TEACHING OF STATISTICS
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Statistical Investigations as a Tool in Undergraduate Mathematics Research

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Abstract

Many widely believed conjectures have little numerical evidence. With the increasing availability of cheap computational power, many interesting cases are within the reach of undergraduates. Further, the algorithms necessary to investigate these problems are often interesting in their own right, and provide an excellent opportunity to introduce students to research problems.

This provides an ideal situation to involve undergraduates in cutting edge research. They numerically and theoretically investigate active areas of mathematics, seeing both what we do, and what it is like to do it. For the faculty, instead of teaching another standard cookbook class, they are able to lecture and guide work in their research areas.

We will discuss the new Undergraduate Mathematics Laboratory at Princeton / Junior Research Seminar (design and implementation, from both the faculty and student perspectives), as well as some of the results obtained.

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Evolution of Jr Research Seminar

$2000 - 2001$:

- 1 **Senior Faculty (Peter Sarnak)**
- 1 **Graduate Student (Steven Miller)**
- 8 **Undergraduates**
- **Topics: Random Matrix Theory, Random Graphs, Circle Method, Elliptic Curves, $\{n^2\alpha\}$, Primes.**
- **Staff to Student Ratio: $1 : 4$.**

$2001 - 2002$:

- 1 **Senior Faculty (Andrew Wiles)**
- 1 **Junior Faculty (Steven Miller)**
- 11 **Undergraduates**
- **Topics in Elliptic Curves (Analytic Rank, Geometric Rank, One-Parameter Families, Distribution of Signs, Spacings of Zeros, Cryptography).**
- **Staff to Student Ratio: $1 : 5.5$.**

Evolution of Jr Research Seminar (Cont)

Fall 2002:

- 2 **Junior Faculty (Steven Miller, Ramin Takloo-Bighash)**
- 2 **Graduate Students (Harald Helfgott, Florin Spinu)**
- 1 **Undergraduate Computer TA (Salman Butt)**
- 16 **Undergraduates**
- **Diophantine Problems and Roth's Theorem (Continued Fractions, Poissonian Behavior, Rational Relations, Lone Runner).**
- **Staff to Student Ratio:** $\mathbf{1 : 3.2}$.

Spring 2003:

- 1 **Senior Faculty (Yakov Sinai)**
- 1 **Junior Faculty (Steven Miller)**
- $1\frac{1}{2}$ **Graduate Students (Lior Silberman full-time; Sasha Bufetov half-time; Joao Boavida in Training)**
- 1 **Undergraduate Computer TA (Atul Pokharel)**
- 16 **Undergraduates**
- **Random Matrix Theory, Random Graphs, Dynamical Systems, Interval Exchanges, Number Theory).**
- **Staff to Student Ratio:** $\mathbf{1 : 3.\overline{5}}$.

Structure of Class

Lecture Series:

1. Main Lectures (current theory, unsolved conjectures);
2. Background Lectures (needed material, ranging from Probability Theory to Complex Analysis).

Computer Labs:

1. Basic Skills lectures (Latex, remotely accessing computers).
2. Simple Projects (small assignments to be done in C, Mathematica, Maple, Matlab, Pari, or whatever the student is comfortable in).

Final Projects:

1. Alone or in small groups, students begin their investigations.
2. Personalized lectures / computer help.

Student / Faculty Perspective

Problem Selection:

Faculty provide suggestions of worthwhile problems (interesting, doable); students have input in choice.

Hands-on Dynamics:

Doing simulations makes one feel involved; very different than just reading passively.

Hard problems, intimidating: numerics makes problems seem accessible.

Motivation.

Vertical Integration:

Grad students learn how to mentor.

Dialogue (as equals, not as students) between undergrads and faculty.

Student / Faculty Perspective (Cont)

Modern Mathematics:

Opportunity to see what research math is like. First time read papers rather than textbooks.

Can go at own pace, freedom to pursue interesting avenues.

Difficulties / Trade-offs:

The more topics explained / the more details given, the less time for independent work / investigations / understanding.

Background of students: lecturing at the appropriate levels.

Diversity of intent of students.

Danger of not doing *something* quickly.

Varied computer skills.

Introduction to Continued Fractions

Continued Fraction Expansions:

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}, \quad a_i \text{ a positive integer}$$

$$= [a_0, a_1, a_2, \ldots]$$

Determination of the a_is:

1. $a_0 = [x]$**, the greatest integer at most** x**.**

2. $a_1 = \left[\frac{1}{x - a_0}\right]$**. Let** $x_1 = \frac{1}{x-a_0}$**.**

3. $a_2 = \left[\frac{1}{x_1 - a_0}\right]$**.**

And so on...

Properties of Digits

Rational Numbers:

x **is rational if and only if the continued fraction expansion of** x **terminates.**

Quadratic Irrationals:

x **is a quadratic irrational (ie, solves** $ax^2 + bx + c = 0$**) if and only if the continued fraction expansion of** x **is periodic.**

Gauss-Kuzmin Theorem:

For almost all $x \in [0, 1)$ **(in the sense of the Lebesgue measure),**

$$\lim_{n\to\infty} \textbf{Prob}\,(a_n(x) \ = \ k) \ = \ \log_2\left(1 + \frac{1}{k(k+2)}\right).$$

Let $I_{GK} \subset [0,1)$ **denote all** x **which, in the limit, have distribution of digits satisfying the Gauss-Kuzmin Law.**

Structure of $\mathbf{I}_{\mathbf{GK}}$

Algebraic Numbers:

Let $\mathcal{A}_n$ denote the algebraic numbers of degree n; ie, complex numbers x satisfying

$$f(x) \;=\; a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \;=\; 0$$

$$a_i \in \mathbb{Z}, \quad f \text{ irreducible}.$$

Fundamental Question:

Describe the numbers x in $\mathbf{I}_{\mathbf{GK}}$.

What is Known:

1. $\mathbb{Q} \cap \mathbf{I}_{\mathbf{GK}} \;=\; \phi$.
2. $\mathcal{A}_2 \cap \mathbf{I}_{\mathbf{GK}} \;=\; \phi$.
3. $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, \ldots] \notin \mathbf{I}_{\mathbf{GK}}$.

For example, note

$$\pi \;=\; [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, \ldots].$$

Is $\pi \in \mathbf{I}_{\mathbf{GK}}$.

Structure of I_{GK}: Student Investigations

Student Projects:

1. **Looking at $1,000,000$ and more digits of Algebraic Numbers of degree 3 and higher;**
2. **Special Values of Special Functions (especially the Γ-Function and $\zeta(s)$, the Riemann-Zeta Function).**

Quick Summary of Results:

Staying away from numbers known to have special continued fraction expansions, the numbers appear to obey the Gauss-Kuzmin frequencies.

Detailed statistical tests of results were performed.

Student Investigators:

Ulysses Andrews IV, John Blatz, Sarah Kapnick (extensive statistical investigations), Felice Kuan, Jeff Law, Matthew Michelini, and Christopher Taylor.

Continued Fraction Expansions

Investigator:

Dan Fishman

Problem Formulation:

Consider a sequence G_n given by

$$G_{n+1} = aG_n + bG_{n-1}, \quad a, b \in \mathbb{Z}.$$

Let

$$\tau = \lim_{n \to \infty} \frac{G_n}{G_{n-1}}.$$

What can be said about the Continued Fraction of τ, or τ^k?

Remarks:

1. **Note that for any fixed a, b, τ is a quadratic irrational. Hence it has a periodic expansion.**
2. **Note $a = b = 1$ gives the Fibonacci sequence F_n.**
3. **Note $\lim_n \frac{G_n}{G_{n-k}} = \tau^k$.**

Continued Fraction Data ($a = b = 1$, $k = 3$)

$$\frac{F_{19}}{F_{19-3}} = [4, 4, 4, 4, 4, 3]$$

$$\frac{F_{20}}{F_{20-3}} = [4, 4, 4, 4, 4, 5]$$

$$\frac{F_{21}}{F_{21-3}} = [4, 4, 4, 4, 4, 4]$$

$$\frac{F_{22}}{F_{22-3}} = [4, 4, 4, 4, 4, 4, 3]$$

$$\frac{F_{23}}{F_{23-3}} = [4, 4, 4, 4, 4, 4, 5]$$

$$\frac{F_{24}}{F_{24-3}} = [4, 4, 4, 4, 4, 4, 4]$$

The data here led us to a proof (see next page), which we generalized for many a, b and ks.

Closed Form Expansions

$$
\begin{aligned}
\frac{F_n}{F_{n-3}} &= \frac{F_{n-1}+F_{n-2}}{F_{n-3}} \\
&= \frac{2F_{n-2}+F_{n-3}}{F_{n-3}} \\
&= 1+2\frac{F_{n-2}}{F_{n-3}} \\
&= 1+\frac{2F_{n-3}+2F_{n-4}}{F_{n-3}} \\
&= 3+2\frac{F_{n-4}}{F_{n-3}} \\
&= 3+\frac{F_{n-4}+F_{n-3}-F_{n-5}}{F_{n-3}} \\
&= 4+\frac{F_{n-4}-F_{n-5}}{F_{n-3}} \\
&= 4+\frac{F_{n-6}}{F_{n-3}} \\
&= 4+\frac{1}{\frac{F_{n-3}}{F_{n-6}}}
\end{aligned}
$$

Signs of Elliptic Curves

Problem Formulation:

Consider an elliptic curve E:

$$y^2 = f(x) = x^3 + ax + b, \; a, b \in \mathbb{Z}.$$

1. **How was a deep problem in number theory made accessible through statistics to an undergraduate with one semester of lectures and basic mathematics background (real analysis, complex analysis, algebra)?**
2. **Did we really need to use statistics?**
3. **How is this different from simply number crunching?**
4. **What was the exercise useful for?**

I will try to interlace the talk with some of the results, and highlight the deeper connections where they come up.

NOTE: Not a talk about Number Theory, but about how elementary Statistics can make some problems in Number Theory accessible. Will gloss over technical details for the sake of staying on topic.

Signs of Elliptic Curves (cont)

The Problem:

We go back to our innocent friend:

An elliptic curve E:

$$y^2 = f(x) = x^3 + ax + b, \ a, b \in \mathbb{Z}.$$

Class taught us Number Theory Techniques.

Step 1: Change variables:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

Step 2: The L-Function

We now define

$$a_p = p + 1 - N_p$$

where $\mathrm{N_p}$ is the number of solutions mod p. (i.e. including the solution at infinity)

The L-function of E is defined to be

$$L(E, s) = \prod_{p|\triangle}(1 - a_p p^{-s})^{-1} \prod_{p\dagger\triangle}(1 - a_p p^{-s} + p^{1-2s})^{-1}$$

Main Point:

We need to take away from this three things: The notion of a_p, N_p and what an L-function is.

Signs of Elliptic Curves (cont)

Functional Equation:

Using something called the modularity theorem, we can prove:

Theorem: $L(E,s)$ **satisfies the following functional equation:**

$$N^{\frac{2-s}{2}}(2\pi)^{s-2}\Gamma(2-s)L(E,2-s) = W(E)N^{\frac{3}{2}}(2\pi)^{-s}\Gamma(s)L(E,s)$$

Root Number:

$W(E)$ **is called the root number of E and** $W(E) = +1$ **or** -1**.**

So we will rewrite this as

$$\Lambda(s) = \epsilon_E \Lambda(2-s)$$

Where $\epsilon_E = \pm 1$**.**

Signs of Elliptic Curves (cont)

Main Problem:

Sign-Parity Conjecture: In a family of elliptic curves, in the limit, the sign of the functional equation of a curve in the family is equally likely to be +1 or -1.

This conjecture comes from deeper unproven conjectures(BSD) in number theory, and so is quite interesting.

To access the problem statistically:

1. **find out what a family is;**
2. **determine the signs;**
3. **perform the analysis.**

In order to get numerical evidence for the sign parity conjecture, we consider a one parameter family of elliptic curves of the form

$$y^2 = x^3 + A(t)x + B(t)$$

For every value of t, we get an elliptic curve in the family. (Where the A's and B's are polynomials in t.)

Signs of Elliptic Curves (cont)

Idea of Experiment:

1. take a family of curves;
2. go through t's to some large number;
3. find the sign for each curve thereby getting a bunch of +1's and -1's;
4. see what it looks like (see if you can statistically model it).

Idea of Experiment:

Cannot prove anything by statistical analysis, we can use it to gain some intuition and perhaps identify possible interesting patterns.

Compared to a random walk.

Grouped the data into blocks, took the sums of each block, and plotted a histogram.

Two curves that we looked at were

$$y^2 = x^3 - t^2 x + t^4$$

and

$$y^2 = x^3 - t^2 x + t^2$$

among others.

Signs of Elliptic Curves (cont)

Usefulness of Statistics:

I’ve explained how a difficult conjecture was made accessible (or "how we could find a way to play around with it"). And this is precisely because it made predictions that were statistically observable.

And we also saw how some elementary statistics made it possible to access the problem and play around with it (complimented of course by theoretical lectures)

More than just Number Crunching:

Because the aim being learning math, most of the time was spent understanding the theory and the number crunching was a sort of "experimentation".

Utility of Course:

The usefulness of this exercise, as we talked about, and will talk about came from the obvious things- becoming familiar with Number Theory programming packages, mathematical analysis, statistics, and an appreciation of deeper theory.

Equidistribution of Roots

Take any polynomial f with integer coefficients, look at its roots modulo p, p a prime. For example, say

$$f(x) = x^2 + 2$$

By trial and error, we find $x = 3$ and $x = 8$ are roots modulo 11 (ie, $f(3)$ and $f(8)$ are both multiples of 11).

We normalize the roots by dividing by p. In our example, we have for $p = 11$ the normalized roots are $\frac{3}{11}$ and $\frac{8}{11}$.

For irreducible quadratics, the normalized roots (as we range over primes) are equidistributed (Duke-Friedlander-Iwaniec). We will talk about conjectures and experimental results for higher degree polynomials.