MAT 202 – Linear Algebra with Applications
FINAL EXAM – May 23, 2008

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(1) For each real number c, let A_c be the 3×4 matrix defined by

$$A_c = \begin{bmatrix} 1 & 0 & 1 & c^3 \\ 1 & c-1 & c & c^2 \\ 2 & 0 & 2 & 2c \end{bmatrix} .$$

(a) For which values of c is the rank of A_c equal to 2?
(b) For $c = 0$, solve the equation $A_0 v = b$, where

$$b = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} .$$

(2) Let V be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix} .$$

(a) Find the matrix P of orthogonal projection onto V.
(b) Find the matrix M of orthogonal projection onto the orthogonal complement, $V^\perp$.
(c) Find an orthonormal basis for $V^\perp$.

(3) Let v_1, v_2, v_3 be a basis of $\mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ a linear transformation such that:

$$T(v_1) = v_1 + 2v_2 + 3v_3 \ , \quad T(v_2) = 2v_1 + 3v_2 + 4v_3 \ , \quad T(v_3) = 3v_1 + 2v_2 + 4v_3 \ .$$

(a) What is the matrix of T with respect to the basis v_1, v_2, v_3?
(b) What is the volume of the parallelepiped defined by the vectors $T(e_1)$, $T(e_2)$, $T(e_3)$, where e_1, e_2, e_3 are the standard basis vectors in $\mathbb{R}^3$?

(4) Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} .$$

(a) Find the characteristic polynomial and all the eigenvalues (real and complex) of A.
Is A diagonalizable over the complex numbers?
(b) Calculate A^{2009}.

(5) Let A be a 3×3 symmetric matrix with diagonal entries $a_{11} = -1$, $a_{22} = -1$, $a_{33} = 1$ (we make no claims on the off-diagonal entries of A), two different eigenvalues $\lambda > \mu$ and corresponding eigenspaces $E_\lambda = \operatorname{span}(v_1, v_2)$ and $E_\mu = \operatorname{span}(v_3)$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(a) Is A positive (semi)definite, negative (semi)definite or indefinite?
Hint: What is the value of the quadratic form $q(v) = v^T A v$ on each of the standard basis vectors e_1, e_2, e_3?
What do you conclude about the signs of λ and μ?
(b) For what initial states $v(0)$ does the corresponding solution of the continuous dynamical system $\frac{dv}{dt} = Av$ have $\lim_{t \to +\infty} v(t) = 0$?
(c) Assuming that $\lambda = 1$ and $\mu = -3$, find the matrix A.

(6) Consider the continuous dynamical system

$$\begin{cases} \frac{dx}{dt} &= -x - 2y \\ \frac{dy}{dt} &= 3x + 4y \end{cases}.$$

(a) Sketch a phase portrait of the solutions of this system.
(b) Find the solution of the system with the initial condition

$$\begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

(7) Mark whether the following statements are true or false.
If you mark the correct answer, you earn 2 points (per part). If you mark the wrong answer, you **lose 1 point**, so you may get overall negative points for this problem. If you mark no answer or both answers, you get no points.
For this problem only, no justifications are needed.
(a) If A is an $m \times n$ matrix and $v_1, v_2 \in \mathbb{R}^n$ are linearly independent vectors that are not in the kernel of A, then Av_1, Av_2 must be linearly independent.
(b) The determinant of a reflection is always 1.
(c) If A and B are $n \times n$ diagonalizable matrices with the same eigenvalues and the same algebraic and geometric multiplicities, then A and B must be similar.
(d) If an $n \times n$ matrix A admits an eigenbasis, then the matrix $A - 7I_n$ must also admit an eigenbasis.
(e) All upper triangular matrices are diagonalizable over the complex numbers.
(f) If a 2×2 matrix A has two negative eigenvalues, then all solutions $v(t)$ of the discrete dynamical system $v(t+1) = Av(t)$ must have $\lim_{t \to +\infty} v(t) = 0$.
(g) For any 3×3 real matrix A there is a real number λ such that $A + \lambda I_n$ is not invertible.
(h) The eigenvalues of an orthogonal matrix are always real.
(i) Any symmetric $n \times n$ matrix A with $A^2 = A$ represents an orthogonal projection onto a subspace of $\mathbb{R}^n$.
(j) If for a real 2×2 matrix A the dynamical system $\frac{dx}{dt} = Ax$ has a trajectory that spirals around the origin, then A must be similar to a rotation-scaling matrix.

(8) Answer the following two questions with justification.
No credit is given for simply stating "Yes" or "No" even if correct.
(a) Does there exist a 2×2 matrix A with $A^3 = 0$ but $A^2 \neq 0$?
(b) Does there exist a 3×3 real matrix B such that $B^2 = A$ where $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$?