Name Solutions Class Time

MATH 202 - QUIZ # 1
Wednesday, October 6, 2010
Covers Chapters 1 and 2 of the text
Time: 45 minutes

Please show all work. Unsupported answers will receive no credit. Books, notes, calculators, are not permitted on this quiz. As part of your obligations under the Honor Code, do not discuss this quiz with anyone until after 1:30PM.

WRITE OUT AND SIGN THE PLEDGE:
I pledge my honor that I have not violated the Honor Code during this examination.

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1. *(10 points)* Consider A, the matrix of orthogonal projection onto the line L in $\mathbf{R}^2$ defined by the equation $5x_1 = 12x_2$.

   (a) Find A.

   (b) If $B = A^2 + A - I_2$, then interpret B geometrically.

L is generated by the vector $\begin{bmatrix} 12 \\ 5 \end{bmatrix}$ which has length $\sqrt{144+25} = \sqrt{169} = 13$.

$$\text{Proj}_L \vec{x} = \frac{\vec{x} \cdot \begin{bmatrix} 12 \\ 5 \end{bmatrix}}{\begin{bmatrix} 12 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 12 \\ 5 \end{bmatrix}} \begin{bmatrix} 12 \\ 5 \end{bmatrix} = \frac{12x_1 + 5x_2}{169} \begin{bmatrix} 12 \\ 5 \end{bmatrix} = \frac{1}{169} \begin{bmatrix} 144 & 60 \\ 60 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 144/169 & 60/169 \\ 60/169 & 25/169 \end{bmatrix}}$$

Because A is a matrix of orthogonal projection,

$$A(A\vec{x}) = A\vec{x} \implies A^2 + A - I = 2A - I$$

$B\vec{x} = 2(\text{Proj}_L \vec{x}) - \vec{x} =$ $\boxed{\text{Reflection of } \vec{x} \text{ across the line } L.}$

2. *(10 points)* Consider a 2×2 matrix A where

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\-1\end{bmatrix} \text{ and } A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}.$$

(a) Find A.

(b) Find A^{10}. (Hint: Think geometrically.)

There are various methods to find A – perhaps the most straightforward is

$$\underbrace{A\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}}_{\text{invert}} = \begin{bmatrix}3 & 1\\-1 & 3\end{bmatrix} \Rightarrow A = \begin{bmatrix}3 & 1\\-1 & 3\end{bmatrix}\begin{bmatrix}1/2 & 1/2\\1/2 & -1/2\end{bmatrix}$$

$$\frac{1}{-2}\begin{bmatrix}-1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}1/2 & 1/2\\1/2 & -1/2\end{bmatrix}$$

$$= \begin{bmatrix}3/2+1/2 & 3/2-1/2\\-1/2+3/2 & -1/2-3/2\end{bmatrix}$$

$$\boxed{A = \begin{bmatrix}2 & 1\\1 & -2\end{bmatrix}}$$

Geometrically A is a reflection-dilation.

$$A = \sqrt{5}\underbrace{\begin{bmatrix}2/\sqrt{5} & 1/\sqrt{5}\\1/\sqrt{5} & -2/\sqrt{5}\end{bmatrix}}_{\text{reflection}}$$

Reflection is its own inverse

so $A^2 = 5 \cdot I_2$

$$\Rightarrow \boxed{A^{10} = 5^5 \cdot I_2 = \begin{bmatrix}3125 & 0\\0 & 3125\end{bmatrix}}$$

3. *(8 points)* Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 2 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ -1 & 2 & k^2-12 & k \end{bmatrix}$$

(a) What is the smallest rank that A can have? Which values of k give that rank?

(b) For what values of k, if any, will the matrix A be invertible?

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & k^2-12 & k \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 2 & -2 & -1 \\ 0 & 2 & k^2-11 & k \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & k^2-9 & k+1 \end{bmatrix}$$

$\Rightarrow$ rank ≥ 3

$r = 3$ when $k^2 = 9$

$\boxed{\text{when } k = \pm 3}$

$\boxed{\text{The smallest possible rank is 3.}}$

rank $= 4$ provided $k^2 - 9 \neq 0$.

$\boxed{\text{A invertible when rank} = 4, \text{ when } k \text{ is any real \# except } 3, -3.}$

continuing the row reduction gives

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & k^2-9 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can always use the 1 in row 3, col 4 above to eliminate the $k+1$ entry there.

If $k^2 - 9 \neq 0$ we get rank 4, reduced form = I_4

If $k^2 - 9 = 0$ reduced form will be

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ of rank 3.

4. *(12 points)* Determine whether the given statement is true or false. If the statement is always true, then explain why. If the statement is (at least sometimes) false, then give an example to show this. You will be graded on the clarity and completeness of your answers.

(a) If A is a rotation matrix that commutes with the matrix of reflection across the line $x_1 = x_2$ in $\mathbf{R}^2$ then $A = \pm I_2$.

TRUE

$$\left.\begin{array}{l}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}\end{array}\right\} \Rightarrow \begin{array}{l} b = c \\ a = d \end{array}$$

$$\underbrace{\begin{bmatrix} a & b \\ b & a \end{bmatrix}}_{\text{Rotation matrix}}$$

$\Rightarrow a^2 + b^2 = 1 \quad b = -b$

$\Rightarrow b = 0, \ a = \pm 1$

(One could also argue geometrically.)

(b) If A is a 6×3 matrix of rank 3 and if C is any matrix satisfying $AC = 0$, then all the entries of C are zero.

TRUE

A 6×3 of rank 3 $\Rightarrow$ rref A has 3 lead 1's $\Rightarrow$ the matrix eqn $A\vec{x} = \vec{0}$ has only $\vec{x} = \vec{0}$ as a solution.
all variables are lead

$A\left[\vec{c}_1 | \dots\right] = \vec{0}$

$\Leftrightarrow A\vec{c}_i = \vec{0} \Rightarrow \vec{c}_i = \vec{0}$.

(c) There exists an invertible $n \times n$ matrix A where the sum of all the columns of A is the zero vector.

False

$$A\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{matrix}\text{col } 1 \\ \text{of } A\end{matrix} + \begin{matrix}\text{col } 2 \\ \text{of } A\end{matrix} + \dots + \begin{matrix}\text{col } n \\ \text{of } A\end{matrix} = \vec{0}$$

$\Rightarrow$ A is not a 1-1 transformation of $\mathbb{R}^n \Rightarrow A$ not invertib

$\Rightarrow A\vec{x} = \vec{0}$ has at least two solns $\Rightarrow$ ∞'ly many; free var
($\vec{x} = \vec{0}$ also works)

$\Rightarrow$ rank $A < n$
(at least one col. with missing lead 1)

(d) If $T(\vec{x}) = A\vec{x}$ is a transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$ where

$$T(\vec{e}_1) = \vec{e}_2, \quad T(\vec{e}_2) = \vec{e}_3 \quad \text{and} \quad T(\vec{e}_3) = \vec{e}_1$$

then $T^2 = T^{-1}$.

True

$$\begin{array}{ccccccc} \vec{e}_1 & \rightarrow & \vec{e}_2 & \rightarrow & \vec{e}_3 & \rightarrow & \vec{e}_1 \\ \vec{e}_2 & \rightarrow & \vec{e}_3 & \rightarrow & \vec{e}_1 & \rightarrow & \vec{e}_2 \\ \vec{e}_3 & \rightarrow & \vec{e}_1 & \rightarrow & \vec{e}_2 & \rightarrow & \vec{e}_3 \\ & T & & T & & T & \end{array}$$

$\Rightarrow$ $T^3(\vec{x}) = \vec{x}$ for all $\vec{x}$

$\Rightarrow T^2(T(\vec{x})) = \vec{x}$

So $T^2 = T^{-1}$