1. *(14 points)* Consider the system of differential equations

$$\frac{dx_1}{dt} = x_1 + 2x_2, \qquad \frac{dx_2}{dt} = kx_1 - 4x_2.$$

   (a) Solve the system if $k = -2$, $x_1(0) = -3$ and $x_2(0) = 3$.

   (b) Sketch the phase portrait for this system when $k = -2$.

   (c) For which values of k will the trajectories in the phase portrait be spirals into the origin? spirals out of the origin? Explain.

2. *(14 points)* Let R be the region in the plane defined by the inequality

$$\frac{x_1^2}{16} + \frac{x_2^2}{4} \le 1.$$

   (a) Sketch the region R.

   (b) If $T(\mathbf{x}) = A\mathbf{x}$ where $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, then sketch the transformed region $T(R)$. (No calculations are needed here. Just think about the geometric meaning of the transformation T.)

   (c) Calculate A^{10}.

   (d) If $B = \begin{bmatrix} -1 & 1 \\ -5 & 3 \end{bmatrix}$, then calculate B^{10}.

3. *(14 points)* Use the method of least squares to determine the coefficients a and b of the parabola $y = ax + bx^2$ that best fits the three data points

$$(x_1, y_1) = (1, 1), \quad (x_2, y_2) = (2, 0), \quad \text{and } (x_3, y_3) = (-1, 1).$$

4. *(14 points)* Consider the quadratic form in 3 variables

$$Q(x, y, z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2kyz.$$

   (a) Find a real symmetric matrix A so that $Q(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

   (b) If $k = 0$ then what are the eigenvalues of A?

   (c) Find the points (x, y, z) that are closest to the origin on the surface S defined by the equation $2x^2 + 2y^2 + 2z^2 + 2xy = 1$.

5. *(14 points)* Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ denote the standard basis of $\mathbf{R}^3$ and let $\mathcal{B}$ denote the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$$\vec{v}_1 = \vec{e}_1 + \vec{e}_2, \quad \vec{v}_2 = -2\vec{e}_1 - \vec{e}_2 + \vec{e}_3, \quad \vec{v}_3 = \vec{e}_1 - 2\vec{e}_3.$$

   Let T be the linear transformation on $\mathbf{R}^3$ determined by

$$T(\vec{v}_1) = \vec{v}_1, \quad T(\vec{v}_2) = \vec{v}_1 + 2\vec{v}_2, \quad T(\vec{v}_3) = \vec{v}_1 - 2\vec{v}_2 + \vec{v}_3.$$

(a) What is the matrix B of the transformation T with respect to the basis $\mathcal{B}$?

(b) Is there a basis $\mathcal{E}$ for $\mathbf{R}^3$ so that the matrix of T with respect to $\mathcal{E}$ will be diagonal? Explain.

(c) Find the volume of the parallelepiped with edges

$$\vec{w}_1 = T^{10}(\vec{e}_1), \quad \vec{w}_2 = T^{10}(\vec{e}_2 + \vec{e}_3), \quad \vec{w}_3 = T^{10}(\vec{e}_2 - \vec{e}_3).$$

6. *(15 points)* The matrix $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 2 & -1 & 0 & 1 \\ 1 & 2 & -5 & 3 \\ 1 & 1 & -3 & 2 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & \alpha & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) Find α and β.

(b) Find a basis for the kernel of A.

(c) Compute the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto the kernel of A.

(d) Let B denote the matrix of reflection across the kernel of A. Find the area of the triangle whose vertices are

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \text{and} \quad B\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(e) Let C be the matrix of orthogonal projection onto the orthogonal complement of the kernel of A.

True or False: $AC\vec{x} = A\vec{x}$ for every $\vec{x}$ in $\mathbf{R}^4$. (Justify your answer.)

7. *(15 points)* Determine whether the following statements are true or false. As usual, briefly justify your answer. Your answer will be graded on its clarity and completeness.

(a) If $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ are linearly independent vectors in $\mathbf{R}^3$ then so are $\vec{w}_1 = \vec{v}_1 + \vec{v}_2 + 2\vec{v}_3$, $\vec{w}_2 = \vec{2}v_1 - \vec{v}_2 + \vec{v}_3$ and $\vec{w}_3 = -\vec{v}_1 + 5\vec{v}_2 + 4\vec{v}_3$.

(b) There is a (real) 2×2 matrix B satisfying $BB^T = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$.

(c) There is a 3×3 matrix A where both A and $A - I_3$ have a two dimensional kernel.

(d) The matrix $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

(e) Let L denote the line parametrized by (t, t) in the plane. Suppose that A is a 2×2 matrix whose image is L and whose kernel is the perpendicular line $L^\perp$. Then A is the matrix of orthogonal projection onto L.