Some problems in Mathematical Physics

Separatrix splitting for a quasi periodically forced pendulum

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Abstract: The degeneracy of the pendulum separatrix is generically removed by a perturbation. The splitting may be hard to compute: this is exemplified by the fact that, in certain simple cases in which the force is quasiperiodic and with a fast frequency (say $\eta^{-1/2}$ times that of the pendulum small oscillations with η small), it can be determined by a series (in the expansion in powers of the perturbation strength ε) convergent in a domain which may be much larger than the domain of perturbation strengths in which one can derive an expression asymptotic as the rapidity of the forcing frequency tends to ∞ (i.e. $\eta\to 0$). This is so because the splitting is determined by taking the asymptotic form as $\eta\to 0$ of each perturbation order in ε and the power series of the leading terms is *only* known to be *convergent in a domain smaller* than the estimate on the radius of convergence itself. The problem discussed here deals with finding an aymptotic expression for the splitting in a region of values of the coupling ε in which the series in ε for the splitting still converges, but no asymptotics as $\eta\to 0$ can be immediately derived from it, because the series of the terms leading at each order does not (seem to) converge.

Consider the following "three time scales" Hamiltonian

$$H=\underline{\omega}\cdot\underline{A}+\frac{1}{2J}I^2+Jg^2(\cos\varphi-1)+\varepsilon f(\underline{\alpha})\,(\cos\varphi-1)+\varepsilon' m(2\alpha_2)(\cos\varphi-1)$$

where $\underline{A}\in R^2$, $\underline{\alpha}=(\alpha_1,\alpha_2)$ are angles, f,m are trigonometric polynomials with 0–average and $\underline{\omega}$ is a constant vector with one "fast" and one "slow" component: $\underline{\omega}=(\eta^{\frac12}\overline{\omega}_1,\eta^{-\frac12}\overline{\omega}_2)$; I,φ describe an ordinary pendulum with inertia J and frequency $g/2\pi$. The parameter $\eta\le 1$ is meant to tend to 0 so that the unperturbed system with $\varepsilon,\varepsilon'=0$ has three time scales $O(\eta^{\frac12})$ (the slow α_1), $O(1)$ (the pendulum) and $O(\eta^{-\frac12})$ (the fast α_2).

This type of Hamiltonian with ε small compared to η and $\varepsilon'=O(1)$, *i.e.* with a large "fast" perturbation $m(2\alpha_2)(\cos\varphi-1)$, arises certain celestial mechanics problems where it is important that the perturbation has a "large" part depending only on the fast angle α_2 and on φ and containing only Fourrier harmonics in α_2 larger than the minimal.

The system above is simplified because it is linear in $\underline{A}$ ("isochronous") and because it admits an exact solution $t\to\underline{A}=const,\underline{\alpha}+\underline{\omega}t$ and $I=0,\varphi=0$, for all ε,ε'. It covers a 2–dimensional torus (*i.e.* $\underline{\alpha}$ is arbitrary).

It is known that the stable and unstable manifolds of the above motions (*i.e.* of the above invariant torus) are generically non degenerate for $\varepsilon,\varepsilon'\ne 0$ (obviously they coincide if $\varepsilon,\varepsilon'=0$). One can look at the splitting of the two surfaces on the surface $\varphi=\pi$. Supposing that the energy H is 0, to fix the ideas, the section $\varphi=\pi, H=0$ has 4 dimensions in phase space and the stable an unstable manifolds are 2–dimensional on it. They intersect, for ε small, in isolated *homoclinic* points: if f,m are even, as we suppose for simplicity and as it is the case in the mentioned applications, then $\underline{\alpha}=\underline{0}$ is one of them by symmetry considerations.

The homoclinic splitting is described by a Jacobian matrix, 2×2, and it is defined to be its determinant $\Delta(\varepsilon,\varepsilon',\eta)$: if $\underline{A}=\underline{A}^u(\underline{\alpha})$ and $\underline{A}=\underline{A}^s(\underline{\alpha})$ are the parametric equations of the manifolds at $\varphi=\pi$ then $\underline{Q}(\underline{\alpha})=\underline{A}^u(\underline{\alpha})-\underline{A}^s(\underline{\alpha})$ is the splitting and the matrix $\underline{\partial}\underline{Q}(\underline{\alpha})|_{\underline{\alpha}=\underline{0}}$ is the Jacobian in question. It is a quantity that can be expanded into a convergent power series in ε,ε' if $\underline{\omega}$ verifies a Diophantine property. An algorithm, with a strong "field theory flavor", for this is described in [1]. Consider from now on that η is in a sequence $\eta=\eta_n$, fixed once for all, tending to 0 with $n\to\infty$ and such that the vector $\underline{\omega}(\eta)$ has fixed Diophantine constants a,τ, defined by $|\underline{\omega}\cdot\underline{\nu}|>\eta^a|\underline{\nu}|^{-\tau}$ for all integer components non zero vectors $\underline{\nu}$, with τ and $a>0$

fixed (note that the power must be $a > 0$ if we want the Diophantine property to hold for all $\eta = \eta_n$ in the sequence).

A (easy) result is that the radii of convergence of the perturbative series for $\Delta(\varepsilon, \varepsilon', \eta)$ in powers of $\varepsilon, \varepsilon'$ are (large) $\geq O(\eta^b), O(\eta^{-b'})$, respectively, for some $b, b' > 0$ (*e.g.* $b = 2$, $b' = \frac{1}{2} - \delta$ are possible choices, for any prefixed $\delta > 0$, [1], for all η small.

However an asymptotic form in $\eta \to 0$ of the sum of the series Δ is harder and it can be found only if $|\varepsilon| < O(\eta^c), |\varepsilon'| < O(\varepsilon^{c'})$ (*i.e.* small) with some $c, c' > 0$ (the optimal values are not not known) *and ε is not too small compared to ε'*: and the problem is rather delicate because the result depends on cancellations between quantities that are not as small as the final result. The matrix elements of the Jacobian matrix whose determinant is the splitting are much larger than the determinant itself, and this is true to all orders of perturbation theory, see [1] where the problem is solved for $|\varepsilon| = |\varepsilon'| < \eta^c$ and some $c > 0$ depending on the Diophantine constant a. The leading asymptotic value is, generically in the choice of f, $\Delta = C(f)\varepsilon^2\eta^{-1}e^{-b\eta^{-1/2}})$ with $C(f) \neq 0$ does not depend on m and the constant b is $\frac{1}{2}\pi\overline{\omega}_2 g^{-1}$: "exponentially small" (while the matrix elements of the Jacobian matrix are generically bounded below by a power of η, to all orders of the perturbation expansion in $\varepsilon, \varepsilon'$).

To avoid a too wide case by case analysis of possibilities I fix $\varepsilon = \eta^c$ (allowing fixing c as large as wished but once and for all). Question: given a large $c > 0$, and setting $\varepsilon = \eta^c$, can one find the asymptotics in η, as $\eta\tau 00$, or at least a positive lower bound for $|\Delta(\varepsilon, \varepsilon', \eta)|$ valid for $|\varepsilon'| = O(1)$ and for $\eta \to 0$? This is particularly interesting because the radius of convergence of the series is much larger than the radius of the domain in which the first term of the series gives the correct asymptotics as $\eta \to 0$.

Similar questions have recently been successfully studied with techniques based on Lazutkin's work, [2], for periodically forced pendulums (*e.g.* $\varepsilon = 0$: for $\varepsilon \neq 0$ the above is quasi periodically forced) or $2D$ maps. The results seem to hint that the phenomenon of the splitting has analogies with the critical point universality mechanisms (one can find "equivalent" or "reference" problems and express the quantity Δ in terms of the solutions to such reference problems, analogous (it seems to me) to the "fixed points" of universality nechanisms). Recalling that we keep $\varepsilon = \eta^c$ with c fixed and large so that ε is not a parameter, and taking (to fix the ideas) $\varepsilon' = \eta^T$ there is a "critical value" of T below which the asymptotics changes, and this change seems to have some universality features. It is remarkable that this transition happens within the domain where perturbation expansions converge and this is rather unusual in other fields.

An interesting special case (as difficult as the above general one) would be $f(\underline{\alpha}) = (\cos\alpha_1 + \cos\alpha_2)$, $m(\alpha_2) = \cos 2\alpha_2$.

[1] Gallavotti, G.: Reviews on Mathematical Physics, **6** (1994), 343–411. Gallavotti, G., Gentile, G., Mastropietro, V.: [Preprint in chao-dyn #9801004]. Gallavotti, G. , Gentile, G., Mastropietro, V.: [Preprint chao-dyn 9709004].

[2] Neishtadt, A.I.: Journal of Applied Mathematics and Mechanics **48** (1984), 133–139. Lazutkin, V.F.,: [Preprint mp_arc #98–421, (translation of a 1984 paper)]. Gelfreich, V.: Nonlinearity, **10**, 175–193, 1997.