MATH 202 - MIDTERM EXAM
Wednesday, October 27, 2010
Covers Chapters 1,2 and 3 and Sections 5.1, 5.2
(90 minutes)

1. *(20 points)* Consider $A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 5 & 4 \\ 3 & 2 & 8 & 7 \end{bmatrix}$.

   (a) Compute $rref(A)$.

   (b) Find a basis for the kernel of A.

   (c) Find a basis for the image of A.

   (d) Determine whether or not the vector $\mathbf{b} = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 1 \end{bmatrix}$ is in the image of A. If yes, find the coordinates of $\mathbf{b}$ with respect to your basis in part (c).

2. *(24 points)* The independent vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

form a basis $\mathcal{B}$ for $\mathbf{R}^3$. Let $\mathcal{P}$ denote the plane spanned by $\mathbf{v}_2$ and $\mathbf{v}_3$.

(a) Compute S^{-1} where S is the 3×3 matrix whose ith column is $\mathbf{v}_i$.

(b) If T is the linear transformation on $\mathbf{R}^3$ determined by

$$T(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \quad T(\mathbf{v}_2) = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and} \quad T(\mathbf{v}_3) = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

then find B, the matrix of T with respect to the basis $\mathcal{B}$.

(c) Find A, the matrix of T with respect to the standard basis.

(d) If $\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$ and $\mathbf{x}$ lies on the plane $\mathcal{P}$, then what equation must the standard coordinates x_1, x_2 and x_3 of $\mathbf{x}$ satisfy?

(e) If $\mathbf{x} = c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + c_3\,\mathbf{v}_3$ lies on $\mathcal{P}$, then what equation must the $\mathcal{B}$-coordinates c_1, c_2 and c_3 of $\mathbf{x}$ satisfy?

3. *(15 points)* Let V be the subspace of $\mathbf{R}^4$ defined by the homogeneous system of equations:

$$x_1 + x_2 - x_3 - x_4 = 0 \quad \text{and} \quad 2x_1 + x_2 - 2x_3 - 2x_4 = 0.$$

(a) Find a basis for $V^\perp$.

(b) Find an orthonormal basis for V.

4. *(15 points)* Consider the matrix

$$A = \begin{bmatrix} 1 & a & 0 & 1 \\ 0 & 1 & b & 0 \\ c & 0 & 1 & d \end{bmatrix}$$

(a) For which values of a, b, c and d will the rank of A be largest? What is the largest possible rank of A?

(b) For which values of a, b, c and d will the kernel of A have the largest dimension? What is the rank of A in this case?

5. *(16 points)* Determine whether the following statements are true or false. Justify your answers.

(a) There is a 3×3 matrix A whose image is the plane $x - y - z = 0$ and whose kernel is the line defined by $x = y = z$. (If true, find A. If false, explain why no such A exists.)

(b) If A is a 2×2 matrix and $A^2 = I_2$ then either $A = \pm I_2$ or A is the matrix of reflection across some line L. (Hint: work algebraically.)

(c) The matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are similar.

(d) If $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = SAS^{-1} = TAT^{-1}$ then S must be a scalar multiple of T.