MATH 202 - FINAL EXAM
Friday, May 14, 2010 (3 hours)

1. *(16 points)*

   (a) Find the QR decomposition of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$.

   (b) Find the volume of $T(\Omega)$ where T is the linear transformation whose matrix is A and Ω is the unit cube in $\mathbf{R}^3$ (that is, Ω is the parallelepiped with edges $\vec{e}_1, \vec{e}_2$ and $\vec{e}_3$ in $\mathbf{R}^3$).

   (c) What is the area of $T(\Phi)$ if Φ is the face of Ω that lies in the $x_1\, x_2$-plane?

2. *(20 points)* The matrix $A = \begin{bmatrix} -7 & 3 \\ -18 & 8 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

   (a) Find the solution of the differential equation $\dfrac{d\vec{x}}{dt} = A\vec{x}(t)$ which satisfies the initial condition $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (b) Solve if $\dfrac{d\vec{x}}{dt} = (A + I)\vec{x}(t)$ and we have the same initial condition.

   (c) Solve if $\dfrac{d\vec{x}}{dt} = (A + 2I)\vec{x}(t)$ and we have the same initial condition.

   (d) Draw the phase portrait for each of these dynamical systems on the coordinate axes provided. In each case consider the following question about how the long-term behavior of the system depends on the initial conditions:

   For which initial values $\vec{x}(0)$ will it be true that $\lim_{t\to\infty} x_1(t) = \lim_{t\to\infty} x_2(t) = +\infty$?

   Indicate your conclusions by shading in the region in your phaseportraits that contains those initial values.

3. *(14 points)* Suppose that A is a 2×2 matrix with the property that

$$A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$$

   (a) Is A similar to $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$? If no, explain why not. If yes, find a matrix S so that $A = SBS^{-1}$.

   (b) Is A similar to $C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$? If no, explain why not. If yes, find a matrix $\tilde{S}$ so that $A = \tilde{S}C\tilde{S}^{-1}$.

   (c) Compute A^{10}.

4. *(12 points)* Find $x_1(t)+x_2(t)$ if

$$x_1(t+1)=x_1(t)+5x_2(t) \quad \text{and} \quad x_2(t+1)=-2x_1(t)-x_2(t) \quad \text{and} \quad x_1(0)=x_2(0)=15.$$

5. *(18 points)* Consider the quadratic form

$$q(x_1,x_2,x_3)=\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ where } A=\begin{bmatrix} 1 & -1 & b \\ -1 & 1 & c \\ b & c & 2 \end{bmatrix}$$

Assume that the matrix A has rank 3.

(a) How does the requirement that A have rank 3 restrict the possibilities for b and c?

(b) How many positive eigenvalues will the matrix A have? For which choices of b and c will the quadratic form be positive definite? negative definite? indefinite?

(c) Find all choices of A for which $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ will be an eigenvector for A.

(d) With A as in part (c) compute all the eigenvalues of A.

6. *(20 points)* Consider V, the subspace of $\mathbf{R}^5$ spanned by

$$\vec{v}_1=\begin{bmatrix} 7 \\ 8 \\ 1 \\ 2 \\ 5 \end{bmatrix}, \quad \vec{v}_2=\begin{bmatrix} 2 \\ 1 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \quad \vec{v}_3=\begin{bmatrix} 3 \\ 6 \\ -3 \\ -2 \\ 7 \end{bmatrix}$$

(a) What is the dimension of V?

(b) If A is the matrix of orthogonal projection onto V, then what is the trace of A?

(c) If B is the matrix of reflection across V, then what is the trace of B?

(d) Let C be the matrix $3A+2B$. Can C be orthogonally diagonalized? Explain.

(e) What is the determinant of C?

7. *(20 points)* Give an example of a matrix A with the given property or explain why no such matrix exists.

(a) A is a 2×2 matrix with the property that the image of A is the same as the kernel of A.

(b) A is a 3×3 matrix with the property that A is diagonalizable but A^2 is not.

(c) A is a 3×3 matrix whose image is the orthogonal complement of its kernel.

(d) A is a real symmetric matrix that is similar to $B=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 5 & -7 & 3 \end{bmatrix}$.

8. *(30 points)* Indicate your conclusions by circling **T** or **F** on the left and briefly explain the reasoning behind your opinion. Answers without supporting explanations will NOT be considered. If you have no idea, just leave it blank. We reserve the right to subtract points for verbose nonsense in extreme cases.

T or F If A and B are 2×2 matrices with the same determinant and the same trace then A is similar to B.

T or F There is a 4×4 real symmetric matrix A with eigenvalues $1, 2, 3+i$ and $3-i$.

T or F If A is an $n \times n$ matrix and $A^2 = A$ then A must be the matrix of orthogonal projection onto some subspace V of $\mathbf{R}^n$.

T or F There is a 3×3 matrix A whose entries are all either 1 or -1 and $\det A = 6$.

T or F If A is square and if the kernel of A^2 is the same as the kernel of A^3 then it must also be true that the kernel of A^3 is the same as the kernel of A^4.

T or F If A is a 2×2 matrix and all the entries of A are distinct prime numbers, then A must be invertible.

T or F If A and A^{-1} are matrices whose entries are all integers, then the determinant of A is the same as the determinant of A^{-1}.

T or F If A is a symmetric 3×3 matrix with real entries and $A^2 = 0$ then $A = 0$.

T or F If A and B are positive definite $n \times n$ real matrices, then $A + B$ is also positive definite.

T or F If A is a 2×2 matrix whose characteristic polynomial $f_A(\lambda)$ has the graph shown below, then every solution of the dynamical system $\frac{d\vec{x}}{dt} = A\vec{x}(t)$ will satisfy $\lim_{t\to\infty} \vec{x}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.