MATH 202 - MIDTERM EXAM
Wednesday, October 27, 2010
Covers Chapters 1,2 and 3 and Sections 5.1, 5.2
(90 minutes)

1. *(20 points)* Consider $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 4 \\ 3 & 2 & 8 & 7 \end{bmatrix}$.

   (a) Compute $rref(A)$.

   (b) Find a basis for the kernel of A.

   (c) Find a basis for the image of A.

   (d) Determine whether or not the vector $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ is in the image of A. If yes, find the coordinates of $\mathbf{b}$ with respect to your basis in part (c).

a) $\text{rref } A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

b) $\ker A = \left\{ \begin{bmatrix} -2x_3 - x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix} \right\}$ Basis $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

c) Basis for image A = $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$.

d) $\vec{b}$ is not in the image of A.

2. *(24 points)* The independent vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

form a basis $\mathcal{B}$ for $\mathbf{R}^3$. Let $\mathcal{P}$ denote the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$.

(a) Compute S^{-1} where S is the 3×3 matrix whose ith column is $\mathbf{v}_i$.

(b) If T is the linear transformation on $\mathbf{R}^3$ determined by

$$T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad T(\mathbf{v}_2) = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{v}_3) = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

then find B, the matrix of T with respect to the basis $\mathcal{B}$.

(c) Find A, the matrix of T with respect to the standard basis.

(d) If $\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$ and $\mathbf{x}$ lies on the plane $\mathcal{P}$, then what equation must the standard coordinates x_1, x_2 and x_3 of $\mathbf{x}$ satisfy?

(e) If $\mathbf{x} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,\mathbf{v}_3$ lies on $\mathcal{P}$, then what equation must the $\mathcal{B}$-coordinates c_1, c_2 and c_3 of $\mathbf{x}$ satisfy?

a) $S^{-1} = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -1 & 1 \\ 3 & -2 & 1 \end{bmatrix}$

b) $AS = SB = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$

c)

$\Rightarrow B = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & -1 \\ 7 & -3 & 2 \end{bmatrix}$

d) $2x_1 - 2x_2 + x_3 = 0$

e) $S\vec{c} = \vec{x}$ and substitute into d) $\quad$ <u>or</u> $\quad$ since $\mathcal{P}$ is spanned by $\vec{v}_2$ and $\vec{v}_3$ we simply need $c_1 = 0$.

3. *(15 points)* Let V be the subspace of $\mathbf{R}^4$ defined by the homogeneous system of equations:

$$x_1 + x_2 - x_3 - x_4 = 0 \quad \text{and} \quad 2x_1 + x_2 - 2x_3 - 2x_4 = 0.$$

(a) Find a basis for $V^\perp$.

(b) Find an orthonormal basis for V.

a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \\ -2 \end{bmatrix} \right\}$ (Our eqns say $\vec{x} \in V \Leftrightarrow \vec{x} \perp$ each of these)

b) Solving the homogeneous system,

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for V

Do G-S to find

$\vec{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$ an orthonormal basis for V.

4. *(15 points)* Consider the matrix

$$A = \begin{bmatrix} 1 & a & 0 & 1 \\ 0 & 1 & b & 0 \\ c & 0 & 1 & d \end{bmatrix}$$

(a) For which values of a, b, c and d will the rank of A be largest? What is the largest possible rank of A?

(b) For which values of a, b, c and d will the kernel of A have the largest dimension? What is the rank of A in this case?

$$A \longrightarrow \begin{bmatrix} 1 & a & 0 & 1 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1+abc & d-c \end{bmatrix} \Longrightarrow$$

rank is always ≥ 2.
rank $= 3$ if either $1+abc$ or $d-c$ is nonzero since then we get a lead 1 in row 3 as well.

$abc \neq -1$ OR $d \neq c$
gives rank 3, the largest possible.

$$\text{nullity } A = 4 - \text{rank } A = 1 \text{ if rank } A = 3$$
$$= 2 \text{ if rank } A = 2.$$

Largest kernel when rank $= 2$
i.e. when $abc = -1$ AND $c = d$.

5. *(16 points)* Determine whether the following statements are true or false. Justify your answers.

True (a) There is a 3×3 matrix A whose image is the plane $x - y - z = 0$ and whose kernel is the line defined by $x = y = z$. (If true, find A. If false, explain why no such A exists.)

False (b) If A is a 2×2 matrix and $A^2 = I_2$ then either $A = \pm I_2$ or A is the matrix of reflection across some line L. (Hint: work algebraically.)

True (c) The matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are similar.

False (d) If $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = SAS^{-1} = TAT^{-1}$ then S must be a scalar multiple of T.

a) $A = \begin{bmatrix} 1 & 1 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}$ is an example. (there are many others).

b) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & d^2+bc \end{bmatrix} = I \Rightarrow \underbrace{a=-d}_{\cancel{b, c \text{ can be chosen}}}$ or $\underbrace{b=c=0}_{\text{leads to } \pm I}$

Ex $\begin{bmatrix} 1 & b \\ 0 & -1 \end{bmatrix}$ will work for any $b \neq 0$.

these are not reflections (unless $b=0$)

c) True $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

d) A = projection onto $L = \{t\vec{u}\}$

$L^\perp = \{t\vec{v}\}$

$\mathcal{B} = \{\vec{u}, \vec{v}\}$

$\downarrow$ orthogonal projection onto L matrix wrt $\mathcal{B}$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $S = [\vec{u} | \vec{v}]$

$\mathcal{B} = \{5\vec{u}, 3\vec{v}\}$ (different choice of $L, L^\perp$ bases)

$\downarrow$ orthogonal projection onto L

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $T = [5\vec{u} | 3\vec{v}]$