Prerequisites from Differential Geometry

appendix to the Symplectic Geometry lecture notes

Ana Cannas da Silva*

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1 Isotopies and Vector Fields

Let M be a manifold, and $\rho : M \times \mathbb{R} \to M$ a map, where we set $\rho_t(p) := \rho(p, t)$.

Definition 1.1 *The map ρ is an* **isotopy** *if each $\rho_t : M \to M$ is a diffeomorphism, and $\rho_0 = \mathrm{id}_M$.*

Given an isotopy ρ, we obtain a **time-dependent vector field**, that is, a family of vector fields v_t, $t \in \mathbb{R}$, which at $p \in M$ satisfy

$$v_t(p) = \left.\frac{d}{ds}\rho_s(q)\right|_{s=t} \qquad \text{where} \qquad q = \rho_t^{-1}(p) \ ,$$

i.e.,

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \ .$$

Conversely, given a time-dependent vector field v_t, if M is compact or if the v_t's are compactly supported, there exists an isotopy ρ satisfying the previous ordinary differential equation.

Suppose that M is compact. Then we have a one-to-one correspondence

$$\begin{array}{ccc} \{\text{isotopies of } M\} & \longleftrightarrow & \{\text{time-dependent vector fields on } M\} \\ \rho_t,\ t \in \mathbb{R} & \longleftrightarrow & v_t,\ t \in \mathbb{R} \end{array}$$

Definition 1.2 *When $v_t = v$ is independent of t, the associated isotopy is called the* **exponential map** *or the* **flow** *of v and is denoted $\exp tv$; i.e., $\{\exp tv : M \to M \mid t \in \mathbb{R}\}$ is the unique smooth family of diffeomorphisms satisfying*

$$\exp tv|_{t=0} = \mathrm{id}_M \quad \text{and} \quad \frac{d}{dt}(\exp tv)(p) = v(\exp tv(p)) \ .$$

*E-mail: acannas@math.ist.utl.pt or acannas@math.princeton.edu

Definition 1.3 *The* **Lie derivative** *along v is the operator*

$$\mathcal{L}_v : \Omega^k(M) \longrightarrow \Omega^k(M) \qquad \text{defined by} \qquad \mathcal{L}_v \omega := \frac{d}{dt}(\exp tv)^* \omega|_{t=0} \ .$$

When a vector field v_t is time-dependent, its flow, that is, the corresponding isotopy ρ, still locally exists by Picard's theorem. More precisely, in the neighborhood of any point p and for sufficiently small time t, there is a one-parameter family of local diffeomorphisms ρ_t satisfying

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t \qquad \text{and} \qquad \rho_0 = \mathrm{id} \ .$$

Hence, we say that the **Lie derivative** along v_t is

$$\mathcal{L}_{v_t} : \Omega^k(M) \longrightarrow \Omega^k(M) \qquad \text{defined by} \qquad \mathcal{L}_{v_t} \omega := \frac{d}{dt}(\rho_t)^* \omega|_{t=0} \ .$$

Exercise. Prove the **Cartan magic formula**,

$$\mathcal{L}_v \omega = \imath_v d\omega + d\imath_v \omega \ ,$$

and the formula

$$\frac{d}{dt}\rho_t^* \omega = \rho_t^* \mathcal{L}_{v_t} \omega \ , \qquad (\star)$$

where ρ is the (local) isotopy generated by v_t. A good strategy for each formula is to follow the steps:

(a) Check the formula for 0-forms $\omega \in \Omega^0(M) = C^\infty(M)$.

(b) Check that both sides commute with d.

(c) Check that both sides are derivations of the algebra $(\Omega^*(M), \wedge)$. For instance, check that

$$\mathcal{L}_v(\omega \wedge \alpha) = (\mathcal{L}_v \omega) \wedge \alpha + \omega \wedge (\mathcal{L}_v \alpha) \ .$$

(d) Notice that, if $\mathcal{U}$ is the domain of a coordinate system, then $\Omega^\bullet(\mathcal{U})$ is generated as an algebra by $\Omega^0(\mathcal{U})$ and $d\Omega^0(\mathcal{U})$, i.e., every element in $\Omega^\bullet(\mathcal{U})$ is a linear combination of wedge products of elements in $\Omega^0(\mathcal{U})$ and elements in $d\Omega^0(\mathcal{U})$.

$\diamondsuit$

We will need the following improved version of formula $(\star)$.

Proposition 1.4 *For a smooth family ω_t, $t \in \mathbb{R}$, of k-forms, we have*

$$\frac{d}{dt}\rho_t^* \omega_t = \rho_t^* \left(\mathcal{L}_{v_t} \omega_t + \frac{d\omega_t}{dt} \right) \ .$$

Proof. If $f(x,y)$ is a real function of two variables, by the chain rule we have

$$\frac{d}{dt}f(t,t) = \frac{d}{dx}f(x,t)\bigg|_{x=t} + \frac{d}{dy}f(t,y)\bigg|_{y=t} .$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\rho_t^*\omega_t &= \underbrace{\frac{d}{dx}\rho_x^*\omega_t\bigg|_{x=t}}_{\rho_x^*\mathcal{L}_{v_x}\omega_t\big|_{x=t} \text{ by } (\star)} + \underbrace{\frac{d}{dy}\rho_t^*\omega_y\bigg|_{y=t}}_{\rho_t^*\frac{d\omega_y}{dy}\big|_{y=t}} \\ &= \rho_t^*\left(\mathcal{L}_{v_t}\omega_t + \frac{d\omega_t}{dt}\right) .\end{aligned}$$

□

2 Submanifolds

Let M and X be manifolds with $\dim X < \dim M$.

Definition 2.1 *A map $i : X \to M$ is an* **immersion** *if $di_p : T_pX \to T_{i(p)}M$ is injective for any point $p \in X$.*

An **embedding** *is an immersion which is a homeomorphism onto its image.*[1]

A **closed embedding** *is a proper*[2] *injective immersion.*

Exercise. Show that a map $i : X \to M$ is a closed embedding if and only if i is an embedding and its image $i(X)$ is closed in M.

Hint:

- If i is injective and proper, then for any neighborhood $\mathcal{U}$ of $p \in X$, there is a neighborhood $\mathcal{V}$ of $i(p)$ such that $f^{-1}(\mathcal{V}) \subseteq \mathcal{U}$.
- On a Hausdorff space, any compact set is closed. On any topological space, a closed subset of a compact set is compact.
- An embedding is proper if and only if its image is closed.

◇

Definition 2.2 *A* **submanifold** *of M is a manifold X with a closed embedding $i : X \hookrightarrow M$.*[3]

Given a submanifold, we may regard the embedding $i : X \hookrightarrow M$ as an inclusion, in order to identify points and tangent vectors:

$$p = i(p) \quad \text{and} \quad T_pX = di_p(T_pX) \subset T_pM .$$

[1]The image has the topology induced by the target manifold.

[2]A map is **proper** if the preimage of any compact set is compact.

[3]When X is an open subset of a manifold M, we refer to it as an *open* submanifold.

3 Tubular Neighborhood Theorem

Let M be an n-dimensional manifold, and let X be a k-dimensional submanifold where $k < n$ and with inclusion map $i : X \hookrightarrow M$. At each $x \in X$, the tangent space to X is viewed as a subspace of the tangent space to M via the linear inclusion $di_x : T_xX \hookrightarrow T_xM$, where we denote $x = i(x)$. The quotient $N_xX := T_xM/T_xX$ is an $(n-k)$-dimensional vector space, known as the **normal space** to X at x. The **normal bundle** of X is

$$NX = \{(x,v) \mid x \in X \ , \ v \in N_xX\} \ .$$

The set NX has the structure of a vector bundle over X of rank $n-k$ under the natural projection, hence as a manifold NX is n-dimensional.

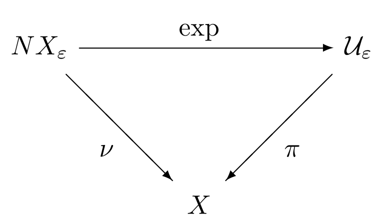
Exercise. Let M be $\mathbb{R}^n$ and let X be a k-dimensional compact submanifold of $\mathbb{R}^n$.

(a) Show that in this case N_xX can be identified with the usual "normal space" to X in $\mathbb{R}^n$, that is, the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x.

(b) Given $\varepsilon > 0$ let $\mathcal{U}_\varepsilon$ be the set of all points in $\mathbb{R}^n$ which are at a distance less than ε from X. Show that, for ε sufficiently small, every point $p \in \mathcal{U}_\varepsilon$ has a *unique* nearest point $\pi(p) \in X$.

(c) Let $\pi : \mathcal{U}_\varepsilon \to X$ be the map just defined for ε sufficiently small. Show that, if $p \in \mathcal{U}_\varepsilon$, then the line segment $(1-t) \cdot p + t \cdot \pi(p)$, $0 \le t \le 1$, joining p to $\pi(p)$ lies in $\mathcal{U}_\varepsilon$.

(d) Let $NX_\varepsilon = \{(x,v) \in NX$ such that $|v| < \varepsilon\}$. Let $\exp : NX \to \mathbb{R}^n$ be the map $(x,v) \mapsto x + v$, and let $\nu : NX_\varepsilon \to X$ be the map $(x,v) \mapsto x$. Show that, for ε sufficiently small, $\exp$ maps NX_ε diffeomorphically onto $\mathcal{U}_\varepsilon$, and show also that the following diagram commutes:



(e) Suppose now that the manifold X is not compact. Prove that the assertion about $\exp$ is still true provided we replace ε by a continuous function $\varepsilon : X \to \mathbb{R}^+$ which tends to zero fast enough as x tends to infinity. You have thus proved the *tubular neighborhood theorem in* $\mathbb{R}^n$.

◊

In general, the zero section of NX,

$$i_0 : X \hookrightarrow NX \ , \qquad x \mapsto (x, 0) \ ,$$

embeds X as a closed submanifold of NX. A neighborhood $\mathcal{U}_0$ of the zero section X in NX is called **convex** if the intersection $\mathcal{U}_0 \cap N_x X$ with each fiber is convex.

Theorem 3.1 (Tubular Neighborhood Theorem) *Let M be an n-dimensional manifold, X a k-dimensional submanifold, NX the normal bundle of X in M, $i_0 : X \hookrightarrow NX$ the zero section, and $i : X \hookrightarrow M$ inclusion. Then there exist a convex neighborhood $\mathcal{U}_0$ of X in NX, a neighborhood $\mathcal{U}$ of X in M, and a diffeomorphism $\varphi : \mathcal{U}_0 \to \mathcal{U}$ such that*

$$\begin{array}{ccc} NX \supseteq \mathcal{U}_0 & \xrightarrow[\simeq]{\varphi} & \mathcal{U} \subseteq M \\ & {\scriptstyle i_0} \nwarrow \quad \nearrow {\scriptstyle i} & \\ & X & \end{array}$$

commutes.

Outline of the proof.

- *Case of $M = \mathbb{R}^n$, and X is a compact submanifold of $\mathbb{R}^n$.*

  Theorem 3.2 (ε-Neighborhood Theorem)

  Let $\mathcal{U}^\varepsilon = \{p \in \mathbb{R}^n : \ |p - q| < \varepsilon$ for some $q \in X\}$ be the set of points at a distance less than ε from X. Then, for ε sufficiently small, each $p \in \mathcal{U}^\varepsilon$ has a unique nearest point $q \in X$ (i.e., a unique $q \in X$ minimizing $|q - x|$).

  Moreover, setting $q = \pi(p)$, the map $\mathcal{U}^\varepsilon \stackrel{\pi}{\to} X$ is a (smooth) submersion with the property that, for all $p \in \mathcal{U}^\varepsilon$, the line segment $(1 - t)p + tq$, $0 \leq t \leq 1$, is in $\mathcal{U}^\varepsilon$.

  Here is a sketch. At any $x \in X$, the *normal* space $N_x X$ may be regarded as an $(n - k)$-dimensional subspace of $\mathbb{R}^n$, namely the orthogonal complement in $\mathbb{R}^n$ of the tangent space to X at x:

  $$N_x X \simeq \{v \in \mathbb{R}^n : \ v \perp w \ , \text{ for all } w \in T_x X\} \ .$$

  We define the following open neighborhood of X in NX:

  $$NX^\varepsilon = \{(x, v) \in NX : \ |v| < \varepsilon\} \ .$$

  Let

  $$\begin{array}{rccc} \exp : & NX & \longrightarrow & \mathbb{R}^n \\ & (x, v) & \longmapsto & x + v \ . \end{array}$$

  Restricted to the zero section, $\exp$ is the identity map on X.

Prove that, for ε sufficiently small, $\exp$ maps NX^ε diffeomorphically onto $\mathcal{U}^\varepsilon$, and show also that the diagram

$$\begin{array}{ccc} NX^\varepsilon & \xrightarrow{\exp} & \mathcal{U}^\varepsilon \\ & {\scriptstyle \pi_0} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & X & \end{array}$$

commutes.

- *Case where X is a compact submanifold of an arbitrary manifold M.*

  Put a riemannian metric g on M, and let $d(p,q)$ be the riemannian distance between $p, q \in M$. The ε-neighborhood of a compact submanifold X is

  $$\mathcal{U}^\varepsilon = \{p \in M \mid d(p,q) < \varepsilon \text{ for some } q \in X\} \ .$$

  Prove the ε-neighborhood theorem in this setting: for ε small enough, the following assertions hold.

  – Any $p \in \mathcal{U}^\varepsilon$ has a unique point $q \in X$ with minimal $d(p,q)$. Set $q = \pi(p)$.

  – The map $\mathcal{U}^\varepsilon \xrightarrow{\pi} X$ is a submersion and, for all $p \in \mathcal{U}^\varepsilon$, there is a unique geodesic curve γ joining p to $q = \pi(p)$.

  – The normal space to X at $x \in X$ is naturally identified with a subspace of T_xM:

  $$N_xX \simeq \{v \in T_xM \mid g_x(v,w) = 0 \ , \text{ for any } w \in T_xX\} \ .$$

  Let $NX^\varepsilon = \{(x,v) \in NX \mid \sqrt{g_x(v,v)} < \varepsilon\}$.

  – Define $\exp : NX^\varepsilon \to M$ by $\exp(x,v) = \gamma(1)$, where $\gamma : [0,1] \to M$ is the geodesic with $\gamma(0) = x$ and $\frac{d\gamma}{dt}(0) = v$. Then $\exp$ maps NX^ε diffeomorphically to $\mathcal{U}^\varepsilon$.

- *General case.*

  When X is not compact, adapt the previous argument by replacing ε by an appropriate continuous function $\varepsilon : X \to \mathbb{R}^+$ which tends to zero fast enough as x tends to infinity.

□

Restricting to the subset $\mathcal{U}^0 \subseteq NX$ from the tubular neighborhood theorem, we obtain a submersion $\mathcal{U}_0 \xrightarrow{\pi_0} X$ with all fibers $\pi_0^{-1}(x)$ convex. We can carry this fibration to $\mathcal{U}$ by setting $\pi = \pi_0 \circ \varphi^{-1}$:

$$\begin{array}{ll} \mathcal{U}_0 & \subseteq NX \quad \text{is a fibration} \\ \pi_0 \downarrow & \\ X & \end{array} \quad \Longrightarrow \quad \begin{array}{ll} \mathcal{U} & \subseteq M \quad \text{is a fibration} \\ \pi \downarrow & \\ X & \end{array}$$

This is called the **tubular neighborhood fibration**.

4 Homotopy Formula

Let $\mathcal{U}$ be a tubular neighborhood of a submanifold X in M. The restriction $i^* : H^\ell_{\mathrm{deRham}}(\mathcal{U}) \to H^\ell_{\mathrm{deRham}}(X)$ by the inclusion map is surjective. As a corollary of the tubular neighborhood fibration, i^* is also injective: this follows from the homotopy-invariance of de Rham cohomology.

Corollary 4.1 *For any degree ℓ, $H^\ell_{\mathrm{deRham}}(\mathcal{U}) \simeq H^\ell_{\mathrm{deRham}}(X)$.*

At the level of forms, this means that, if ω is a closed ℓ-form on $\mathcal{U}$ and $i^*\omega$ is exact on X, then ω is exact. We will need the following related result.

Proposition 4.2 *If a closed ℓ-form ω on $\mathcal{U}$ has restriction $i^*\omega = 0$, then ω is exact, i.e., $\omega = d\mu$ for some $\mu \in \Omega^{d-1}(\mathcal{U})$. Moreover, we can choose μ such that $\mu_x = 0$ at all $x \in X$.*

Proof. Via $\varphi : \mathcal{U}_0 \xrightarrow{\simeq} \mathcal{U}$, it is equivalent to work over $\mathcal{U}_0$. Define for every $0 \le t \le 1$ a map

$$\begin{array}{rccc} \rho_t : & \mathcal{U}_0 & \longrightarrow & \mathcal{U}_0 \\ & (x,v) & \longmapsto & (x,tv) \end{array} .$$

This is well-defined since $\mathcal{U}_0$ is convex. The map ρ_1 is the identity, $\rho_0 = i_0 \circ \pi_0$, and each ρ_t fixes X, that is, $\rho_t \circ i_0 = i_0$. We hence say that the family $\{\rho_t \mid 0 \le t \le 1\}$ is a **homotopy** from $i_0 \circ \pi_0$ to the identity fixing X. The map $\pi_0 : \mathcal{U}_0 \to X$ is called a **retraction** because $\pi_0 \circ i_0$ is the identity. The submanifold X is then called a **deformation retract** of $\mathcal{U}$.

A (de Rham) **homotopy operator** between $\rho_0 = i_0 \circ \pi_0$ and $\rho_1 = \mathrm{id}$ is a linear map

$$Q : \Omega^d(\mathcal{U}_0) \longrightarrow \Omega^{d-1}(\mathcal{U}_0)$$

satisfying the **homotopy formula**

$$\mathrm{Id} - (i_0 \circ \pi_0)^* = dQ + Qd \ .$$

When $d\omega = 0$ and $i_0^*\omega = 0$, the operator Q gives $\omega = dQ\omega$, so that we can take $\mu = Q\omega$. A concrete operator Q is given by the formula:

$$Q\omega = \int_0^1 \rho_t^*(\imath_{v_t}\omega)\ dt \ ,$$

where v_t, at the point $q = \rho_t(p)$, is the vector tangent to the curve $\rho_s(p)$ at $s = t$. The proof that Q satisfies the homotopy formula is below.

In our case, for $x \in X$, $\rho_t(x) = x$ (all t) is the constant curve, so v_t vanishes at all x for all t, hence $\mu_x = 0$. □

To check that Q above satisfies the homotopy formula, we compute

$$\begin{aligned} Qd\omega + dQ\omega &= \int_0^1 \rho_t^*(\imath_{v_t}d\omega)dt + d\int_0^1 \rho_t^*(\imath_{v_t}\omega)dt \\ &= \int_0^1 \rho_t^*(\underbrace{\imath_{v_t}d\omega + d\imath_{v_t}\omega}_{\mathcal{L}_{v_t}\omega})dt \ , \end{aligned}$$

where $\mathcal{L}_v$ denotes the Lie derivative along v, and we used the Cartan magic formula: $\mathcal{L}_v\omega = \imath_v d\omega + d\imath_v\omega$. The result now follows from

$$\frac{d}{dt}\rho_t^*\omega = \rho_t^*\mathcal{L}_{v_t}\omega$$

and from the fundamental theorem of calculus:

$$Qd\omega + dQ\omega = \int_0^1 \frac{d}{dt}\rho_t^*\omega \; dt = \rho_1^*\omega - \rho_0^*\omega \; .$$

5 Whitney Extension Theorem

Theorem 5.1 (Whitney Extension Theorem) *Let M be an n-dimensional manifold and X a k-dimensional submanifold with $k < n$. Suppose that at each $p \in X$ we are given a linear isomorphism $L_p : T_pM \stackrel{\simeq}{\to} T_pM$ such that $L_p|_{T_pX} = \mathrm{Id}_{T_pX}$ and L_p depends smoothly on p. Then there exists an embedding $h : \mathcal{N} \to M$ of some neighborhood $\mathcal{N}$ of X in M such that $h|_X = \mathrm{id}_X$ and $dh_p = L_p$ for all $p \in X$.*

The linear maps L serve as "germs" for the embedding.

Sketch of proof for the Whitney theorem.

Case $M = \mathbb{R}^n$: For a compact k-dimensional submanifold X, take a neighborhood of the form

$$\mathcal{U}^\varepsilon = \{p \in M \mid \text{ distance } (p, X) \leq \varepsilon\} \; .$$

For ε sufficiently small so that any $p \in \mathcal{U}^\varepsilon$ has a unique nearest point in X, define a projection $\pi : \mathcal{U}^\varepsilon \to X$, $p \mapsto$ point on X closest to p. If $\pi(p) = q$, then $p = q + v$ for some $v \in N_qX$ where $N_qX = (T_qX)^\perp$ is the normal space at q; see Appendix A. Let

$$\begin{array}{rccc} h : & \mathcal{U}^\varepsilon & \longrightarrow & \mathbb{R}^n \\ & p & \longmapsto & q + L_qv \; , \end{array}$$

where $q = \pi(p)$ and $v = p - \pi(p) \in N_qX$. Then $h_X = \mathrm{id}_X$ and $dh_p = L_p$ for $p \in X$. If X is not compact, replace ε by a continuous function $\varepsilon : X \to \mathbb{R}^+$ which tends to zero fast enough as x tends to infinity.

General case: Choose a riemannian metric on M. Replace distance by riemannian distance, replace straight lines $q + tv$ by geodesics $\exp(q, v)(t)$ and replace $q + L_qv$ by the value at $t = 1$ of the geodesic with initial value q and initial velocity L_qv. $\square$