Name Solutions Class Time

MATH 202 - QUIZ # 2
Wednesday, December 1, 2010
Covers Sections 5.3-5.4, 6.1-6.3 and 7.1-7.2 of the text
Time: 45 minutes

Please show all work. Unsupported answers will receive no credit. Books, notes, calculators, are not permitted on this quiz. As part of your obligations under the Honor Code, do not discuss this quiz with anyone until Friday's class.

WRITE OUT AND SIGN THE PLEDGE:
I pledge my honor that I have not violated the Honor Code during this examination.

(a)

1. *(10 points)* Consider $A = \begin{bmatrix} 8 & -15 \\ 2 & -3 \end{bmatrix}$.

$\det = -24 + 30 = 6$

$tr = 5$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

$\boxed{\lambda_1 = 2 \quad \lambda_2 = 3.}$

(a) Find the eigenvalues of A.

(b) For each eigenvalue of A find a corresponding eigenvector $\vec{v}$.

(c) If $\vec{x}(t+1) = A\vec{x}(t)$, $x_1(0) = 1$ and $x_2(0) = 0$ then find $x_2(10)$.

(b) $E_2 = \ker \begin{bmatrix} 6 & -15 \\ 2 & -5 \end{bmatrix} = \ker \begin{bmatrix} 2 & -5 \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 5 \\ 2 \end{bmatrix} t \right\}$

$2x_1 = 5x_2$

$E_3 = \ker \begin{bmatrix} 5 & -15 \\ 2 & -6 \end{bmatrix} = \ker \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix} = \left\{ \begin{bmatrix} 3t \\ t \end{bmatrix} \right\}$

$x_1 = 3x_2$

$\boxed{\vec{v}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \text{ for eigenvalue } \lambda = 2 \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ for eigenvalue } \lambda = 3.}$

c) $\begin{bmatrix} 1 \\ 0 \end{bmatrix} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$2c_1 + c_2 = 0$

$5c_1 + 3c_2 = 1 \Rightarrow 5c_1 - 6c_1 = 1$

$c_1 = -1$

$c_2 = 2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} - \begin{bmatrix} 5 \\ 2 \end{bmatrix} \checkmark = -1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$

$A^{10} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 3^{10} \begin{bmatrix} 6 \\ 2 \end{bmatrix} - 2^{10} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$\boxed{x_2(10) = 2(3^{10} - 2^{10})}$

2. *(10 points)* Let W denote the subspace of $\mathbf{R}^4$ defined by the equations

$$x_1 - x_2 + x_3 = 0 \quad \text{and} \quad x_1 - x_4 = 0.$$

Let V denote the orthogonal complement of W.

(a) Find a matrix A so that V is the image of A.

(b) Find the orthogonal projection of $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto V.

(c) Is the point $(1,1,1,1)$ in $\mathbf{R}^4$ closer to V or to W? (Justify your answer of course.)

a) $W = \ker \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \Rightarrow V = W^\perp = \text{im} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}}_{A.}$

b) $\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^* = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{inverse} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 2/5 \\ -1/5 \end{bmatrix}.$$

Projection is $A\vec{x}^* = \begin{bmatrix} 2/5 \\ -2/5 \\ 2/5 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/5 \\ 0 \\ 0 \\ -1/5 \end{bmatrix} = \boxed{\begin{bmatrix} 1/5 \\ -2/5 \\ 2/5 \\ 1/5 \end{bmatrix}}$

c) length of projection onto V is $\frac{1}{5}\sqrt{1+4+4+1} = \frac{\sqrt{10}}{5}$

~~xxxx~~ $=$ distance from $(1,1,1,1)$ to $V^\perp = W$

length of $\vec{b} = 2$.

distance from $(1,1,1,1)$ to $V = \sqrt{\|\vec{b}\|^2 - \left\|\frac{\sqrt{10}}{5}\right\|^2} = \sqrt{4 - \frac{10}{25}} = \sqrt{}$ $= \frac{3\sqrt{10}}{5}$

$= \frac{3\sqrt{10}}{5}$ $\Rightarrow$ ~~xxxxx~~ three times as far to V as to W.

3. *(10 points)* Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 2 & 4 & 2 & -2 \\ 3 & 6 & 1 & -1 \\ -1 & 1 & -3 & 0 \end{bmatrix}$$

(a) Compute the determinant of A.

(b) Let $\vec{v}_1$ and $\vec{v}_2$ denote the first two columns of the matrix A. What is the area of the parallelogram in $\mathbf{R}^4$ spanned by $\vec{v}_1$ and $\vec{v}_2$?

(c) What is the area of the parallelogram spanned by $2\vec{v}_1 - \vec{v}_2$ and $\vec{v}_1 + 3\vec{v}_2$?

$$\det A = 2 \det \begin{bmatrix} 0 & -3 & 1 & 0 \\ 1^3 & 2^6 & 1^3 & 1^3 \\ 3 & 6 & 1 & -1 \\ -1 & 1 & -3 & 0 \end{bmatrix} = 2 \det \begin{bmatrix} 0 & -3 & 1 & 0 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & -2 & -4 \\ 0 & 3 & -2 & 1 \end{bmatrix}$$

$$= -4 \det \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & -2 & 1 \end{bmatrix} = -4 \det \begin{bmatrix} 0 & 0 & 0 & 3 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 3 & -2 & 1 \end{bmatrix}$$

$$= -12 \det \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \boxed{-36}$$

(b) $V_1 \cdot V_1 = 4 + 4 + 9 + 1 = 18$

$V_1 \cdot V_2 = 2 + 8 + 18 - 1 = 27$

$V_2 \cdot V_2 = 1 + 16 + 36 + 1 = 54$

$$\sqrt{\det \begin{bmatrix} 18 & 27 \\ 27 & 54 \end{bmatrix}} = \sqrt{9 \cdot 9 \underbrace{\det \begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}}_{3}}$$

$\boxed{\text{Area} = 9\sqrt{3}}$ or $\sqrt{243}$

(c) $$\begin{bmatrix} V_1 & V_2 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}}_{\text{magnifies area by } 7} = \begin{bmatrix} 2V_1 - V_2 & V_1 + 3V_2 \end{bmatrix}$$

new area $= \boxed{9 \cdot 7\sqrt{3}}$ or $\sqrt{11,907}$

4. (10 points) As usual, justify your answers.

(a) **T** or **F**? If A is a 4×4 matrix whose determinant is -5 then the matrix Q of the QR-factorization of A must have determinant -1.

T

$A = QR \quad \det A = \det Q \det R$

$(-5) = (\det Q)\underbrace{(\text{volume})}_{=5}$

$\Rightarrow \underline{\det Q = -1}$

(b) **T** or **F**? If A is an $n \times n$ orthogonal matrix then all its n eigenvalues must be either 1 or -1.

F

2×2 rotation $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

has no preserved lines

(unless $\theta = \pi$ or 2π)

$\Rightarrow$ complex eigenvalues.

(c) **T** or **F**? If A is a singular (i.e. not invertible) matrix of the form $\begin{bmatrix} 1-a & b & 1 \\ c & a & 2 \\ 3 & b & 1 \end{bmatrix}$, then either $a = 2b$ or $a = -2$.

T

$\det \begin{bmatrix} 1-a & b & 1 \\ c & a & 2 \\ 3 & b & 1 \end{bmatrix} = \det \begin{bmatrix} 1-a & 0 & 1 \\ c & a-2b & 2 \\ 3 & 0 & 1 \end{bmatrix}$

$= (a-2b) \underbrace{\det \begin{bmatrix} 1-a & 1 \\ 3 & 1 \end{bmatrix}}_{\substack{1-a-3 \\ -2-a}} = (a-2b)(-2-a) = 0$

$\Leftrightarrow a = 2b$ or $a = -2$