You have 180 minutes to complete this exam. Please show all work. Books, notes and calculators are not permitted, but you can cite directly the results from the textbook or the homework assignments. Have a look at all problems before starting.

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WRITE OUT AND SIGN THE PLEDGE:
I pledge my honor that I have not violated the Honor Code during this exam.
1. (11 points) Consider the vector space $V$ of all $3 \times 3$ real skew-symmetric matrices.

(a) (3 points) Find a basis of $V$. (Write down explicit matrices.)

(b) (8 points) Let $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$ with $a_1a_2a_3 \neq 0$, and $T$ be the linear transformation $T : V \rightarrow V$ defined by $T(M) = AMAT^T$. Find the matrix of $T$ with respect to the basis that you found in (a).
2. *(12 points)* For an integer \( n \geq 2 \) and numbers \( a, b, c \) with \( abc \neq 0 \), consider the \( n \times n \) matrix

\[
A = \begin{bmatrix}
  a & b & b & \cdots & b \\
  c & a & 0 & \cdots & 0 \\
  c & 0 & a & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c & 0 & 0 & \cdots & a 
\end{bmatrix}.
\]

Find the eigenvalues, a basis for each eigenspace, and the determinant of \( A \).
3. (10 points) Consider the recursion formula

\[ x_{n+1} = 2x_n + 3x_{n-1} \]

with \( x_1 = x_0 = 1 \). Find an explicit formula for \( x_n \).
4. (12 points) Let

\[ A(t) = \begin{bmatrix} 1 + t & 1 \\ -t^2 & 1 - t \end{bmatrix}, \quad B(t) = A(0) + tA'(0), \]

where \( A'(0) \) is the (entry-by-entry) derivative of \( A(t) \) at \( t = 0 \).

(a) (4 points) Show that \( B(t) \) is similar to \( A(0) \) only for \( t = 0 \).

(b) (8 points) What is the normalized Jordan form \( J \) of \( A(t) \)? Find an invertible \( 2 \times 2 \) matrix \( S = S(t) \) (depending on \( t \)) such that \( A(t) = SJS^{-1} \).
5. (20 points) True or False question. (As usual, if you think a statement is correct, prove it or give an example for an existence statement; if you think a statement is wrong, give a counterexample or prove that it’s impossible. NO POINTS WITHOUT JUSTIFICATION.)

(a) There exists an $n \times n$ real matrix $A$ such that $e^{tA} - I_n$ is singular.
(b) If $A$ and $B$ are both positive definite real symmetric matrices, then $A + B$ must be positive definite.
(c) There exists square matrices $A$ and $B$ such that $A$ is invertible and $AB - BA = A$.
(d) All the matrices in the collection

$$S = \{5 \times 5 \text{ matrices } A : A^4 = 0 \text{ but } A^3 \neq 0\}$$

are similar with each other.
(e) Assume $A$ is a $3 \times 3$ matrix that is simultaneously unitary and skew-Hermitian, then $A$ must have determinant $i$ or $-i$. 

6. (15 points) Consider the real symmetric matrix

\[
A = \begin{bmatrix}
-4 & 2 & 2 \\
2 & -1 & 4 \\
2 & 4 & a
\end{bmatrix}.
\]

Assume that \(-5\) is an eigenvalue of \(A\) with multiplicity 2.

(a) (3 points) Find the value of \(a\). (Double check your answer!)

(b) (6 points) Find a diagonal matrix \(\Lambda\) and an orthogonal matrix \(Q\) such that \(A = Q\Lambda Q^T\).

(c) (6 points) Consider the differential equation

\[
\frac{d}{dt} \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}
\]

with initial values \(x(0) = \sqrt{2}, y(0) = e^\pi, z(0) = \ln 3\). Find the limit

\[
\lim_{t \to +\infty} \frac{z(t)}{x(t)}.
\]
7. (20 points) Consider the $3 \times 3$ real skew-symmetric matrix

$$A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$$

with $a, b, c \in \mathbb{R}$ not all equal to 0.

(a) (6 points) Show that the eigenvalues of $A$ are $0$, $i\rho$ and $-i\rho$, where $i = \sqrt{-1}$ and $\rho = \sqrt{a^2 + b^2 + c^2} > 0$. Find an eigenvector of $A$ with eigenvalue $0$. Denote it by $\vec{u}$.

(b) (8 points) Let $\vec{w} + i\vec{v}$ be an eigenvector of $A$ with eigenvalue $i\rho$, where $\vec{v}, \vec{w} \in \mathbb{R}^3$. Prove that $\vec{u}$, $\vec{v}$ and $\vec{w}$ are mutually orthogonal (i.e. any two of them are perpendicular), and that they form a basis of $\mathbb{R}^3$. (You don’t have to find the explicit expression of $\vec{w} + i\vec{v}$.)

(c) (6 points) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\vec{x}) = e^A \vec{x}$, where

$$e^A = I_3 + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots.$$  

Find the matrix of $T$ with respect to the basis $\mathcal{B} = \{\vec{u}, \vec{v}, \vec{w}\}$. What’s the geometric meaning of the linear transformation $T$?