MAT202 Final Fall 2010
(The average on this exam was 64 percent.)

1. *(14 points)* Consider the system of differential equations

$$\frac{dx_1}{dt} = x_1 + 2x_2, \quad \frac{dx_2}{dt} = kx_1 - 4x_2.$$

(a) Solve the system if $k = -2$, $x_1(0) = -3$ and $x_2(0) = 3$.

(b) Sketch the phase portrait for this system when $k = -2$.

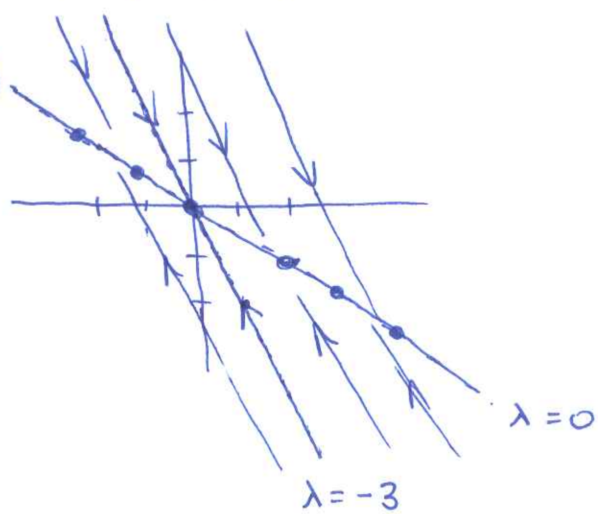
(c) For which values of k will the trajectories in the phase portrait be spirals into the origin? spirals out of the origin? Explain.

a) e'vals $0, -3$ $\Rightarrow$ $\vec{x}(t) = \begin{bmatrix} -2 - e^{-3t} \\ 1 + 2e^{-3t} \end{bmatrix}$

e'vec's $\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\begin{bmatrix} -3 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

b)



c) eigenvalues will be complex when $k < -25/8$; then the real part of the e'vals will always be $-3/2$ so the trajectories will spiral into the origin, never out.

2. *(14 points)* Let R be the region in the plane defined by the inequality

$$\frac{x_1^2}{16} + \frac{x_2^2}{4} \leq 1.$$

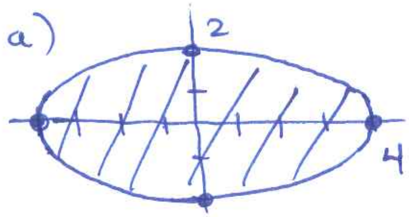
(a) Sketch the region R.

(b) If $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then sketch the transformed region $T(R)$. (No calculations are needed here. Just think about the geometric meaning of the transformation T.)

(c) Calculate A^{10}.

(d) If $B = \begin{bmatrix} -1 & 1 \\ -5 & 3 \end{bmatrix}$, then calculate B^{10}.

a)



b) Rotate CCW by $\pi/4$ or 45° semimajor axis along the line $x_1 = x_2$, length increases from 4 to $4\sqrt{2} \cong 6$; semiminor axis along $x_1 = -x_2$, length $2\sqrt{2} \cong 3$.

c) $A = \sqrt{2} \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix} \Rightarrow A^{10} = 2^5 \begin{bmatrix} \cos 5\pi/2 & -\sin 5\pi/2 \\ \sin 5\pi/2 & \cos 5\pi/2 \end{bmatrix}$

$= \begin{bmatrix} 0 & -32 \\ 32 & 0 \end{bmatrix}.$

d) B has eigenvalues $1 \pm i$, eigenvectors $\begin{bmatrix} 1 \\ 2 \pm i \end{bmatrix} \Rightarrow B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} A \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

$\Rightarrow B^{10} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -32 \\ 32 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1}$

$= \begin{bmatrix} -64 & 32 \\ -160 & 64 \end{bmatrix}$

3. *(14 points)* Use the method of least squares to determine the coefficients a and b of the parabola $y = ax + bx^2$ that best fits the three data points

$$(x_1, y_1) = (1, 1), \quad (x_2, y_2) = (2, 0), \quad \text{and } (x_3, y_3) = (-1, 1).$$

We want the best approximate solution to

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

multiply through by A^T and solve.

$$\begin{bmatrix} 6 & 8 \\ 8 & 18 \end{bmatrix} \begin{bmatrix} a^* \\ b^* \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

$$\Rightarrow \quad a^* = -4/11, \quad b^* = 3/11$$

$\boxed{y = \frac{3x^2}{11} - \frac{4x}{11}}$ is the best-fit parabola.

4. *(14 points)* Consider the quadratic form in 3 variables

$$Q(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2kyz.$$

(a) Find a real symmetric matrix A so that $Q(x,y,z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(b) If $k = 0$ then what are the eigenvalues of A?

(c) Find the points (x, y, z) that are closest to the origin on the surface S defined by the equation $2x^2 + 2y^2 + 2z^2 + 2xy = 1$.

a) $A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & k \\ 0 & k & 2 \end{bmatrix}$

b) $k=0$ char. equ

$(2-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$

$\boxed{\lambda = 1, 2, 3}$

c) The diagonalized form is $u^2 + 2v^2 + 3w^2$;

the new eqn for S is $u^2 + 2v^2 + 3w^2 = 1$.

closest pt to origin in (u,v,w)-coords is

$(0, 0, \pm 1/\sqrt{3})$. w-axis pts along $\pm \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$\boxed{\text{closest } (x,y,z) = \pm (1/\sqrt{6}, 1/\sqrt{6}, 0)}$

5. *(14 points)* Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ denote the standard basis of $\mathbf{R}^3$ and let $\mathcal{B}$ denote the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$$\vec{v}_1 = \vec{e}_1 + \vec{e}_2, \quad \vec{v}_2 = -2\vec{e}_1 - \vec{e}_2 + \vec{e}_3, \quad \vec{v}_3 = \vec{e}_1 - 2\vec{e}_3.$$

Let T be the linear transformation on $\mathbf{R}^3$ determined by

$$T(\vec{v}_1) = \vec{v}_1, \quad T(\vec{v}_2) = \vec{v}_1 + 2\vec{v}_2, \quad T(\vec{v}_3) = \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3.$$

(a) What is the matrix B of the transformation T with respect to the basis $\mathcal{B}$?

(b) Is there a basis $\mathcal{E}$ for $\mathbf{R}^3$ so that the matrix of T with respect to $\mathcal{E}$ will be diagonal? Explain.

(c) Find the volume of the parallelepiped with edges

$$\vec{w}_1 = T^{10}(\vec{e}_1), \quad \vec{w}_2 = T^{10}(\vec{e}_2 + \vec{e}_3), \quad \vec{w}_3 = T^{10}(\vec{e}_2 - \vec{e}_3).$$

a) $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

b) $E_1 = \ker \underbrace{\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}}_{\substack{\text{rank } 2 \\ \text{nullity } 1}}$ is 1-d.

$\Rightarrow$ no eigenbasis; T only has 2 independent eigenvectors.

not diagonalizable.

c) $\vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\downarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$\vec{e}_1, \vec{e}_2 + \vec{e}_3, \vec{e}_2 - \vec{e}_3$

det = -2 $\Rightarrow$ doubles volume.

$$\downarrow T^{10}$$

det B = 2 $\Rightarrow$ T doubles volume as well

$T^{10}(\vec{e}_1), T^{10}(\vec{e}_2 + \vec{e}_3), T^{10}(\vec{e}_2 - \vec{e}_3)$

unit cube of vol 1 $\longrightarrow$ parallelepiped of volume $\boxed{2^{11}}$

6. *(15 points)* The matrix $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & -5 & 3 \\ 1 & 1 & -3 & 2 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & \alpha & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find α and β.

(b) Find a basis for the kernel of A.

(c) Compute the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto the kernel of A.

(d) Let B denote the matrix of reflection across the kernel of A. Find the area of the triangle whose vertices are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad B\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(e) Let C be the matrix of orthogonal projection onto the orthogonal complement of the kernel of A.

True or False: $AC\vec{x} = A\vec{x}$ for every $\vec{x}$ in $\mathbf{R}^4$. (Justify your answer.)

a) $\alpha = -2, \ \beta = 1$

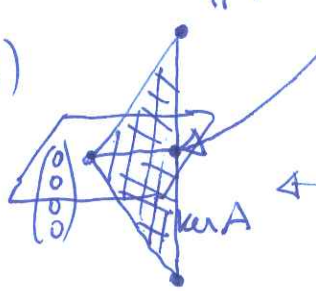
b) $\ker A = \left\{ \begin{bmatrix} r-s \\ 2r-s \\ r \\ s \end{bmatrix} \right\}$ Basis $\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

c) One method: Solve $\begin{bmatrix} 1 & -1 \\ 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ by least-squares

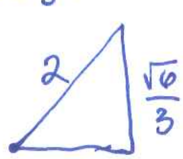
length 2

$\Rightarrow$ Desired Projection is $\begin{bmatrix} 1/3 \\ 4/3 \\ 3/3 \\ 2/3 \end{bmatrix}$. $\leftarrow$ length $= \frac{\sqrt{30}}{3}$

d)



$B\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$



(from Pythagorean theorem)

$\boxed{\text{Area} = 2\sqrt{5}/3}$

e) $\vec{x} = \vec{p} + \vec{n} \quad \vec{p} \in \ker A, \ \vec{n} \in (\ker A)^{\perp}$

$\Rightarrow C\vec{x} = \vec{n} \ ; \ A\vec{x} = A\vec{p} + A\vec{n} = A\vec{n}$

$\Rightarrow AC\vec{x} = A\vec{n} = A\vec{x}$ so $\boxed{\text{true}}$

7. *(15 points)* Determine whether the following statements are true or false. As usual, briefly justify your answer. Your answer will be graded on its clarity and completeness.

F (a) If $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are linearly independent vectors in $\mathbf{R}^3$ then so are $\vec{w}_1 = \vec{v}_1 + \vec{v}_2 + 2\vec{v}_3$, $\vec{w}_2 = 2\vec{v}_1 - \vec{v}_2 + \vec{v}_3$ and $\vec{w}_3 = -\vec{v}_1 + 5\vec{v}_2 + 4\vec{v}_3$.

F (b) There is a (real) 2×2 matrix B satisfying $BB^T = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$.

F (c) There is a 3×3 matrix A where both A and $A - I_3$ have a two dimensional kernel.

F (d) The matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

F (e) Let L denote the line parametrized by (t, t) in the plane. Suppose that A is a 2×2 matrix whose image is L and whose kernel is the perpendicular line $L^\perp$. Then A is the matrix of orthogonal projection onto L.

a) $\left[\vec{w}_1 | \vec{w}_2 | \vec{w}_3\right] = \left[\vec{v}_1 | \vec{v}_2 | \vec{v}_3\right] \underbrace{\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 5 \\ 2 & 1 & 4 \end{bmatrix}}_{\det = 0} \Rightarrow \det \underbrace{\left[\vec{w}_1 | \vec{w}_2 | \vec{w}_3\right]}_{\Rightarrow \text{ dependent}} = 0$

b) $\det BB^T = (\det B)^2 = \det \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = -5$

$\Rightarrow \det B = \sqrt{5}\, i \quad \rightarrow\leftarrow$

c) e'vals of A are $0, 0, \lambda$ $\quad \lambda \neq 0$

$\Rightarrow$ e'vals of $A - I$ are $-1, -1, \lambda - 1$ $\Rightarrow$ 0 can't be a double eigenvalue of $A - I$.

d) They have different determinants so they can't be similar.

e) If A = orthogonal projection onto L then $\operatorname{im} A = L$, $\ker A = L^\perp$

This is also true for projection-dilation $5A$ for example.