1. (20 points) Consider \( A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 4 \\ 3 & 2 & 8 & 7 \end{bmatrix} \).

(a) Compute \( \text{rref}(A) \).
(b) Find a basis for the kernel of \( A \).
(c) Find a basis for the image of \( A \).
(d) Determine whether or not the vector \( b = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 1 \end{bmatrix} \) is in the image of \( A \). If yes, find the coordinates of \( b \) with respect to your basis in part (c).
2. (24 points) The independent vectors
\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}
\]
form a basis \( \mathcal{B} \) for \( \mathbb{R}^3 \). Let \( \mathcal{P} \) denote the plane spanned by \( \mathbf{v}_2 \) and \( \mathbf{v}_3 \).

(a) Compute \( S^{-1} \) where \( S \) is the \( 3 \times 3 \) matrix whose \( i \)th column is \( \mathbf{v}_i \).

(b) If \( T \) is the linear transformation on \( \mathbb{R}^3 \) determined by
\[
T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad T(\mathbf{v}_2) = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{v}_3) = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}
\]
then find \( B \), the matrix of \( T \) with respect to the basis \( \mathcal{B} \).

(c) Find \( A \), the matrix of \( T \) with respect to the standard basis.

(d) If \( \mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3 \) and \( \mathbf{x} \) lies on the plane \( \mathcal{P} \), then what equation must the standard coordinates \( x_1, x_2 \) and \( x_3 \) of \( \mathbf{x} \) satisfy?

(e) If \( \mathbf{x} = c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + c_3 \mathbf{v}_3 \) lies on \( \mathcal{P} \), then what equation must the \( \mathcal{B} \)-coordinates \( c_1, c_2 \) and \( c_3 \) of \( \mathbf{x} \) satisfy?
3. (15 points) Let $V$ be the subspace of $\mathbb{R}^4$ defined by the homogeneous system of equations:

$$x_1 + x_2 - x_3 - x_4 = 0 \quad \text{and} \quad 2x_1 + x_2 - 2x_3 - 2x_4 = 0.$$ 

(a) Find a basis for $V^\perp$.

(b) Find an orthonormal basis for $V$. 
4. (15 points) Consider the matrix

\[ A = \begin{bmatrix}
1 & a & 0 & 1 \\
0 & 1 & b & 0 \\
c & 0 & 1 & d
\end{bmatrix} \]

(a) For which values of \(a, b, c\) and \(d\) will the rank of \(A\) be largest? What is the largest possible rank of \(A\)?

(b) For which values of \(a, b, c\) and \(d\) will the kernel of \(A\) have the largest dimension? What is the rank of \(A\) in this case?
5. (16 points) Determine whether the following statements are true or false. Justify your answers.

(a) There is a $3 \times 3$ matrix $A$ whose image is the plane $x - y - z = 0$ and whose kernel is the line defined by $x = y = z$. (If true, find $A$. If false, explain why no such $A$ exists.)

(b) If $A$ is a $2 \times 2$ matrix and $A^2 = I_2$ then either $A = \pm I_2$ or $A$ is the matrix of reflection across some line $L$. (Hint: work algebraically.)

(c) The matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are similar.

(d) If $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = SAS^{-1} = TAT^{-1}$ then $S$ must be a scalar multiple of $T$. 