

FOLD-FORMS FOR FOUR-FOLDS

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ABSTRACT. This paper explains an application of Gromov's h-principle to prove the existence, on any orientable 4-manifold, of a folded symplectic form. That is a closed 2-form which is symplectic except on a separating hypersurface where the form singularities are like the pull-back of a symplectic form by a folding map. We use the h-principle for folding maps (a theorem of Eliashberg) and the h-principle for symplectic forms on open manifolds (a theorem of Gromov) to show that, for orientable even-dimensional manifolds, the existence of a stable almost complex structure is necessary and sufficient to warrant the existence of a folded symplectic form.

1. INTRODUCTION

One says that a differential problem satisfies the h-principle if any formal solution (i.e., a solution for the associated algebraic problem) is homotopic to a genuine (i.e., differential) solution. Therefore, when the h-principle holds, one may concentrate on a purely topological question in order to prove the existence of a differential solution.

Differential problems are equations, inequalities or, more generally, relations [12] involving derivatives of maps. The following are examples of problems known to satisfy the h-principle: existence of immersions in strictly positive codimension (theorems of Whitney [30], Nash [24], Kuiper [15], Smale [26], Hirsch [13] and Poénaru [25]), existence of symplectic forms on open manifolds (theorem of Gromov [11], who built the general machinery of the h-principle as an obstruction theory for the sheaves of germs of maps) and existence of maps whose only singularities are folds (theorem of Eliashberg [6, 7]).

This paper explains an application of the h-principle to prove the existence, on any compact orientable 4-manifold, of a folded symplectic form, that is, a closed 2-form with only fold singularities as defined below. According to the h-principle philosophy, this proof is divided in two steps:

- (1) show that the h-principle holds for this problem,
- (2) show that a formal solution exists.

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For the first step, the basic ingredients are the h-principle for maps whose only singularities are folds [6, 7] and the h-principle for symplectic forms on open manifolds [11]. This combination is a shortcut based on an idea contained in a book by Eliashberg and Mishachev [8]. We thus avoid dealing with the h-principle in its generality.

Here is the flavor of Eliashberg's result for maps with fold singularities. Let Z be a hypersurface in a manifold M , that is, a codimension 1 embedded submanifold (this is the meaning of *hypersurface* throughout this paper). A map $f : M \rightarrow N$ between manifolds of the same dimension is called a Z -immersion (or said to *fold along the submanifold Z*) if it is regular (i.e., its derivative is invertible) on $M \setminus Z$, and if near any $p \in Z$ and near its image $f(p)$ there are coordinates centered at those points where f becomes

$$(x_1, x_2, \dots, x_n) \longmapsto (x_1^2, x_2, \dots, x_n) .$$

A homomorphism $F : TM \rightarrow TN$ between tangent bundles is called a Z -monomorphism, if it is injective on $T(M \setminus Z)$ and on TZ , and if there exists a fiber involution $\tau : \mathcal{T} \rightarrow \mathcal{T}$ on a tubular neighborhood \mathcal{T} of Z whose set of fixed points is Z and such that $F \circ d\tau = F$. The differential $df : TM \rightarrow TN$ of a Z -immersion is a Z -monomorphism. Eliashberg [6] proved that, if every connected component of $M \setminus Z$ is open, then any Z -monomorphism $TM \rightarrow TN$ is homotopic (within Z -monomorphisms $TM \rightarrow TN$) to the differential of a Z -immersion. Recall that a manifold is *open* if there are no closed manifolds (i.e., compact and without boundary) among its connected components. In the language of [12], the theorem says that, when $M \setminus Z$ is open, Z -immersions satisfy the (everywhere C^0 -dense) h-principle; a Z -monomorphism is then called a *formal solution*. For the present application, we require a more general statement [7] dealing with foliated target manifolds.

A *folded symplectic form* on a $2n$ -dimensional manifold M is a closed 2-form ω which is nondegenerate except on a hypersurface Z called the *folding hypersurface* where, centered at every point $p \in Z$, there are coordinates for M adapted to Z where the form ω becomes

$$x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \dots + dx_{2n-1} \wedge dx_{2n} .$$

The pullback of a symplectic form by a Z -immersion is a folded symplectic form with folding hypersurface Z .

A formal solution for the problem of existence of a folded symplectic form turns out to be a stable almost complex structure. Let M be a $2n$ -dimensional manifold with a structure of complex vector bundle on $TM \oplus \mathbb{R}^2$, where \mathbb{R}^2 denotes the trivial rank 2 real vector bundle over M . We will show that M admits folded symplectic forms.

Here is how Gromov's theorem comes in. We embed M as level zero in $M \times \mathbb{R}$. The given stable almost complex structure on M yields a complex

hyperplane field on $M \times \mathbb{R}$ and hence an almost complex structure on $M \times \mathbb{R}^2$. Since this manifold is open, Gromov's application of the h-principle [11] guarantees the existence of a symplectic form on $M \times \mathbb{R}^2$ inducing almost complex structures in the same homotopy class as the given one. Since $M \times \mathbb{R}$ sits here as a codimension one submanifold, the restriction ω_0 of the symplectic form to this submanifold has maximal rank, i.e., has exactly a one-dimensional kernel at every point. Let \mathcal{L} be the one-dimensional foliation determined by the kernel L of ω_0 . The projection of ω_0 to $T(M \times \mathbb{R})/L$ is well-defined and nondegenerate. Suppose that we could immerse M in $M \times \mathbb{R}$ in a *good* way, meaning that locally the composition of that immersion with the projection to the local leaf space of \mathcal{L} is a Z -immersion, for some hypersurface Z in M . Since this leaf space is symplectic, by pullback we would obtain a folded symplectic form on M . Hence, we concentrate on deforming the initial embedding at level zero into a good immersion in order to prove:

Theorem A *Let M be a $2n$ -dimensional manifold with a stable almost complex structure J . Then M admits a folded symplectic form consistent with J in any degree 2 cohomology class.*

In each connected component of M , the folding hypersurface can be chosen to be the boundary of a disjoint collection of balls or the boundary of such a collection contained in a larger ball, as shown in § 4.

The notion of consistency is explained in § 2. The existence of a stable almost complex structure is a necessary condition for the existence of a folded symplectic form on an orientable manifold (see § 2). Theorem A is then saying that it is also sufficient. This contrasts with the case of a (honest) symplectic form, for whose existence an almost complex structure is necessary, but only sufficient if the manifold is open [11]. The sphere S^6 is a trivial example (thanks to Stokes' theorem) and $\mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ is an important example (thanks to Seiberg-Witten invariants [28]) of almost complex manifolds without any symplectic form.

To produce a formal solution for 4-manifolds is easily accomplished. Hirzebruch and Hopf [14] showed that the integral Stiefel-Whitney class W_3 vanishes for any compact orientable 4-manifold, or, in other words, such manifolds always have stable almost complex structures. (This is the same reason why such manifolds are spin-c [16, Thm.D.2].) Since we are in the stable range, it is enough to add a trivial \mathbb{R}^2 bundle to TM for this to admit a structure of complex vector bundle. All this is also true when M is not compact [10, §5.7]. We thus obtain the following relevant special case of Theorem A:

Theorem B *Let M be an orientable 4-manifold. Then M admits a folded symplectic form consistent with any given stable almost complex structure and in any degree 2 cohomology class.*

In higher dimensions, there are plenty of orientable manifolds which have no stable almost complex structures ($S^1 \times \text{SU}(3)/\text{SO}(3)$, for instance [16]), and hence cannot have folded symplectic forms. The condition $W_3(M) = 0$ is necessary and sufficient in dimensions 6 (since the next obstruction W_7 vanishes for dimensional reasons) and 8 (where Massey [19] proved that W_7 always vanishes). According to [4, 29], until 1998 it was still not known general necessary and sufficient conditions (in terms of invariants such as characteristic classes and the cohomology ring) for the existence of a stable almost complex structure on manifolds of dimension ≥ 10 .

As for the contents of this paper: § 2 reviews folded symplectic manifolds and some *folded* tangent bundles associated to them; § 3 describes the application of Gromov's theorem to guarantee a symplectic form starting from a structure of complex vector bundle; § 4 proves the existence of an isomorphism between a *folded* tangent bundle and a suitable complex vector bundle; § 5 describes the application of Eliashberg's theorem to produce folded symplectic forms; § 6 contains the conclusion of the proof of Theorems A and B.

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Y. Eliashberg has pointed out that Theorem A can be alternatively deduced from the Singular h-Principle Theorem of Gromov [12, p.112] following a hint in Exercise (b) of page 113 in Gromov's book to overcome the lack of microflexibility.

2. FOLDED SYMPLECTIC MANIFOLDS

Let M be an oriented manifold of dimension $2n$, and let ω be a closed 2-form on M . The highest wedge power ω^n is a section of the (trivial) orientation bundle $\wedge^{2n}T^*M$.

Definition. *A folded symplectic form is a closed 2-form ω such that ω^n intersects the 0-section of $\wedge^{2n}T^*M$ transversally, and such that $i^*\omega$ has maximal rank everywhere, where $i : Z \hookrightarrow M$ is the inclusion of the zero-locus, Z , of ω^n .*

By transversality, Z is a codimension-1 submanifold of M , called the *folding hypersurface*. A *folded symplectic manifold* is a pair (M, ω) where ω is a

folded symplectic form on M . The folding hypersurface Z of a folded symplectic manifold (M, ω) separates M into the regions M^+ and M^- , where the form matches or is opposite to the given orientation, respectively. Hence, Z has a co-orientation depending on ω and on the choice of orientation on M . (The notion of folded symplectic form extends to arbitrary even-dimensional manifolds, not necessarily orientable, but we will not deal with those in this paper.)

The Darboux theorem for folded symplectic forms states that, if (M, ω) is a folded symplectic manifold and p is any point on the folding hypersurface Z , then there is a coordinate chart $(\mathcal{U}, x_1, \dots, x_{2n})$ centered at p such that on \mathcal{U}

$$\omega = x_1 dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + \dots + dx_{2n-1} \wedge dx_{2n} \quad \text{and} \quad Z \cap \mathcal{U} = \{x_1 = 0\}.$$

This follows, for instance, from a folded analogue of Moser's trick [3].

Doubles of symplectic manifolds with boundary are easy examples of manifolds with folded symplectic forms. Simplest instances are the spheres S^{2n} , where a folded symplectic form is obtained by pulling back the standard symplectic form on \mathbb{R}^{2n} via the folding map $S^{2n} \rightarrow D^{2n}$.

Starting in dimension 4, folded symplectic forms are not generic in the set of closed 2-forms. Let M be a (compact) oriented 4-manifold, and let ω be a closed 2-form on M . If γ is a given volume form on M , then $\omega \wedge \omega = f\gamma$ for some $f \in C^\infty(M)$. A generic ω [17] is never 0, has rank 2 on a (compact) codimension-1 submanifold, Z , and is nondegenerate elsewhere. The hypersurface Z is the 0-locus of f . Its complement $M \setminus Z$ is the disjoint union of the sets $M^+ = \{f > 0\}$ where ω matches the given orientation and $M^- = \{f < 0\}$ where ω induces the opposite orientation. For ω to be folded symplectic, we would need that TZ and the rank 2 bundle over Z given by $\ker \omega$ intersect transversally as subbundles of $TM|_Z$. Yet generically ω is not folded symplectic, since its restriction to Z vanishes along some codimension-2 submanifold C (a union of circles), where $\ker \omega$ is contained in TZ [17]. Although a generic 2-form on a 3-manifold vanishes only at isolated points, here the 3-manifold already depends on the 2-form. Moreover, generically there are isolated *parabolic* points on those lines (circles), where the tangent space to those lines is contained in $\ker \omega$. There is at least one continuous family of inequivalent neighborhoods of parabolic points [1, 9].

Now let M be an m -dimensional manifold with a separating hypersurface Z . By *separating* we mean that the complement of Z in M is the disjoint union of open sets M^+ and M^- . For instance, M could be an oriented manifold equipped with a folded symplectic form and Z its folding hypersurface.

Over Z , the tangent bundle has a trivial line subbundle V , spanned by a vector field transverse to Z pointing from M^- to M^+ . The quotient TM/V is isomorphic to TZ , so that $TM|_Z \simeq TZ \oplus V$.

Definition. *The Z -tangent bundle of M is the rank m real vector bundle ${}^Z TM$ over M obtained by gluing $TM|_{M \setminus M^-}$ to $TM|_{M \setminus M^+}$ by the constant diagonal map $\text{Id} \oplus (-1) : Z \rightarrow \text{GL}(TZ \oplus V)$.*

There are analytic and algebraic approaches to ${}^Z TM$, which enhance its geometry [3]. From its definition it follows that:

Lemma 1. *Let M be an m -dimensional manifold with a separating hypersurface Z . Then there is an isomorphism of real vector bundles*

$$TM \oplus \mathbb{R} \simeq {}^Z TM \oplus \mathbb{R} .$$

A *complex structure* on a vector bundle E over a manifold M is a bundle homomorphism $J : E \rightarrow E$ such that $J^2 = -\text{Id}$. If E is an orientable rank $2m$ vector bundle, the existence of a complex structure on E is equivalent to the existence of a section of the associated $(\text{SO}(2m)/\text{U}(m))$ -bundle. A *stable complex structure* on a vector bundle E over M is an equivalence class of complex structures on the vector bundles $E \oplus \mathbb{R}^k$ ($k \in \mathbb{Z}_0^+$), two complex structures, J_1 on $E \oplus \mathbb{R}^{k_1}$ and J_2 on $E \oplus \mathbb{R}^{k_2}$, being *equivalent* when there exist $m_1, m_2 \in \mathbb{Z}_0^+$ such that $((E \oplus \mathbb{R}^{k_1}) \oplus \mathbb{C}^{m_1}, J_1 \oplus i)$ and $((E \oplus \mathbb{R}^{k_2}) \oplus \mathbb{C}^{m_2}, J_2 \oplus i)$ are homotopic complex vector bundles, i.e., correspond to homotopic sections of the associated $(\text{SO}(2n)/\text{U}(n))$ -bundle. A *stable almost complex structure* on M is a stable complex structure on TM .

The Z -tangent bundle for the folding hypersurface Z of a folded symplectic form ω has a canonical complex structure J_0 [3] *consistent* with ω . We say that a folded symplectic form ω is *consistent* with a stable almost complex structure on M if $({}^Z TM \oplus \mathbb{C}, J_0 \oplus i)$ belongs to the given equivalence class of complex structures on $TM \oplus \mathbb{R}^{2k}$, $k \in \mathbb{Z}_0^+$.

3. FIRST INSTANCE OF THE H-PRINCIPLE

Let M be a $2n$ -dimensional manifold with a stable almost complex structure. The homotopy groups $\Pi_q(\text{SO}(2m)/\text{U}(m))$ are isomorphic for fixed q and variable m such that $q < 2m - 1$ (this is the so-called *stable range* [18]). Hence, if there exists a complex structure on $TM \oplus \mathbb{R}^{2k}$, then there exists a complex structure on $TM \oplus \mathbb{R}^2$.

Let J be a complex structure on $TM \oplus \mathbb{R}^2$. Let

$$\begin{array}{ccc} i : M & \hookrightarrow & M \times \mathbb{R} & \text{and} & \pi : M \times \mathbb{R} & \rightarrow & M \\ & & p & \mapsto & (p, 0) & & \\ & & & & (p, t) & \mapsto & p \end{array}$$

be the embedding at level zero and the projection to the first factor. By pullback, i induces an isomorphism in cohomology.

Via the identification $T(M \times \mathbb{R}) \simeq \pi^*(TM) \oplus \mathbb{R}$, the structure J induces a structure of complex vector bundle, still called J , on $T(M \times \mathbb{R}) \oplus \mathbb{R} \simeq$

$\pi^*(TM) \oplus \mathbb{R}^2$. Then the complex subbundle

$$H_0 = T(M \times \mathbb{R}) \cap J(T(M \times \mathbb{R})) \subset T(M \times \mathbb{R}) \oplus \mathbb{R}$$

is a complex hyperplane field over $M \times \mathbb{R}$. Let ω_1 be a 2-form of maximal rank in $M \times \mathbb{R}$ compatible with J , that is,

$$\omega_1(u, v) = g(Ju, v), \quad \forall u, v \in H_0, \quad \text{and } \omega_1(u, \cdot) = 0, \quad \forall u \in H_0^\perp,$$

for some riemannian metric g on $TM \times \mathbb{R}$, where H_0^\perp denotes the orthocomplement of H_0 with respect to g . A *regular homotopy* of two 2-forms of maximal rank is a homotopy within 2-forms of maximal rank.

Lemma 2. *Let M be a manifold with a structure J of complex vector bundle on $TM \oplus \mathbb{R}^2$. Then, in each degree 2 cohomology class, there exists in $M \times \mathbb{R}$ a closed 2-form of maximal rank which is regularly homotopic to any 2-form of maximal rank compatible with J .*

This is a consequence of the following proposition which was originally proved by McDuff [20]. The proof below is from Eliashberg-Mishachev [8]. We reproduce it since this result is not as widely known as the other applications of the h-principle and since the idea in this proof is crucial for the present paper's strategy. The key to this proof is Gromov's theorem [11] saying that, for every degree 2 cohomology class on any open manifold, any nondegenerate 2-form is regularly homotopic to a symplectic form in that class; moreover, if two symplectic forms are regularly homotopic, then they are homotopic within symplectic forms.

Proposition. [20] *For any 2-form of maximal rank on an odd-dimensional manifold and any degree 2 cohomology class, there exists a closed 2-form of maximal rank in that class which is regularly homotopic to the given form.*

Proof. Let ω_1 be a 2-form of maximal rank on a $(2n + 1)$ -dimensional manifold N and let α be a degree 2 cohomology class in N . By homotopy, the projection to the first factor $\pi : N \times \mathbb{R} \rightarrow N$ induces an isomorphism in cohomology.

If N is orientable, then ω_1 extends in a homotopically unique way compatible with orientations to a nondegenerate 2-form, ω_2 , in $N \times \mathbb{R}$. Gromov's result [11] cited above guarantees the existence, in the class $\pi^*\alpha$, of a homotopically unique symplectic form ω_3 in $N \times \mathbb{R}$ regularly homotopic to ω_2 . The restriction of ω_3 to the zero level M is a closed 2-form of maximal rank.

If N is not orientable, we replace $N \times \mathbb{R}$ in the previous argument by the total space of the real line bundle given by the kernel of ω_1 . \square

4. VECTOR BUNDLE ISOMORPHISM

Let $\tilde{\omega}$ be a closed 2-form of maximal rank in $M \times \mathbb{R}$, and let L be the line field on $M \times \mathbb{R}$ given by the kernel of $\tilde{\omega}$ at each point. By orientability of M , the line bundle L is trivializable. Let \mathcal{L} be the 1-dimensional foliation corresponding to L . Choose a complementary hyperplane field H so that $T(M \times \mathbb{R}) \simeq H \oplus L$.

Let Z_0 be a separating hypersurface in M with a coorientation. Since by Lemma 1 we have that

$${}^{Z_0}TM \oplus \mathbb{R} \simeq TM \oplus \mathbb{R} \simeq i^*(H \oplus L) ,$$

the restriction i^*H is stably isomorphic to ${}^{Z_0}TM$. The Stiefel-Whitney classes are stable invariants, and the mod 2 reduction of the Euler class of an orientable rank m real vector bundle E coincides with the m th Stiefel-Whitney class of E (see, for instance, [23]). Therefore the Euler numbers (i.e., the evaluations of the Euler classes over the fundamental homology class) of i^*H and of ${}^{Z_0}TM$ differ by an even integer, let us say

$$\chi(i^*H) = \chi({}^{Z_0}TM) + 2k .$$

If two stably isomorphic orientable rank $2n$ real vector bundles over an $2n$ -dimensional connected manifold have the same Euler number, then they are isomorphic. This was contained in the work of Dold and Whitney when the base is a 4-manifold [5]. In general, this follows from observing in the diagram

$$\begin{array}{ccc} & & S^{2n} \hookrightarrow \mathrm{SO}/\mathrm{SO}(2n) \\ & \nearrow & \downarrow \\ M^{2n} & \rightrightarrows & \mathrm{BSO}(2n) \\ & \searrow & \downarrow \\ & & \mathrm{BSO} \end{array}$$

that the fiber $\mathrm{SO}/\mathrm{SO}(2n)$ of $\mathrm{BSO}(2n) \rightarrow \mathrm{BSO}$ is $(2n - 1)$ -connected, that $[M^{2n}, S^{2n}] \simeq \mathbb{Z}$ where the homotopy type is detected by the degree, and that the pullback of the Euler class to S^{2n} is nontrivial (since $S^{2n} \rightarrow \mathrm{BSO}(2n)$ is the classifying map for TS^{2n}).

Consider the following operation on rank m real vector bundles over m -dimensional manifolds. If E is such a bundle and D^m is a small disk in the base manifold M , let $E\sharp TS^m$ be the bundle obtained by gluing $E|_{M \setminus \mathrm{Int} D^m}$ to the trivial bundle \mathbb{R}^m over D^m by the characteristic map of TS^m , i.e., by the map $S^{m-1} \rightarrow \mathrm{SO}(m)$ which characterizes the tangent bundle of S^m as the gluing over the equator of northern and southern trivial bundles [27, §18.1]. For an integer k , the bundle $E\sharp kTS^m$ is built analogously by taking the k th power of the characteristic map of S^m . By counting with orientations the vanishing points of a section transverse to zero, we see that $E\sharp kTS^m$ has

Euler characteristic $\chi(E) + 2k$. We conclude that

$$i^*H \simeq {}^Z_0TM \# kTS^{2n} .$$

For k positive, let Z be the union of Z_0 with k homologically trivial spheres S^{2n-1} contained in the negative part of $M \setminus Z_0$ with respect to the given coorientation. For k negative, define Z similarly but with the spheres in the positive part of $M \setminus Z_0$. It follows from the computations in [6, §3.9] that i^*H and Z_0TM have the same Euler number, and hence are isomorphic. It is possible to start from the empty hypersurface, in which case a coorientation is not defined. Yet the same argument holds by taking Z to be a union of spheres (as many as half of the absolute value of the difference of the Euler numbers of TM and of i^*H) whose coorientation is determined by the sign of k above. We have thus proved the following:

Lemma 3. *Let H be a coorientable hyperplane field in $M \times \mathbb{R}$ and $i : M \hookrightarrow M \times \mathbb{R}$ the inclusion at level zero. The restriction i^*H is isomorphic to Z_0TM , where Z is a separating hypersurface as described in the previous paragraph.*

5. SECOND INSTANCE OF THE H-PRINCIPLE

Throughout this section, let M be an m -dimensional manifold with a hypersurface Z , and let N be an $(m + 1)$ -dimensional manifold with a 1-dimensional foliation \mathcal{L} . The following notions are from Eliashberg [7].

Definition. *A map $f : M \rightarrow N$ is a Z -immersion relative to \mathcal{L} , if near any point $p \in M \setminus Z$ there are coordinates y_1, \dots, y_{m+1} in N adapted to the foliation (i.e., each leaf is a level set of the first m coordinates) where the induced map to each level set of y_{m+1} is regular, and if near any $p \in Z$ and near its image there are coordinates centered at those points and adapted to the foliation where f becomes*

$$(x_1, x_2, \dots, x_m) \longmapsto (x_1^2, x_2, \dots, x_m, 0) .$$

In the adapted coordinates x_i , the hypersurface Z is given by $x_1 = 0$. Loosely speaking, a Z -immersion relative to \mathcal{L} is a Z -immersion to the leaf space of \mathcal{L} . The definition extends to higher-dimensional foliations whose codimension is equal to the dimension of M .

Lemma 4. *Let $\tilde{\omega}$ be a closed 2-form of maximal rank in N whose kernel is the tangent space to the leaves of \mathcal{L} . If $f : M \rightarrow N$ is a Z -immersion relative to \mathcal{L} , then $f^*\tilde{\omega}$ is a folded symplectic form on M with folding hypersurface Z .*

The reason is simply that the form $\tilde{\omega}$ induces a symplectic form in the local leaf spaces and that the composition of f with the local quotient maps is a Z -immersion.

Proof. Let $p \in M$. There is a neighborhood \mathcal{U} of $f(p)$ where we have a trivialization $\mathcal{U} \simeq \mathcal{F}_{\mathcal{U}} \times \mathcal{L}_{\mathcal{U}}$, given in local coordinates centered at $f(p)$ by $(x_1, \dots, x_{m+1}) \mapsto ((x_1, \dots, x_m), x_{m+1})$, the set $\mathcal{F}_{\mathcal{U}}$ being a leaf space (say the level zero of x_{m+1}), and $\mathcal{L}_{\mathcal{U}}$ a typical leaf (say the level zero of (x_1, \dots, x_m)). The restriction of $\tilde{\omega}$ to $\mathcal{F}_{\mathcal{U}}$ is a symplectic form, $\omega_{\mathcal{U}}$. The composition $g_{\mathcal{U}} : f^{-1}(\mathcal{U}) \rightarrow \mathcal{F}_{\mathcal{U}}$ of f with the projection to $\mathcal{F}_{\mathcal{U}}$ is a $(Z \cap \mathcal{U})$ -immersion, so that $g_{\mathcal{U}}^* \omega_{\mathcal{U}}$ is a folded symplectic form with folding hypersurface $Z \cap \mathcal{U}$. The result follows from the fact that $f^* \tilde{\omega}$ on $f^{-1}(\mathcal{U})$ coincides with $g_{\mathcal{U}}^* \omega_{\mathcal{U}}$. \square

We now turn to the formal analogue of a Z -immersion.

Definition. A bundle map $F : TM \rightarrow TN$ is a Z -monomorphism relative to \mathcal{L} , if $F|_{T(M \setminus Z)}$ is fiberwise transverse to \mathcal{L} , and if each $p \in Z$ admits a neighborhood \mathcal{U} where $F|_{T\mathcal{U}}$ is the differential of some $(Z \cap \mathcal{U})$ -immersion relative to \mathcal{L} .

The following lemma is a direct consequence of Eliashberg's result in [7, §6.3], where he extends to the case of foliations the result described in the introduction.

Lemma 5. Let $N = M \times \mathbb{R}$ be equipped with a decomposition $TN \simeq H \oplus L$, where L is a line field, and let \mathcal{L} be the corresponding 1-dimensional foliation. Let the hypersurface Z be such that every connected component of $M \setminus Z$ is open. Then, for every Z -monomorphism $F : TM \rightarrow TN$ relative to \mathcal{L} , there exists a Z -immersion $f : M \rightarrow N$ relative to \mathcal{L} whose differential df is homotopic to F through Z -monomorphisms relative to \mathcal{L} .

Part of the work to prove Theorem A consists in showing a (general) procedure to deform by homotopy a weaker bundle map into a Z -monomorphism relative to \mathcal{L} . The weaker map is of the following type:

Definition. A bundle map $F : TM \rightarrow TN \simeq H \oplus L$ is an almost Z -monomorphism relative to L , if $\pi_L \circ F|_{T(M \setminus Z)}$ and $\pi_L \circ F|_{TZ}$ are fiberwise injective, $\pi_L : TN \rightarrow H$ being the projection along L , and if there is a tubular neighborhood \mathcal{T} of Z in M , with a fiber involution $\tau : \mathcal{T} \rightarrow \mathcal{T}$ whose set of fixed points is Z , where $F \circ d\tau = F$.

6. CONCLUSION OF THE PROOF

Let M be a compact $2n$ -dimensional manifold with a stable almost complex structure J . Then J is representable by a structure of complex vector

bundle on $TM \oplus \mathbb{R}^2$, and any two such representatives are homotopic, by Bott periodicity [2]. Let $N = M \times \mathbb{R}$ and denote still by J an induced structure of complex vector bundle on $TN \oplus \mathbb{R}$ as in § 3.

By Lemma 2, there exists on N , in any degree 2 cohomology class, a closed 2-form $\tilde{\omega}$ of maximal rank compatible with a complex vector bundle structure homotopic to J . Let $\tilde{\omega}$ be such a form and let L be the line field given by its kernel, with associated foliation \mathcal{L} .

By Lemma 4, the existence of a folded symplectic form on M with some folding hypersurface Z is guaranteed by the existence of a Z -immersion $f : M \rightarrow N$ relative to \mathcal{L} . Since $\tilde{\omega}$ is compatible with J up to homotopy, we have that $f^*\tilde{\omega}$ is consistent with the complex vector bundle structure on $f^*(TN \oplus \mathbb{R})$. We will seek such a Z -immersion which is homotopic to the embedding at level zero $i : M \hookrightarrow N$, so that $f^* = i^*$ in cohomology and $f^*\tilde{\omega}$ is consistent with J on $i^*(TN \oplus \mathbb{R}) \simeq TM \oplus \mathbb{R}^2$. If M is connected and Z is nonempty, then $M \setminus Z$ is open.

By Lemma 5, in order to produce a Z -immersion f relative to \mathcal{L} for $M \setminus Z$ open, it suffices to show that there exists a Z -monomorphism $F : TM \rightarrow TN$ relative to \mathcal{L} . So that f is homotopic to i , we search for an F covering a map $M \rightarrow N$ homotopic to i .

By Lemma 3, we have a vector bundle isomorphism $F_0 : {}^Z TM \rightarrow i^*H$ for some hypersurface Z , which may be chosen so that each connected component of $M \setminus Z$ is open.

The map F_0 may be translated into a fiberwise injective bundle map $F_1 : {}^Z TM \rightarrow H$ covering the immersion $i : M \rightarrow N$. Change F_1 by a homotopy in M collapsing a tubular neighborhood \mathcal{T} of Z . This map guarantees the existence of a (canonically unique up to homotopy) almost Z -monomorphism $F_2 : TM \rightarrow H \oplus L$ relative to L , covering a map homotopic to i , defined by the following recipe:

Choose a trivial line bundle V over Z spanned by a vector field on M transverse to Z pointing from M^- to M^+ . The quotient ${}^Z TM/V$ is isomorphic to TZ , so that ${}^Z TM|_Z \simeq TZ \oplus V$. We obtain TM by gluing ${}^Z TM|_{M \setminus M^-}$ to ${}^Z TM|_{M \setminus M^+}$ by the constant diagonal map $\text{Id} \oplus (-1) : Z \rightarrow \text{GL}(TZ \oplus V)$. Using this recovery of TM from ${}^Z TM$, we may define F_2 equal to $F_1 \oplus 0$ outside the tubular neighborhood \mathcal{T} of Z in M , and on \mathcal{T} set

$$F_2(u \oplus v) = F_1(u \oplus \psi v) \oplus 0 ,$$

with respect to the decomposition ${}^Z TM|_{\mathcal{T}} \simeq \pi^*(TZ) \oplus \pi^*V$, where $\pi : \mathcal{T} \rightarrow Z$ is the tubular projection, and $\psi : \mathcal{T} \rightarrow [0, 1]$ is equal to 1 outside a narrower tubular neighborhood of Z and vanishes exactly over Z . By choosing ψ symmetric with respect to an involution $\tau : \mathcal{T} \rightarrow \mathcal{T}$ whose set of fixed points is Z , we obtain F_2 invariant under τ .

For each $p \in Z$, choose a connected neighborhood \mathcal{U} whose image $i(\mathcal{U})$ is contained in a connected trivialization $\mathcal{N}_{\mathcal{U}} \simeq \mathcal{F}_{\mathcal{U}} \times \mathcal{L}_{\mathcal{U}}$ of the foliation \mathcal{L} , the set $\mathcal{F}_{\mathcal{U}}$ being a local leaf space and $\mathcal{L}_{\mathcal{U}}$ a leaf segment. Let $\pi_{\mathcal{U}} : \mathcal{N}_{\mathcal{U}} \rightarrow \mathcal{F}_{\mathcal{U}}$ be the projection to the first factor. The composition $F_{2,\mathcal{U}} = d\pi_{\mathcal{U}} \circ F_2|_{\mathcal{U}} : T\mathcal{U} \rightarrow T\mathcal{F}_{\mathcal{U}}$ is a $(Z \cap \mathcal{U})$ -monomorphism.

$$\begin{array}{ccc} & & T\mathcal{N}_{\mathcal{U}} \\ & \nearrow^{F_2} & \downarrow d\pi_{\mathcal{U}} \\ T\mathcal{U} & \xrightarrow{F_{2,\mathcal{U}}} & T\mathcal{F}_{\mathcal{U}} \end{array}$$

By [6, §2.2], the composition $F_{2,\mathcal{U}}$ is homotopic, through $(Z \cap \mathcal{U})$ -monomorphisms, to the differential $dg_{\mathcal{U}}$ of a Z -immersion $g_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{F}_{\mathcal{U}}$. Moreover, if over a closed subset $\mathcal{W} \subset \mathcal{U}$, the composition $F_{2,\mathcal{U}}$ was already the differential of a map, then there is a homotopy which is constant on \mathcal{W} . Let $G_t : T\mathcal{U} \rightarrow T\mathcal{F}_{\mathcal{U}}$, $1 \leq t \leq 2$, be a homotopy such that $G_1 = dg_{\mathcal{U}}$ and $G_2 = F_{2,\mathcal{U}}$.

Choose a $(Z \cap \mathcal{U})$ -immersion $\tilde{g}_{\mathcal{U}} : \mathcal{U} \rightarrow \mathcal{N}_{\mathcal{U}}$ relative to \mathcal{L} such that $\pi_{\mathcal{U}} \circ \tilde{g}_{\mathcal{U}} = g_{\mathcal{U}}$. We can always pick a $\tilde{g}_{\mathcal{U}}$ extending a sensible preassigned lift over a closed subset \mathcal{W} of \mathcal{U} .

By the covering homotopy property for the fibering $T\mathcal{N}_{\mathcal{U}} \rightarrow T\mathcal{F}_{\mathcal{U}}$, there is a lifted homotopy $\tilde{G}_t : T\mathcal{U} \rightarrow T\mathcal{N}_{\mathcal{U}}$, $1 \leq t \leq 2$, through almost Z -monomorphisms relative to L such that $\tilde{G}_1 = d\tilde{g}_{\mathcal{U}}$ and $d\pi_{\mathcal{U}} \circ \tilde{G}_t = G_t$ for all t . If G_t was constant on a closed subset \mathcal{W} , then we may choose \tilde{G}_t also constant on \mathcal{W} .

$$\begin{array}{ccc} & & T\mathcal{N}_{\mathcal{U}} \\ & \nearrow^{\tilde{G}_t} & \downarrow d\pi_{\mathcal{U}} \\ T\mathcal{U} & \xrightarrow{G_t} & T\mathcal{F}_{\mathcal{U}} \end{array}$$

Since $d\pi_{\mathcal{U}} \circ \tilde{G}_2 = G_2 = F_{2,\mathcal{U}} = d\pi_{\mathcal{U}} \circ F_2$, the difference $\tilde{G}_2 - F_2$ takes values in $L = \ker d\pi_{\mathcal{U}}$. By fiberwise homotopy, we may deform the vertical component of \tilde{G}_2 to make it equal to F_2 . Without loss of generality, we hence assume that \tilde{G}_t also satisfies $\tilde{G}_2 = F_2$, and that all maps are invariant with respect to the same involution τ .

Take a riemannian metric symmetric with respect to τ . For a point $p \in Z$, choose spherical neighborhoods \mathcal{U}_1 and \mathcal{U}_2 in \mathcal{T} , consisting of points at a riemannian distance less than ε and 4ε from p , with $\varepsilon > 0$ small enough for the exponential map to be injective and for the closure of \mathcal{U}_2 to be contained in the neighborhood \mathcal{U} above. Choose a smooth function $\rho : \mathcal{U}_2 \rightarrow [1, 2]$ satisfying $\rho(q) = 2$ if the distance from p to q is greater than 3ε , and $\rho(q) = 1$ if the distance from p to q is less than 2ε . Define $F_3 : TM \rightarrow TN$ by

$$F_3 = \begin{cases} F_2 & \text{on } M \setminus \mathcal{U}_2 \\ \tilde{G}_{\rho(q)} & \text{over points } q \in \mathcal{U}_2 \setminus \mathcal{U}_1 \\ d\tilde{g}_{\mathcal{U}} & \text{on } \mathcal{U}_1 . \end{cases}$$

Then F_3 is an almost Z -monomorphism relative to L whose restriction to \mathcal{U}_1 is the differential of a $(Z \cap \mathcal{U}_1)$ -immersion relative to \mathcal{L} .

Since Z is compact, take a subcover of Z in M by a finite number of the \mathcal{U}_1 's. Apply iteratively the construction of the previous paragraph to an ordering of the \mathcal{U}_1 's, starting first from F_2 and then from its replacements F_3 , etc. At each stage, the homotopy should be taken constant over the closure \mathcal{W} of the previous \mathcal{U}_1 's.

We have thus concluded the proof of Theorem A in the compact case by showing the existence of a Z -monomorphism relative to \mathcal{L} covering a map homotopic to i .

Remark. If M is a compact oriented 2-dimensional manifold, folded symplectic forms on M are generic 2-forms. The cohomology class of a 2-form is determined by its total integral. The homotopy classes of complex structures on $TM \oplus \mathbb{R}^2$ are determined by the Euler number, which is an even integer. By changing Z as in § 4, any even number may be obtained as Euler number for ${}^Z TM$, thus fitting any given stable complex structure. Let ω be a 2-form which vanishes transversally on an appropriate Z . By changing the values of ω over $M \setminus Z$, any real number may be obtained as total integral of ω . Hence, Theorem A holds easily (and not interestingly) for compact 2-manifolds.

For the noncompact case, a statement stronger than Theorem A is true. If a $2n$ -dimensional manifold M is orientable, connected, not compact and $TM \oplus \mathbb{R}^2$ has a complex structure, then M has an almost complex structure because it retracts to a $(2n - 1)$ -dimensional cell complex [22, Thm.8.1] and $\Pi_q(\mathrm{SO}(2n)/\mathrm{U}(n)) \simeq \Pi_q(\mathrm{SO}(2n + 2)/\mathrm{U}(n + 1))$ for $q \leq 2n - 2$. By Gromov's theorem [11], M admits a compatible symplectic form in any degree 2 cohomology class.

Let E be a rank $2m$ oriented real bundle over M . The condition $W_3(E) = 0$ ensures the existence over the 3-skeleton of M of a section for the associated $(\mathrm{SO}(2m)/\mathrm{U}(m))$ -bundle. By Bott's periodicity, $\Pi_q(\mathrm{SO}(6)/\mathrm{U}(3)) = 0$ for $q < 5, q \neq 2$. Therefore, the Hirzebruch-Hopf fact [14] that $W_3(M) = 0$ for any orientable 4-manifold, asserts the existence of a stable complex structure on any such manifold.

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