*

$$\lim_{i\to\infty}\int_a^b\int_{L^i_s\cap B_R(0)}\left(\left|\mathbf{x}^{\perp}\right|^2+|H|^2\right)\,d\mathcal{H}^n ds=0.$$

Proof. From Huisken's monotonicity formula we have that, for all $i\in\mathbb{N}$,

$$\frac{d}{ds}\int_{L^i_s}\theta^2_{i,s}\Phi\,d\mathcal{H}^n=\int_{L^i_s}\left(-2|H|^2-\left|H-\frac{\mathbf{x}^{\perp}}{2s}\right|^2\theta^2_{i,s}\right)\Phi\,d\mathcal{H}^n$$

and

$$\frac{d}{ds}\int_{L^i_s}\Phi\,d\mathcal{H}^n=\int_{L^i_s}-\left|H-\frac{\mathbf{x}^{\perp}}{2s}\right|^2\Phi\,d\mathcal{H}^n.$$

Using the scale invariance properties of the backward heat kernel, we obtain that

$$\lim_{i\to\infty}2\int_a^b\int_{L^i_s}|H|^2\Phi\,d\mathcal{H}^n ds\le\lim_{i\to\infty}\left(\int_{L^i_a}\theta^2_{i,a}\Phi\,d\mathcal{H}^n-\int_{L^i_b}\theta^2_{i,b}\Phi\,d\mathcal{H}^n\right)=0$$

and

$$\lim_{i\to\infty}\int_a^b\int_{L^i_s}\left|H-\frac{\mathbf{x}^\perp}{2s}\right|^2\Phi\, d\mathcal{H}^n ds=\lim_{i\to\infty}\left(\int_{L^i_a}\Phi\, d\mathcal{H}^n-\int_{L^i_b}\Phi\, d\mathcal{H}^n\right)=0.$$

Therefore

$$\lim_{i\to\infty}\int_a^b\int_{L^i_s}\left|\frac{\mathbf{x}^\perp}{2s}\right|^2\Phi\, d\mathcal{H}^n ds$$
$$\leq\lim_{i\to\infty}\int_a^b\int_{L^i_s}\left(\left|H-\frac{\mathbf{x}^\perp}{2s}\right|^2+|H|^2\right)\Phi\, d\mathcal{H}^n ds=0$$

and so the result follows. □

Pick $a<0$ for which

$$\lim_{i\to\infty}\int_{L^i_a\cap B_R(0)}\left(\left|\mathbf{x}^\perp\right|^2+|H|^2\right)=0$$

for all positive R.

The maximum principle implies that the Lagrangian angle θ_t is uniformly bounded and hence, by scale invariance, the same is true for the Lagrangian angle of L^i_a. Lemma B.1 implies the existence of a constant D_0 for which

$$\mathcal{H}^n\left(L^i_a\cap B_R(0)\right)\leq D_0R^n$$

for all positive R. We can, therefore, apply Proposition 5.1 to the sequence (L^i_a) and, after a simple diagonalization argument, obtain a subsequence for which there are integral Special Lagrangian currents

$$L_1,\ldots,L_N$$

and a finite set $\{\bar\theta_1,\ldots,\bar\theta_N\}$ such that, for every smooth function ϕ compactly supported and every f in $C^2(\mathbb{R})$,

$$\lim_{i\to\infty}\int_{L^i_a}f(\theta_{i,a})\phi\, d\mathcal{H}^n=\sum_{j=1}^N m_jf(\bar\theta_j)\mu_j(\phi),$$

where μ_j and m_j denote the Radon measure and the multiplicity of L_j respectively. The fact that

$$\lim_{i\to\infty}\int_{L^i_a\cap B_R(0)}\left|\mathbf{x}^\perp\right|^2 d\mathcal{H}^n=0$$

for all positive R implies that the Special Lagrangians L_j are all cones.

Next, we want to show that, for all $b<0$,

$$\lim_{i\to\infty}\int_{L^i_b}f(\theta_{i,b})\phi\, d\mathcal{H}^n=\lim_{i\to\infty}\int_{L^i_a}f(\theta_{i,a})\phi\, d\mathcal{H}^n=\sum_{j=1}^N m_jf(\bar\theta_j)\mu_j(\phi).$$

This comes from

$$\frac{d}{ds}\int_{L^i_s} f(\theta_{i,s})\phi\, d\mathcal{H}^n = \int_{L^i_s} f'(\theta_{i,s})\Delta\theta_{i,s}\phi\, d\mathcal{H}^n + \int_{L^i_s} f(\theta_{i,s})\langle H, D\phi\rangle\, d\mathcal{H}^n - \int_{L^i_s} f(\theta_{i,s})|H|^2\phi\, d\mathcal{H}^n$$

because, after integration with respect to the s variable, all terms on the right hand side vanish when i goes to infinity. We check this for the first term. Integrating by parts (and assuming $a < b$ for simplicity), we obtain

$$\int_a^b\int_{L^i_s} f'(\theta_{i,s})\Delta\theta_{i,s}\phi\, d\mathcal{H}^n ds = -\int_a^b\int_{L^i_s} f''(\theta_{i,s})|\nabla\theta_{i,s}|^2\phi\, d\mathcal{H}^n ds - \int_a^b\int_{L^i_s} f'(\theta_{i,s})\langle\nabla\theta_{i,s}, D\phi\rangle\, d\mathcal{H}^n ds$$

and hence, by Hölders's inequality, there is a constant $C = C(\phi, f, D_0, a, b)$ such that, for all $i \in \mathbb{N}$,

$$\int_a^b\int_{L^i_s} \left|f''(\theta_{i,s})|\nabla\theta_{i,s}|^2\phi\right|\, d\mathcal{H}^n ds \le C\int_a^b\int_{L^i_s} |H|^2\Phi\, d\mathcal{H}^n ds$$

and

$$\int_a^b\int_{L^i_s} \left|f'(\theta_{i,s})\langle\nabla\theta_{i,s}, D\phi\rangle\right|\, d\mathcal{H}^n ds \le C\left(\int_a^b\int_{L^i_s} |H|^2\Phi\right)^{1/2} d\mathcal{H}^n ds.$$

Finally, we show that $\{\bar\theta_1, \ldots, \bar\theta_N\}$ does not depend on the sequence of rescalings chosen. Let

$$\left(\widehat{L}^k_s\right)_{s<0}$$

be another sequence of rescaled flows for which there are Special Lagrangian cones

$$\widehat{L}_1, \ldots, \widehat{L}_P$$

and a finite set $\left\{\hat\theta_1, \ldots, \hat\theta_P\right\}$ such that, for every smooth function ϕ compactly supported, every f in $C^2(\mathbb{R})$, and every $s < 0$

$$\lim_{k\to\infty}\int_{\widehat{L}^k_s} f(\theta_{k,s})\phi\, d\mathcal{H}^n = \sum_{j=1}^P \widehat{m}_j f(\hat\theta_j)\widehat{\mu}_j(\phi),$$

where $\widehat{\mu}_j$ and $\widehat{m}_j$ denote the Radon measure of the support of L_j and its multiplicity respectively.

For any real number y and any integer q, we have the following evolution equation

$$\frac{d}{dt}(\theta_t - y)^{2q} = \Delta(\theta_t - y)^{2q} - 2q(2q-1)(\theta_t - y)^{2p-2}|H|^2.$$

Applying the monotonicity formula to $(\theta_t - y)^{2q}$, we get that

$$\frac{d}{dt}\int_{L_t}(\theta_t - y)^{2q}\Phi_{x_0,T}\,d\mathcal{H}^n \le 0$$

and thus, by scale invariance, we obtain for any $s, \bar{s} < 0$

$$\lim_{i\to\infty}\int_{L^i_s}(\theta_{i,s} - y)^{2q}\Phi\,d\mathcal{H}^n = \lim_{k\to\infty}\int_{\widehat{L}^k_{\bar{s}}}(\theta_{i,\bar{s}} - y)^{2q}\Phi\,d\mathcal{H}^n$$
$$= \lim_{t\to T}\int_{L_t}(\theta_s - y)^{2q}\Phi_{x_0,T}\,d\mathcal{H}^n.$$

Therefore

$$\sum_{j=1}^{N} m_j(\bar{\theta}_j - y)^{2q}\mu_j(\Phi) = \sum_{j=1}^{P}\widehat{m}_j\left(\hat{\theta}_j - y\right)^{2q}\widehat{\mu}_j(\Phi)$$

for all positive integer q and all y in $\mathbb{R}$ and this implies that

$$\left\{\theta_1, \ldots \theta_N\right\} = \left\{\hat{\theta}_1, \ldots \hat{\theta}_P\right\}.$$

□

6. Evolution Equations

Let L_0 be a rational and zero-Maslov Lagrangian submanifold of $\mathbb{C}^n$. We will argue now that the rational condition is preserved by the flow. Denoting by F_t the normal deformation by mean curvature, we have

$$\frac{d}{dt}\int_{F_t(\gamma)}\lambda = \frac{d}{dt}\int_{\gamma}F_t^*\lambda = \int_{\gamma}\mathcal{L}_H F_t^*\lambda$$
$$= \int_{\gamma} dF_t^*(H\lrcorner\lambda) + F_t^*(H\lrcorner 2\omega) = \int_{\gamma} dF_t^*(H\lrcorner\lambda - 2\theta_t) = 0$$

for every $[\gamma]$ in $H_1(L_0)$. Hence

$$[\lambda] = [F_t^*(\lambda)] \quad \text{in } H^1(L_0)$$

for all times where the solution exists smoothly and therefore it follows that

$$\lambda\left(H_1(L_t,\mathbb{Z})\right) = \lambda\left(H_1(L_0,\mathbb{Z})\right) = \{a2k\pi \mid k\in\mathbb{Z}\}.$$

Thus, there is a smooth family of multivalued functions

$$\beta_t : L_t \longrightarrow \mathbb{R}/2\pi a\mathbb{Z}$$

such that

$$\nabla\beta_t(x) = (J\mathbf{x})^{\top} \quad \text{for all } x\in L_t.$$

Proposition 6.1. *The functions β_t can be chosen so that*

$$\frac{d\beta_t}{dt} = \Delta\beta_t - 2\theta_t.$$

Proof. Assume, without loss of generality, that the family of functions β_t is smooth with respect to the time parameter. We have

Lemma 6.2.

$$\Delta\beta_t = H\lrcorner\lambda,$$

Proof. We use a normal coordinate system around the point x and denote the coordinate vectors by $\{\partial_1, \cdots, \partial_n\}$. The result follows from

$$\begin{aligned}\langle \nabla_{\partial_i}(J\mathbf{x})^\top, \partial_j\rangle &= \partial_i\langle J\mathbf{x}, \partial_j\rangle - \langle (J\mathbf{x})^\top, D_{\partial_i}\partial_j\rangle = \langle J\partial_i, \partial_j\rangle + \langle (J\mathbf{x})^\perp, D_{\partial_i}\partial_j\rangle \\ &= \langle J\mathbf{x}, A_{ij}\rangle.\end{aligned}$$

□

Thus,

$$d\left(\frac{d\beta_t}{dt}\right) = \frac{d\lambda}{dt} = \mathcal{L}_H\lambda = d(H\lrcorner\lambda) + H\lrcorner 2\omega = d(\Delta\beta_t - 2\theta_t)$$

and so we can add a time dependent constant to each β_t so that the desired result follows. □

Given any t_0 in $\mathbb{R}$ and any k in $\mathbb{Z}$, the function

$$u_t \equiv \cos\left(\frac{k(\beta_t + 2(t-t_0)\theta_t)}{a}\right)$$

is well defined on L_t. If L_0 is exact, take $a = 1$. A straightforward computation using Proposition 6.1 and

$$J\mathbf{x}^\perp = (J\mathbf{x})^\top$$

gives

Corollary 6.3.

$$\frac{du_t}{dt} = \Delta u_t + u_t\left|\frac{k(\mathbf{x}^\perp + 2(t_0 - t)H)}{a}\right|^2.$$

7. Proof of Compactness Theorem B

Theorem B. *If L_0 is almost-calibrated and rational, then after passing to a subsequence of $(L^i_s)_{s<0}$, the following property holds for all $R > 0$ and almost all $s < 0$.*

For any convergent subsequence (in the Radon measure sense) Σ^i of connected components of $B_{4R}(0) \cap L^i_s$ intersecting $B_R(0)$ there exists a Special Lagrangian cone L in $B_{2R}(0)$ with Lagrangian angle $\bar\theta$ such that

$$\lim_{i\to\infty}\int_{\Sigma^i} f(\theta_{i,s})\phi\, d\mathcal{H}^n = mf(\bar\theta)\mu(\phi),$$

for every f in $C(\mathbb{R})$ and every smooth ϕ compactly supported in $B_{2R}(0)$, where μ and m denote the Radon measure of the support of L and its multiplicity respectively.

Proof. The almost-calibrated condition is preserved by the flow and implies the following lemma.

Lemma 7.1. *There is a constant C_1 such that, for all $s < 0$,*

$$(\mathcal{H}^n(A))^{(n-1)/n} \leq C_1 \mathcal{H}^{n-1}(\partial A),$$

where A is any open subset of L^i_s with rectifiable boundary.

Proof. The Isoperimetric Theorem [10, Theorem 30.1] guarantees the existence of an integral current B with compact support such that $\partial B = \partial A$ and for which

$$(\mathcal{H}(B))^{(n-1)/n} \leq C\mathcal{H}^{n-1}(\partial A),$$

where $C = C(n)$. If T denotes the cone over the current $A - B$ (see [10, page 141]), then $\partial T = A - B$ and thus, because

$$\operatorname{Re}\Omega_{|L^i_s} = \cos\theta_{i,s} \geq \varepsilon_0$$

for some positive ε_0, we obtain

$$\begin{aligned}\mathcal{H}^n(A) &\leq \varepsilon_0^{-1}\int_A \operatorname{Re}\Omega = \varepsilon_0^{-1}\int_B \operatorname{Re}\Omega + \partial T(\operatorname{Re}\Omega)\\ &\leq \varepsilon_0^{-1}\mathcal{H}^n(B) + T(d\operatorname{Re}\Omega) \leq \varepsilon_0^{-1}\left(C\mathcal{H}^{n-1}(\partial A)\right)^{n/(n-1)}.\end{aligned}$$

□

The discussion in Section 6 implies the existence of $a \in \mathbb{R}$ and of a family of multivalued functions

$$\beta_{i,s} : L^i_s \longrightarrow \mathbb{R}/\sigma_i^2 a 2\pi\mathbb{Z}$$

such that

$$\nabla\beta_{i,s}(x) = (J\mathbf{x})^\top$$

for all $x \in L^i_s$ and all $s < 0$. Furthermore, we can choose a bounded sequence (b_i) so that, for any real number s_0,

$$u_{i,s} \equiv \cos\left(\frac{\beta_{i,s} + 2(s - s_0)\theta_{i,s}}{b_i}\right)$$

is a well defined function. After passing to a subsequence, the sequence (b_i) converges to $b \neq 0$ and, for simplicity, we assume that $b = 1$. Furthermore, from Lemma 5.4, we can also assume that

$$\lim_{i\to\infty}\int_{L^i_{-1}\cap B_R(0)} \left(|H|^2 + \left|\mathbf{x}^\perp\right|^2\right) d\mathcal{H}^n = 0$$

for all $R > 0$.

Lemma 7.2. *There is a set*

$$\{(\cos\bar\beta_1, \sin\bar\beta_1), \ldots, (\cos\bar\beta_Q, \sin\bar\beta_Q)\}$$

and integral Special Lagrangian cones

$$P_1, \ldots, P_Q$$

such that, after passing to a subsequence, we have for all smooth ϕ with compact support and all f in $C(\mathbb{R})$,

$$\lim_{i\to\infty}\int_{L^i_{-1}} f(\cos(\beta_{i,-1}/b_i))\phi\, d\mathcal{H}^n = \sum_{k=1}^{Q} p_k f(\cos\bar{\beta}_k)\nu_k(\phi)$$

$$\lim_{i\to\infty}\int_{L^i_{-1}} f(\sin(\beta_{i,-1}/b_i))\phi\, d\mathcal{H}^n = \sum_{k=1}^{Q} p_k f(\sin\bar{\beta}_k)\nu_k(\phi),$$

where ν_k and the positive integer p_k denote the Radon measure of the support of P_k and its multiplicity respectively.

Proof. Let (R_k) denote a sequence of positive numbers going to infinity. We start by arguing the existence of a uniform bound on the number of connected components of $L^i_{-1}\cap B_{4R_k}(0)$ that intersect $B_{R_k}(0)$. For any x in $L^i_{-1}\cap B_{2R_k}(0)$, denote the intrinsic ball of radius r around x by $\widehat{B}_i(x,r)$. Set

$$\psi_i(r)\equiv\mathcal{H}^n\left(\widehat{B}_i(x,r)\right)$$

which has, for almost all r, derivative given by

$$\psi_i'(r)=\mathcal{H}^{n-1}\left(\partial\widehat{B}_i(x,r)\right).$$

We know from Lemma 7.1 that, for all $r<R_k$,

$$(\psi_i(r))^{(n-1)/n}\le C_1\psi_i'(r)$$

and so

$$\mathcal{H}^n\left(\widehat{B}_i(x,r)\right)\ge Kr^n$$

for all x in $L^i_{-1}\cap B_{2R_k}(0)$, where $K=K(n,C_1)$. Hence, each connected component has area bigger than KR^n and so the claim follows from the uniform area bounds for L^i_{-1} (Lemma B.1).

From Proposition 5.1 we know that, after passing to a subsequence, all the connected components of $L^i_{-1}\cap B_{4R_k}(0)$ intersecting $B_{R_k}(0)$ converge to a union of Special Lagrangian cones in $B_{2R_k}(0)$. Moreover,

$$|\nabla\beta_{i,-1}(x)|=\left|(J\mathbf{x})^{\top}\right|=\left|\mathbf{x}^{\perp}\right|$$

and thus the functions $\cos(\beta_{i,-1}/b_i)$ and $\sin(\beta_{i,-1}/b_i)$ satisfy the conditions of Proposition A.1. We can, therefore, apply this result to all the connected components of $L^i_{-1}\cap B_{2R_k}(0)$ intersecting $B_{R_k}(0)$. A standard diagonalization method finds a subsequence that works for all R_k and so the lemma is proved. □

Combining this lemma with Theorem A we obtain that, after a rearrangement of the supports of the Special Lagrangian cones and its multiplicities (which we still denote by

$$L_1,\ldots,L_N$$

and $m_1, \dots, m_N$ respectively), we have for all ϕ with compact support, all f in $C(\mathbb{R})$, and all $y \in \mathbb{R}$,

$$\text{(9)} \quad \lim_{i\to\infty} \int_{L^i_{-1}} f\left(\cos\left(\frac{\beta_{i,-1} + 2y\theta_{i,-1}}{b_i}\right)\right) \phi \, d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\cos(\bar{\beta}_j + 2y\bar{\theta}_j))\mu_j(\phi)$$

where μ_j denotes the Radon measure of the support of L_j and the elements of the set

$$\{(\cos\bar{\beta}_1, \sin\bar{\beta}_1, \bar{\theta}_1), \dots, (\cos\bar{\beta}_N, \sin\bar{\beta}_N, \bar{\theta}_N)\}$$

are all distinct.

Using the evolution equation for $u_{i,s}$ we show

Lemma 7.3. *For all ϕ with compact support, all f in $C^2(\mathbb{R})$, and all $s < 0$,*

$$\lim_{i\to\infty} \int_{L^i_s} f(\cos(\beta_{i,s}/b_i))\phi \, d\mathcal{H}^n = \sum_{j=1}^{N} m_j f(\cos(\bar{\beta}_j - 2(s+1)\bar{\theta}_j))\mu_j(\phi).$$

Proof. Corollary 6.1 implies that, for all ϕ with compact support, all f in $C^2(\mathbb{R})$, and all $s_0 < 0$,

$$\text{(10)} \quad \frac{d}{ds}\int_{L^i_s} f(u_{i,s})\phi \, d\mathcal{H}^n = \int_{L^i_s} f'(u_{i,s})\Delta u_{i,s}\phi \, d\mathcal{H}^n + \int_{L^i_s} f'(u_{i,s})u_{i,s}\left|\frac{\mathbf{x}^\perp + 2(s_0 - s)H}{b_i}\right|^2 \phi \, d\mathcal{H}^n + \int_{L^i_s} f(u_{i,s})\langle H, D\phi\rangle \, d\mathcal{H}^n - \int_{L^i_s} f(u_{i,s})|H|^2\phi \, d\mathcal{H}^n.$$

From Lemma 5.4, we obtain that (assuming $-1 < s_0 < 0$ for simplicity)

$$\lim_{i\to\infty} \int_{-1}^{s_0}\int_{L^i_s\cap B_R(0)} \left|\frac{\mathbf{x}^\perp + 2(s_0 - s)H}{b_i}\right|^2 d\mathcal{H}^n \le \lim_{i\to\infty} 8\int_{-1}^{s_0}\int_{L^i_s\cap B_R(0)} \left(\frac{(s - s_0)^2|H|^2 + |\mathbf{x}^\perp|^2}{b_i^2}\right) d\mathcal{H}^n = 0$$

for all positive R.

This inequality allows us to argue in the same way as it was done in the proof of Theorem A and show that, after integration with respect to the s variable, all terms on the right hand side of (10) converge to zero when i goes to infinity. Thus, because

$$u_{i,s_0} = \cos(\beta_{i,s_0}/b_i) \quad \text{and} \quad u_{i,-1} = \cos\left(\frac{\beta_{i,-1} - 2(1 + s_0)\theta_{i,-1}}{b_i}\right),$$

we obtain from identity (9)

$$\lim_{i\to\infty}\int_{L^i_{s_0}} f(\cos(\beta_{i,s_0}/b_i))\phi\, d\mathcal{H}^n$$
$$= \lim_{i\to\infty}\int_{L^i_{-1}} f\left(\cos\left(\frac{\beta_{i,-1}-2(1+s_0)\theta_{i,-1}}{b_i}\right)\right)\phi\, d\mathcal{H}^n$$
$$= \sum_{j=1}^N m_j f(\cos(\beta_j - 2(1+s_0)\theta_j))\mu_j(\phi).$$

The result follows from the arbitrariness of s_0. □

The proof of the theorem can now be completed. Because the elements of the set

$$\{(\cos\bar\beta_1, \sin\bar\beta_1, \bar\theta_1), \ldots, (\cos\bar\beta_N, \sin\bar\beta_N, \bar\theta_N)\}$$

are all distinct we get that, for all but countably many s, the real numbers

$$\cos(\bar\beta_1 - 2(s+1)\bar\theta_1), \ldots, \cos(\bar\beta_N - 2(s+1)\bar\theta_N)$$

are all distinct. Moreover, Lemma 5.4 implies that, for almost all $s<0$,

$$\lim_{i\to\infty}\int_{L^i_s\cap B_R(0)}\left(|H|^2 + \left|\mathbf{x}^\perp\right|^2\right) d\mathcal{H}^n = 0$$

for all $R>0$.

Pick s so that both conditions described above hold and consider a subsequence of connected components Σ^i of $B_{4R}(0)\cap L^i_s$ intersecting $B_R(0)$ that converges weakly to Σ. The arguments presented in the proof of Lemma 5.4 imply that Σ has positive measure. We first show that Σ is a Special Lagrangian cone.

Proposition A.1 can be applied to the sequence Σ_i and thus, after passing to a subsequence, $(\cos(\beta_{i,s}/b_i))$ converges to a constant γ. Define $f\in C^2(\mathbb{R})$ to be a nonnegative cutoff function that is one in small neighborhood of γ and zero everywhere else.

Denoting by μ_Σ the Radon measure of Σ, we obtain from Lemma 7.3 that for every nonnegative test function ϕ with support in $B_{2R}(0)$

$$\mu_\Sigma(\phi) = \lim_{i\to\infty}\int_{\Sigma_i}\phi\, d\mathcal{H}^n = \lim_{i\to\infty}\int_{\Sigma_i} f(\cos(\beta_{i,s}/b_i))\phi\, d\mathcal{H}^n$$
$$\le \lim_{i\to\infty}\int_{L^i_s} f(\cos(\beta_{i,s}/b_i))\phi\, d\mathcal{H}^n = \sum_{j=1}^N m_j f(\cos(\bar\beta_j - 2(s+1)\bar\theta_j))\mu_j(\phi).$$

Because the support of f can be chosen arbitrarily small and the real numbers

$$\cos(\bar\beta_1 - 2(s+1)\bar\theta_1), \ldots, \cos(\bar\beta_N - 2(s+1)\bar\theta_N)$$

are all distinct, the above inequality implies that

$$\gamma = \cos(\bar\beta_{j_0} - 2(s+1)\bar\theta_{j_0})$$

for a unique j_0. Thus

$$\mu_\Sigma(\phi) \le m_{j_0}\mu_{j_0}(\phi)$$

for every $\phi \ge 0$ and, as a result, the support of Σ must be contained in L_{j_0}.

Finally, suppose there are f continuous and ϕ compactly supported in $B_{2R}(0)$ such that

$$\int_{\Sigma^i} f(\theta_{i,s})\phi \, d\mathcal{H}^n$$

has two distinct convergent subsequences. We can use Proposition 5.1 to get a contradiction because L_0 being almost-calibrated implies that any two Special Lagrangian cones with support contained in the support of Σ have the same Lagrangian angle. □

Appendix A

Suppose we have a sequence of functions (α_i) defined on a sequence of manifolds (N^i) converging weakly to N and such that the L^2-norm of $|\nabla\alpha_i|$ converges to zero. The next proposition gives conditions under which, after passing to a subsequence, (α_i) converges to a constant. Before giving its proof, we comment on the necessity of all the hypothesis.

Proposition A.1. *Let (N^i) and (α_i) be a sequence of smooth k-submanifolds in $\mathbb{R}^n$ and smooth functions on N^i respectively, such that (N^i) converges weakly to an integer rectifiable stationary k-varifold N and*

$$\lim_{i\to\infty} \int_{N^i \cap B_{3R}(0)} \left(|H|^2 + |\nabla\alpha_i|^2\right) \, d\mathcal{H}^n = 0$$

for some $R > 0$.

Assume that the following properties hold.

a) *There exists a constant D_0 such that*

$$\mathcal{H}^k(N^i \cap B_{3R}) \le D_0 R^k$$

for all $i \in \mathbb{N}$, and

$$\left(\mathcal{H}^k(A)\right)^{(k-1)/k} \le D_0 \mathcal{H}^{k-1}(\partial A)$$

for all open subsets A of $N^i \cap B_{3R}$ with rectifiable boundary.

b) *There exists a constant D_1 for which*

$$\sup_{N^i \cap B_{3R}(0)} |\nabla\alpha_i| + R^{-1} \sup_{N^i \cap B_3(0)} |\alpha_i| \le D_1$$

for all $i \in \mathbb{N}$.

c) *For all $i \in \mathbb{N}$,*

$$N^i \cap B_{2R}(0) \quad \textit{is connected}$$

and

$$\partial(N^i \cap B_{3R}(0)) \subset \partial B_{3R}(0).$$

Then, there is a real number α such that, after passing to a subsequence, we have for all ϕ with compact support in $B_R(0)$ and all f in $C(\mathbb{R})$

$$\lim_{i\to\infty}\int_{N^i} f(\alpha_i)\phi = f(\alpha)\mu_N(\phi),$$

where μ_N denotes the Radon measure associated to N.

The first hypothesis is needed in order to ensure lower density bounds on N^i. The second hypothesis is essential because, without the pointwise bounds on $|\nabla\alpha_i|$ and α_i, the result would be false. Finally, the last hypothesis is needed because otherwise the proposition would fail for trivial reasons.

Proof. It suffices to find $\alpha\in\mathbb{R}$ and a sequence (ε_j) converging to zero such that, for some appropriate subsequence, we have for all $j\in\mathbb{N}$

$$\lim_{i\to\infty}\mathcal{H}^k(\{|\alpha_i-\alpha|\le\varepsilon_j\}\cap B_R(0))=\mathcal{H}^k(N\cap B_R(0)).$$

For the rest of this proof, $K=K(D_0,D_1,k)$ will denote a generic constant depending only on the mentioned quantities. Choose any sequence (x_i) in $N^i\cap B_R(0)$. After passing to a subsequence, we have that

$$\lim_{i\to\infty} x_i = x_0 \quad\text{and}\quad \lim_{i\to\infty}\alpha_i(x_i)=\alpha$$

for some $x_0\in B_R(0)$ and $\alpha\in\mathbb{R}$. Furthermore, consider also a sequence (ε_j) converging to zero such that, for all $j\in\mathbb{N}$,

$$\lim_{i\to\infty}\mathcal{H}^{k-1}\big(\{\alpha_i=\alpha\pm\varepsilon_j\}\cap B_{3R}\big)=0.$$

Such a subsequence exists because, by the coarea formula, we have

$$\begin{aligned}\lim_{i\to\infty}\int_{-\infty}^{\infty}\mathcal{H}^{k-1}\big(\{\alpha_i=s\}\cap B_{3R}\big)ds &= \lim_{i\to\infty}\int_{N^i\cap B_{3R}}|\nabla\alpha_i|\,d\mathcal{H}^n\\ &\le \lim_{i\to\infty} KR^{k/2}\left(\int_{N^i\cap B_{3R}}|\nabla\alpha_i|^2\,d\mathcal{H}^n\right)^{1/2}=0.\end{aligned}$$

Define

$$N^{i,\alpha,j}\equiv\{|\alpha_i-\alpha|\le\varepsilon_j\}.$$

The first variation formula yields, for any vector field Y supported in B_{3R},

$$\delta N^{i,\alpha,j}(Y)=-\int_{N^{i,\alpha,j}\cap B_{2R}}\langle H,Y\rangle\,d\mathcal{H}^n+\oint_{\partial\{|\alpha_i-\alpha|\le\varepsilon_j\}\cap B_{2R}}\langle Y,\nu\rangle\,d\mathcal{H}^{n-1}$$

where ν denotes the exterior unit normal. Hence, whenever the sup norm of Y satisfies $|Y|_\infty\le 1$, we get

$$\begin{aligned}|\delta N^{i,\alpha,j}(Y)|\le KR^{k/2}&\left(\int_{N^{i,\alpha,j}\cap B_{2R}}|H|^2\,d\mathcal{H}^n\right)^{1/2}\\ &+\mathcal{H}^{k-1}\big(\{\alpha_i=\alpha\pm\varepsilon_j\}\cap B_{2R}\big).\end{aligned}$$

We can now apply Allard compactness theorem to conclude that, after passing to a subsequence, we have convergence to an integer rectifiable stationary varifold $N^{\alpha,j}$. By a standard diagonalization argument, we can find a subsequence that works for every positive integer j.

Lemma A.2. *For all* $j \in \mathbb{N}$,

$$\mathcal{H}^k\big(N^{\alpha,j} \cap B_R(x_0)\big) \geq KR^k.$$

Proof. Set

$$\psi_i(s) \equiv \mathcal{H}^k\big(\{|\alpha_i - \alpha_i(x_i)| \leq s\} \cap B_s(x_i)\big)$$

which, by the coarea formula, has derivative equal to

$$\psi_i'(s) = \oint_{\partial B_s(x_i)\cap\{|\alpha_i-\alpha_i(x_i)|\leq s\}} \frac{|x - x_i|}{|(\mathbf{x} - \mathbf{x}_i)^\top|}\, d\mathcal{H}^{n-1} + \oint_{B_s(x_i)\cap\partial\{|\alpha_i-\alpha_i(x_i)|\leq s\}} \frac{1}{|\nabla\alpha_i|}\, d\mathcal{H}^{n-1}$$

for almost all s. We can estimate

$$\begin{aligned}\psi_i'(s) \geq \mathcal{H}^{k-1}&\big(\partial B_s(x_i) \cap \{|\alpha_i - \alpha_i(x_i)| \leq s\}\big)\\ &+ K\mathcal{H}^{k-1}\big(B_s(x_i) \cap \partial\{|\alpha_i - \alpha_i(x_i)| \leq s\}\big)\\ &\geq K\mathcal{H}^{k-1}\big(\partial(B_s(x_i) \cap \{|\alpha_i - \alpha_i(x_i)| \leq s\})\big)\end{aligned}$$

and so, using the isoperimetric condition a), we obtain

$$(\psi_i(s))^{(k-1)/k} \leq D_0\mathcal{H}^{k-1}\big(\partial(B_s(x_i) \cap \{|\alpha_i - \alpha_i(x_i)| \leq s\})\big) \leq K\psi_i'(s)$$

for almost all $s \leq R$. This implies that

$$s^{-k}\mathcal{H}^k\big(\{|\alpha_i - \alpha_i(x_i)| \leq s\} \cap B_s(x_i)\big) \geq K$$

for all $s \leq R$. This inequality and the inclusion

$$\{|\alpha_i - \alpha_i(x_i)| \leq \varepsilon_j/2\} \cap B_{\varepsilon_j/2}(x_i) \subset \{|\alpha_i - \alpha| \leq \varepsilon_j\} \cap B_{\varepsilon_j}(x_0),$$

valid for all i sufficiently large, imply that

$$\begin{aligned}\varepsilon_j^{-k}\mathcal{H}^k&\big(N^{i,\alpha,j} \cap B_{\varepsilon_j}(x_0)\big)\\ &\geq \varepsilon_j^{-k}\mathcal{H}^k\big(\{|\alpha_i - \alpha(x_i)| \leq \varepsilon_j/2\} \cap B_{\varepsilon_j/2}(x_i)\big) \geq K\end{aligned}$$

for all i sufficiently large. Taking the limit when i goes to infinity and recalling that $N^{\alpha,j}$ is a stationary varifold we get, by the monotonicity formula, that

$$R^{-k}\mathcal{H}^k\big(N^{\alpha,j} \cap B_R(x_0)\big) \geq \varepsilon_j^{-k}\mathcal{H}^k\big(N^{\alpha,j} \cap B_{\varepsilon_j}(x_0)\big) \geq K$$

for all $j \in \mathbb{N}$. □

Suppose that for some positive integer j we have

$$\mathcal{H}^k\big(N^{\alpha,j} \cap B_R(0)\big) < \mathcal{H}^k(N \cap B_R(0)).$$

Repeating the same type of arguments, we can find y_0 in $B_R(0)$ and a closed interval I disjoint from $[\alpha-\varepsilon_j, \alpha+\varepsilon_j]$ so that, after passing to a subsequence,

$$\lim_{i\to\infty} \mathcal{H}^k\big(\alpha_i^{-1}(I)\cap B_R(y_0)\big) \geq KR^k.$$

Given any positive integer p, pick disjoint closed intervals

$$I_1, \cdots, I_p$$

lying between I and $[\alpha-\varepsilon_j, \alpha+\varepsilon_j]$. The connectedness of $N^i \cap B_{2R}(0)$ implies that all $\alpha_i^{-1}(I_l)\cap B_{2R}(0)$ are nonempty for i sufficiently large. Hence, arguing as before, we find $y_1, \ldots, y_p$ in $B_{2R}(0)$ such that, after passing to a subsequence,

$$\lim_{i\to\infty} \mathcal{H}^k\big(\alpha_i^{-1}(I_l)\cap B_R(y_l)\big) \geq KR^k,$$

for all l in $\{1, \ldots, p\}$. This implies that

$$\begin{aligned}\lim_{i\to\infty} \mathcal{H}^k\big(N^i\cap B_{2R}(0)\big) &\geq \lim_{i\to\infty}\sum_{l=1}^{p} \mathcal{H}^k\big(\alpha_i^{-1}(I_j)\cap B_R(y_l)\big)\\ &\geq pKR^k.\end{aligned}$$

Choosing p sufficiently large we get a contradiction. □

Appendix B

The next lemma is a simple modification of a result that can be found in Ecker's book [5] and Ilmanen's preprint [9]. The proof is the same but we write it here for the sake of completeness.

Lemma B.1. *Let $(M_t)_{t\geq 0}$ be family of k-dimensional submanifolds $(M_t)_{t\geq 0}$ moving by mean curvature flow in $\mathbb{R}^m$. Assume there are constants A_0 and R_0 such that*

$$\mathcal{H}^k\big(M_0\cap B_r(0)\big) \leq A_0 r^k,$$

for all $r \geq R_0$. Then, for all $t \geq t_0$ and $x_0 \in \mathbb{R}^m$, there is a constant $C = C(A_0, R_0/\sqrt{t_0}, |x_0|)$ such that

$$\mathcal{H}^k\big(M_t\cap B_r(x_0)\big) \leq Cr^k$$

for all $r > 0$.

Proof. In what follows, $C = C(A_0, t_0^{-1}, R_0, |x_0|)$ will denote a constant depending only on the mentioned quantities. Using the monotonicity formula we obtain

$$\begin{aligned}
\frac{\mathcal{H}^k\big(M_t \cap B_r(x_0)\big)}{r^k} &\le C\int_{M_t} \frac{1}{(4\pi r^2)^{k/2}} e^{-\frac{|x-x_0|^2}{4r^2}}\, d\mathcal{H}^n\\
&\le C\int_{M_0} \frac{1}{(4\pi (t+r^2))^{k/2}} e^{-\frac{|x-x_0|^2}{4(t+r^2)}}\, d\mathcal{H}^n\\
&\le C\int_{M_0} \frac{1}{(4\pi (t+r^2))^{k/2}} e^{-\frac{|x|^2}{8(t+r^2)}}\, d\mathcal{H}^n\\
&\le C\int_{\lambda M_0} e^{-|x|^2}\, d\mathcal{H}^n,
\end{aligned}$$

where $\lambda \equiv (8(t+r^2))^{-1/2}$. For all $s \ge \lambda R_0$ we have

$$\mathcal{H}^k\big(\lambda M_0 \cap B_s(0)\big) \le A_0 s^k$$

and thus, setting $R_1 \equiv \max\{2, (8t_0)^{-1/2}R_0\}$, the result follows from

$$\begin{aligned}
\int_{\lambda M_0} e^{-|x|^2}\, d\mathcal{H}^n &\le A_0 R_1^k + \int_{\lambda M_0 \setminus B_{R_1}} e^{-|x|^2}\, d\mathcal{H}^n\\
&= A_0 R_1^k + \sum_{j\ge 0}\int_{\lambda M_0 \cap \left(B_{R_1^{j+1}} \setminus B_{R_1^j}\right)} e^{-|x|^2}\, d\mathcal{H}^n\\
&\le A_0 R_1^k + \sum_{j\ge 0} A_0 R_1^{j+1} e^{-R_1^{2j}}.
\end{aligned}$$

□

References

[1] H. Anciaux, Mean curvature flow and self-similar submanifolds, **Séminaire de Théorie Spectrale et Gémométrie. Vol. 21** Année 2002–2003, 43–53.

[2] S. Angenent, Parabolic equations for curves on surfaces. II. Intersections, blow-up and generalized solutions, **Ann. of Math. (2) 133** (1991), 171–215.

[3] J. Chen and J. Li, Singularity of mean curvature flow of Lagrangian submanifolds, **Invent. Math. 156** (2004), 25–51.

[4] J. Chen, J. Li and G. Tian, Two-dimensional graphs moving by mean curvature flow, **Acta Math. Sin. 18**, (2002), 209–224.

[5] K. Ecker, Regularity theory for mean curvature flow, **Progress in Nonlinear Differential Equations and their Applications, 57**, Birkhuser Boston, MA, 2004.

[6] K. Ecker and G. Huisken, Mean curvature evolution of entire graphs, **Ann. of Math. (2) 130** (1989), 453–471.

[7] R. Harvey and H. B. Lawson, H. Calibrated geometries, **Acta Math. 148** (1982), 47–157.

[8] G. Huisken, Asymptotic behavior for singularities of the mean curvature flow, **J. Differential Geom. 31** (1990), 285–299.

[9] T. Ilmanen, Singularities of Mean Curvature Flow of Surfaces, preprint.

[10] L. Simon, Lectures on geometric measure theory, **Proceedings of the Centre for Mathematical Analysis, Australian National University, 3**.

[11] R. Schoen and J. Wolfson, Minimizing area among Lagrangian surfaces: the mapping problem, **J. Differential Geom. 58** (2001), 1–86.

[12] K. Smoczyk, A canonical way to deform a Lagrangian submanifold, preprint.
[13] K. Smoczyk, Harnack inequality for the Lagrangian mean curvature flow, **Calc. Var. Partial Differential Equations 8** (1999), 247–258.
[14] K. Smoczyk, Angle theorems for the Lagrangian mean curvature flow, **Math. Z. 240** (2002), 849–883.
[15] K. Smoczyk, Longtime existence of the Lagrangian mean curvature flow, **Calc. Var. Partial Differential Equations 20** (2004), 25–46.
[16] K. Smoczyk and M.-T. Wang, Mean curvature flows of Lagrangians submanifolds with convex potentials, **J. Differential Geom. 62** (2002), 243–257.
[17] M.-P. Tsui and M.-T. Wang, Mean curvature flows and isotopy of maps between spheres, **Comm. Pure Appl. Math. 57** (2004), 1110–1126.
[18] M.-T. Wang, Mean curvature flow of surfaces in Einstein four-manifolds, **J. Differential Geom. 57** (2001), 301–338.
[19] M.-T. Wang, Deforming area preserving diffeomorphism of surfaces by mean curvature flow, **Math. Res. Lett. 8** (2001), 651–661.
[20] M.-T. Wang, Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension, **Invent. Math. 148** (2002), 525–543.
[21] M.-T. Wang, Gauss maps of the mean curvature flow, **Math. Res. Lett. 10** (2003), 287–299.
[22] B. White, A local regularity theorem for mean curvature flow. **Ann. of Math. 161** (2005), 1487–1519.

E-mail address: aneves@math.stanford.edu

Instituto Superior Tećnico, Lisbon, Portugal, and

Mathematics Department, Stanford University, Stanford, CA 94305, USA