Some problems in Mathematical Physics

One dimensional Fermi gas with spin and attractive interaction

Contributed by G. Gallavotti, (Roma 1, Italy), Nov. 6, 1998

Abstract: the large distance behavior for the single particle reduced density matrix in the case of attractive interaction is still an open problem. It is a simple example of a problem in which the renormalization group approach fails because the large distance behavior is dictated by a nontrivial fixed point. It might be the one case in which a theory is possible as there are many claims about what it should look like.

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The ground state of one dimensional systems of weakly interactin fermionic particles with no spin is quite well understood at small coupling at least. Although a mathematically rigorous theory has been only recently completed it has been long known which its main features should have been. The 1949 work of Tomonaga had laid down all the essentials on the matter. The model should behave as a Luttinger liquid, in the sense that its correlations should have its same properties like, for instance, a non sharp singularity at the Fermi surface for the Fourier transform of the one particle density matrix (*i.e.* an "anomalously slow" decay as large distances). The Hamiltonian is

$$H=\int\psi_+(x)(-\frac{1}{2m}\frac{d^2}{dx^2}-\mu)\psi_+(x)dx+\varepsilon\int\psi_+(x)\psi_-(x)\,v(x-y)\,\psi_+(y)\psi_-(y)\,dxdy$$

where the $\psi_\pm$ are fermionic fields which are scalars or spinorial, m,μ,v,ε are mass, chemical potential, pair potential and coupling strength, respectively; the "coupling constant" ε is supposed to be samll enough. The integrals are extended over a finite interval L with periodic boundary. The one particle reduced density matrix is defined as

$$\rho(x-y)=\lim_{\beta\to\infty}\lim_{L\to\infty}\frac{\mathrm{Tr}\,e^{\beta H}\psi_+(x)\psi_-(y)}{\mathrm{Tr}\,e^{\beta H}}$$

which for ε,ε' small enough is shown to behave as $(x-y)\to\infty$ as const $|x-y|^{-1+\eta}\sin p_F(x-y)$ where p_F is the Fermi momentum (related to the density) and η is a constant (and $\eta>0$ generically), provided $p_F\,(x-y)$ is away from integer multiples of π. It is remarkable that η is analytic in ε if the potential v is nice, *e.g.* if it is smooth and finite ranged. The coefficients of the power series can be computed order by order. The above results rely on the exact solutions of the Luttinger model of Lieb and Mattis which was the first hint that a rigorous mathematical theory of such many body problems was possible.

It has also been shown that the spinning case can be treated as the spinless case *provided* it is repulsive (*i.e.* $v\ge0$) (still at coupling ε small). Although one can easily find claims that the matter is "physically" well understood, as far as I know the problem of which is the asymptotic behavior of $\rho(x-y)$ in *attractive and spinning* systems is open from a mathematical viewpoint (in spite of the one dimensional nature of the problem); *even though* ε *is small*, see [2]; the key question is wheter the anomaly η is ε–dependent: it is not impossible that η has a fixed positive value as soon as $\varepsilon\ne0$, generically in v (if attractive).

A similar problem can be posed for a Fermi system confined in a fixed horizontally infinite and vertically finite strip of width ℓ (with, say, Dirichlet boundary conditions. In this case the behavior of $\rho(x-y)$ is not known, at least if the interaction is attractive (with or without spin) and $p_F>\pi\ell^{-1}$. And even the "simpler" problem of a lattice system of spinless fermions moving on a lattice consisting *just* of two parallel rows and attracting each other is, as far as I can see, very difficult and open. The latter problems are obvious "first steps" towards a 2–dimensional theory of interacting fermions. There the open problems are daunting, but they are also considered to be of fundamental importance, as stressed repeatedly by P. Anderson (for instance).

In fact problem 2 and problem 3 are treated with quite tightly related techniques.

Condensed matter Physics seems to be a rich source for mathematical physics: many problems are thought to be solved at a "physical level" even though they have not yet been even properly formulated as mathematical questions. Many other examples could be mentioned. From a philosophical viewpoint it may seem disturbing that some consider as well understood problems that others consider not even formulated properly. Certainly more efforts to understand each other would alleviate the problem: but I do not think it will disappear, because sometimes the lack of precise mathematical formulations is simply an essential part of the solution. In other words it is often difficult to separate assumptions from conclusions. This should be a temporary situation, we must believe, and as such it is stimulating.

[2] G. Benfatto, G. Gallavotti, A. Procacci, B. Scoppola: Communications in Mathematical Physics, **160**, 93–172, 1994. Bonetto, F., Mastropietro, V.: Communications in Mathematical Physics, **172**, 57–93, 1975, Physical Review B, **56**, 1297-1308, 1997. MPEJ, **2**, 1–36, 1995. Benfatto, G., Gallavotti, G.: Renormalization group, Princeton, 1995.

[3] Benfatto, G, Gentile, G., Mastropietro, V.: J. Statistical Physics, **89**, 655–708, 1997.